Paper Review: Risk Processes with Hawkes Claims Arrivals

Stabile, Torrisi (2010)

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Outline of Presentation

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- Introduction: Risk Processes with Hawkes Claims Arrivals
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Risk Model Background - The Cramér-Lundberg Model

The classical risk model or compound-Poisson risk model was introduced in 1903 by Filip Lundberg and has the following form:

\[ R(t) = u + ct - \sum_{n=1}^{N_t} Z_n \]

where \( u \) denotes the initial capital, \( c \) denotes the (continuous) premium rate and the number of claims in the interval \((0, t]\) is a homogeneous Poisson process \( N_t \) with rate \( \lambda \). The claims \( Z_n \) are i.i.d. positive random variables with distribution function \( G \) and mean \( \mu_G \) independent of \((N_t)\).
Risk Model Background - The Cramér-Lundberg Model

Quantities of interest:

Ruin time:
\[ \tau = \inf \{ t > 0 : R(t) < 0 \} \]

Probability of ruin in \([0, t)\):
\[ \Psi(u, t) = P(\tau \leq t | R(0) = u) = P(\inf_{0 < s \leq t} R(s) < 0) \]

Probability of ultimate ruin:
\[ \Psi(u) = \lim_{t \to \infty} \Psi(u, t) = P(\inf_{t > 0} R(t) < 0) \]

Severity of ruin: \[ |R(\tau)| \]

Distribution of the time to ruin, given that ruin occurs:
\[ P(\tau < t | \tau < \infty) \]

Net Profit Condition:
When is the mean income strictly larger than the mean outflow?

Premium Principle:
How should the premium be set against the insurance risk?
Risk Model Background - The Cramér-Lundberg Model
Depending on the claim distribution $G$, formulas for the probability of ruin are either known in closed form (e.g. for the exponential distribution) or can be approximated by Lundberg's inequality or the Cramér-Lundberg approximation. Accordingly, the net profit condition and premium principle can be explicitly stated or approximated.
Risk Model Background - Extensions of the Cramér-Lundberg Model

**Problem**: The Poisson risk model is a convenient foundation for classical risk theory, but it does not reflect the dependencies of incoming claims in reality.

Several extensions have been proposed:

The **Sparre-Andersen model / renewal risk model** introduced in 1957 replaces the Poisson process with a general renewal process $N_t$, thus allowing for claim inter-arrival times with arbitrary distribution functions. However, the lengths of the intervals between subsequent arrivals stay independent.

More recently, risk models using a **Cox process (with Poisson shot noise)** have been studied as they incorporate time-dependent intensity influenced by exogenously caused jumps ("shocks", environmental factors).
Background - The Hawkes Process
However, it has been observed that effects like contagion and clustering in financial contexts are often caused endogenously. Therefore, Hawkes processes have gained attention due to their ability to reflect endogenously caused jumps of the intensity function.
Hawkes processes have e.g. been successfully applied to construct stock price models including financial contagion, to model mid-price changes in high-frequency trading and in order book flow modelling (Bacry (2015) gives a good overview).
Thus, as we suspect the phenomenon of clustering also occurs in the context of insurance claims, it is interesting to consider a risk model with Hawkes claims arrivals, which was mentioned for the first time in Stabile et al. (2010).
Stabile et al. (2010) - Introduction

Consider the risk model

\[ R(t) = u + ct - \sum_{n=1}^{N_t} Z_n \]

where \( u \) denotes the initial capital, \( c \) denotes the (continuous) premium rate and the number of claims in the interval \((0, t] \) is denoted \( N_t = N_X(0, t] \) and follows a non-stationary Hawkes process \( X \).

The claims \( Z_n \) are i.i.d. positive random variables independent of \( X \).
Stabile et al. (2010) - Introduction
The Hawkes process $X$ is represented as a Poisson cluster process with the following structure:

a) Immigrants (cluster centers) are distributed according to a homog. Poisson process $I$ with points $X_i \in (0, \infty)$ and intensity $\lambda > 0$.

b) Each immigrant (of generation 0) $X_i$ generates a cluster $C_i = C_{X_i}$, a random set with the following branching structure: Given generations $0, 1, \ldots, n \in C_i$, each $Y \in C_i$ (of generation $n$) generates a Poisson process on $(Y, \infty)$ of offspring (of generation $n + 1$) with intensity function $\mu(\cdot - Y)$, where $\mu : (0, \infty) \to (0, \infty]$ is a non-neg. Borel function called fertility rate.

c) Given the immigrants, the centered clusters $C_i - X_i = \{Y - X_i : Y \in C_i\}$ for $X_i \in I$ are i.i.d. and independent of $I$.

d) $X$ is the union of all clusters $\bigcup_i C_i$. 
Stabile et al. (2010) - Introduction

Interpretation in the risk model context:
We observe standard claims which arrive according to the points of \( I \) and trigger other claims according to the branching structure described before.

The fertility rate is mon. decreasing, so the process has a self-exciting structure.

This risk process is closely related to the shot-noise Cox model (doubly stochastic Poisson model) studied e.g. in Albrecher (2006), with the main difference that the other model incorporates time-dependent intensity influenced by exogenously caused jumps ("shocks", environmental factors) as opposed to endogenously caused clustering in the Hawkes model.
Stabile et al. (2010) - Preliminaries

Some notation:

Ruin time: \( \tau_u = \inf\{t \geq 0 : R(t) \leq 0\} \) for \( u > 0 \)

Infinite horizon ruin probability: \( \Psi(u) = P(\tau_u < \infty) \) for \( u > 0 \)

Finite horizon ruin probability: \( \Psi(u, uz) = P(\tau_u \leq uz) \) for \( u, z > 0 \)

Let the following hold for the mean number of points in any offspring process:

\[
\hat{\mu} = \int_0^\infty \mu(t)dt \in (0, 1)
\]  

This avoids the trivial case with a.s. no offspring and prevents explosion of the process.

Equivalently, assume that the total number of points in a cluster \( S \) is finite with \( E[S] = \frac{1}{1-\hat{\mu}} \).
Recall that the Hawkes process $X$ is a simple point process with $\mathcal{F}_t^X$-stochastic intensity

$$
\lambda(t) = \lambda + \int_0^t \mu(t-s)N_X(ds)
$$

$$
= \lambda + \sum_i \mu(t-X_i)1_{(0,t]}(X_i)
$$

where $X_i$ are the points of $X$ and $\{\mathcal{F}_t^X\}$ is the filtration generated by $X$.

We denote the aggregate process by

$$
A(t) = \sum_{n=1}^{N_t} Z_n
$$

and let $\{\mathcal{F}_t^A\}$ the filtration generated by it.

We define $k(\eta) = E[e^{\eta Z}]$. 

Stabile et al. (2010) - Preliminaries
Stabile et al. (2010) - Preliminaries

A remark which will be useful later (and is generally nice to remember):

Consider the Hawkes process $X_e$ with immigrants following a homog. Poisson process with intensity $\lambda > 0$ on $(0, \infty)$ and fertility rate $\mu(\cdot)$. This process is ergodic with intensity $\frac{\lambda}{1-\hat{\mu}}$ and it holds that $N_t \leq N_{X_e}(0, t]$.

Thus it follows

$$E[N_t] \leq E[N_{X_e}(0, t)] = \frac{\lambda t}{1-\hat{\mu}} < \infty$$  \hspace{1cm} (2)

The following theorem allows us to use a measure change from our original measure $P$ to a family of prob. measures - this will be useful (and more concrete) for simulation later.
Stabile et al. (2010) - Preliminaries

Theorem 1 Assume 1 and let \( \{P_\eta\}_{\eta} : \hat{\mu}E[k(\eta)^S] < 1 \) be the family of probability measures s.t. under \( P_\eta \), \( X \) is a non-stat. Hawkes process on \((0, \infty)\) with immigrants distributed acc. to a homog. Poisson process on \((0, \infty)\) with intensity \( \lambda_\eta = \lambda E[k(\eta)^S] \) and fertility rate \( \mu_\eta(t) = \mu E[k(\eta)^S] \). The claims \( \{Z_n\} \) are i.i.d. and indep. of \( X \) and their law is absol. continuous w.r.t. their law under \( P \) with density

\[
\frac{dP_\eta(Z)}{dP(Z)}(z) = \frac{e^{\eta Z}}{k(\eta)}
\]

Then \( P_\eta << P \) on \( \{\mathcal{F}_t^A\} \), for each \( t > 0 \), with Radon-Nikodym derivative

\[
l_t^{P_\eta, P} = e^{\eta A(t)} \exp(\left( \int_0^t (1 - E[k(\eta)^S]) \lambda(s) ds \right) E[k(\eta)^S] \frac{N_t}{k(\eta)^{N_t}})
\]
Stabile et al. (2010) - Large Deviation Principle for CHP

Definition 1 (Large Deviation Principle) A family of probability measures $\{\mu_\alpha\}_{\alpha \in (0,\infty)}$ fulfils a LDP with rate function $J(\cdot)$ if for every $B \in \mathcal{B}$ the following holds:

$$
- \inf_{x \in \text{int}(B)} J(x) \leq \lim_{\alpha \to \infty} \inf\frac{1}{\alpha} \log(\mu_\alpha(B)) \leq \lim_{\alpha \to \infty} \sup\frac{1}{\alpha} \log(\mu_\alpha(B)) \leq - \inf_{x \in \overline{B}} J(x)
$$

(3)

where $\mathcal{B}$ is the Borel-$\sigma$-field on $\mathbb{R}$, $\text{int}(B)$ is the interior of $B$, $\overline{B}$ is the closure of $B$ and $J : \mathbb{R} \to [0, \infty]$ is a lower semi-continuous function, which means that the level sets $\{x \in \mathbb{R} : J(x) \leq a\}$ for $a \geq 0$ are closed. When these sets are compact, the rate function $J(\cdot)$ is called good.
Stabile et al. (2010) - Large Deviation Principle for CHP

We define the function

\[ \Lambda(\eta) = \lambda (E[k(\eta)^S] - 1) \quad \text{for} \quad \eta \in \mathbb{R} \quad (4) \]

Note: The effective domain of \( \Lambda \) is actually \((-\infty, \bar{\eta}]\) (with \( \bar{\eta} \) defined below) as here it is ensured that \( E[k(\eta)^S] < \infty \).

Now we can give a LDP for the compound Hawkes process:

**Proposition 1** Assume Equation (1) and that there exists \( \bar{\eta} > 0 : k(\bar{\eta}) = \frac{e^{\bar{\mu}-1}}{\bar{\mu}} \).

Then \( \left\{ \frac{A(t)}{t} \right\} \) satisfies a LDP on \((\mathbb{R}, \mathcal{B})\) with good rate function:

\[
\Lambda^*(x) = \sup_{\eta \in \mathbb{R}} (\eta x - \Lambda(\eta)) = \begin{cases} 
  x \eta x + \lambda - \frac{\lambda k(\eta x) x}{\lambda k'(\eta x) + \mu k(\eta x) x} & \text{if } x \in (0, \infty). \\
  \lambda & \text{if } x = 0. \\
  +\infty & \text{if } x \in (-\infty, 0).
\end{cases}
\]
Stabile et al. (2010) – Large Deviation Principle for CHP

where $k'(\eta)$ is the first-order derivative of $k(\eta)$ and $\eta = \eta_x$ is the unique solution of

$$\Lambda'(\eta) = x \quad \text{for } x > 0$$
on $(\infty, \bar{\eta})$ or equivalently of

$$E[k(\eta)^S] = \frac{k(\eta)x}{\lambda k'(\eta) + \hat{\mu}k(\eta)x}$$

Note that the mean value theorem for $\Lambda(\cdot)$ can be used to prove the existence and uniqueness of such a solution.
Stabile et al. (2010) - Large Deviation Principle for CHP

The LDP allows to state the following Lemma, which provides a Law of Large Numbers for \( \frac{A(t)}{t} \).

**Lemma 1** Under the assumptions of Proposition (1) we have that

\[
\lim_{t \to \infty} \frac{A(t)}{t} = \Lambda'(0) = \frac{\lambda E[Z]}{1 - \hat{\mu}}
\]

almost surely.
The next chapter deals with the asymptotic behaviour of ruin probabilities.

**Theorem 2** Under assumptions of Prop. (1), if additionally

\[
\frac{\lambda E[Z]}{1 - \hat{\mu}} < c < \frac{\lambda(1 - \hat{\mu})}{\hat{\mu} \eta}
\]  

(5)

then

\[
\lim_{u \to \infty} \frac{1}{u} \log (\Psi(u)) = -w
\]  

(6)

where \( w \in (0, \bar{\eta}) \) is the unique positive solution of \( \Lambda(\eta) - c\eta = 0 \).

Note the similarity to the result we saw for the Cramér-Lundberg model.
Stabile et al. (2010) - Ruin Probabilities for Infinite Horizon
Note that by Lemma 1, the left inequality corresponds to the net profit condition. The right inequality is equivalent to assuming the existence of a unique positive solution $w \in (0, \bar{\eta})$ of $\Lambda(\eta) - c\eta = 0$ which can be seen by applying the mean value theorem on the function $g(\eta) = \Lambda(\eta) - c\eta$ for $\eta \in [0, \bar{\eta}]$. 
Stabile et al. (2010) - Ruin Probabilities for Infinite Horizon

The following Lemma is mainly used to prove Theorem 2. However, it will also be relevant for simulation in the next part, as we can see that under the measure $P_w$ ruin occurs a.s.

**Lemma 2**  
*Under assumptions of Theorem 2 we have*

$$P_w(\tau_u < \infty) = 1 \quad \text{for all } u > 0.$$  \hspace{1cm} (7)

It is worth noting that as part of the proof of Theorem 2, we obtain the Cramér-Lundberg-inequality for risk processes with non-stationary Hawkes claims arrivals, namely

$$\psi(u) \leq e^{-wu} \quad \text{for all } u \geq 0.$$
Stabile et al. (2010) - An Efficient Simulation Law

The first approach to simulation of $\Psi(u)$ would be considering $n$ independent replications of $\tau_u$ under $P$ and using the crude Monte Carlo estimator

$$\hat{\Psi}(u) = \frac{1}{n} \sum_{i=1}^{n} 1\{\tau_u^{(i)} < \infty\}$$

This is problematic for two reasons:
First, under $P$, simulation of $1\{\tau_u < \infty\}$ could be impossible as we have $P(\tau_u = \infty) > 0$.
Second, the relative error of the estimator is

$$\frac{1}{\Psi(u)} \sqrt{\Psi(u)(1 - \Psi(u))} \frac{1}{n}.$$

In order to keep this error fixed, $n$ would have to grow exponentially with $u$ for $u \to \infty$ as by Theorem 2 we have $\frac{1}{\Psi(u)} \sim e^{wu}$. 
Stabile et al. (2010) - An Efficient Simulation Law

As we have seen in Lemma 2, the first problem can be solved by simulating under the new law $P_w$ as we know $P_w(\tau_u < \infty) = 1$. Thus, we consider independent replications $(l_{\tau_u}^{P,P_w}(1), ..., (l_{\tau_u}^{P,P_w}(n)$ and introduce the importance sampling estimator

$$\hat{\Psi}(u)_{P_w} = \frac{1}{n} \sum_{i=1}^{n} (l_{\tau_u}^{P,P_w}(i)$$

which is an unbiased estimator of $\Psi(u)$ under the law $P_w$ with variance

$$Var_{P_w}[\hat{\Psi}(u)_{P_w}] = \frac{EP_w[(l_{\tau_u}^{P,P_w})^2] - \Psi(u)^2}{n}$$
Stabile et al. (2010) - An Efficient Simulation Law

The following Theorem proves that the law $P_w$ is asymptotically efficient for simulation, i.e. the variance of the importance sampling estimator is minimized (Sigmund’s criterion).

**Theorem 3** Under assumptions of Theorem 2 we have that

$$\lim_{u \to \infty} \frac{1}{u} \log E_{P_w}[(l_{\tau_u}^{P,P_w})^2] = -2w$$
Stabile et al. (2010) - Ruin Probabilities for Finite Horizon
We now consider finite horizon ruin probabilities and give two
Theorems (analogously to the Theorems 2 and 3 above) for the
asymptotic behaviour and an efficient simulation law.

**Theorem 4** Under the assumptions of Theorem 2 we have

\[
\lim_{u \to \infty} \frac{1}{u} \log(\Psi(u, uz)) = -w(z) \quad \text{for all } z > 0. \quad (8)
\]

where \( w(z) = \begin{cases} 
\Lambda^*(\frac{1}{z} + c) & \text{if } z \in \left(0, \frac{1}{N'(w)-c}\right) \\
w & \text{if } z \in \left[\frac{1}{N'(w)-c}, \infty\right) 
\end{cases}\)

and \( w \) is defined

as in Theorem 2 and \( \Lambda^*(x) = \sup_{\eta \leq \bar{\eta}} (\eta x - \Lambda(\eta)) \).
Stabile et al. (2010) - Ruin Probabilities For Finite Horizon

For simulation in the finite horizon case, the only issue to be considered is efficiency. Depending on the value of $z$, we sample under two different laws in this case. For $z \in \left(0, \frac{1}{N'(w) - c}\right)$, $\exists \eta(z) \in (w, \bar{\eta})$ such that $N'(\eta(z)) - c = \frac{1}{z}$.

Thus, consider the importance sampling estimator

$$[\hat{\Psi}(u, uz)]_{P_{\eta(z)}} = \frac{1}{n} \sum_{i=1}^{n} (l_{\tau_u}^{P, P_{\eta(z)}} 1\{\tau_u \leq uz\})^{(i)}$$

with $n$ independent replications of $l_{\tau_u}^{P, P_{\eta(z)}} 1\{\tau_u \leq uz\}$ under $P_{\eta(z)}$. The estimator is unbiased and has variance

$$Var_{P_{\eta(z)}}[[\hat{\Psi}(u, uz)]_{P_{\eta(z)}}] = \frac{EP_{\eta(z)}\left[\left(l_{\tau_u}^{P, P_{\eta(z)}}\right)^2 1\{\tau_u \leq uz\}\right] - \Psi(u, uz)^2}{n}$$
Stabile et al. (2010) - Ruin Probabilities For Finite Horizon

For $z \in \left[ \frac{1}{\Lambda'(w) - c}, \infty \right)$, consider the estimator

$$[\hat{\psi}(u, uz)]_{P^w} = \frac{1}{n} \sum_{i=1}^{n} \left( l^{P^w}_{\tau_u} 1\{\tau_u \leq uz\} \right)^{(i)}$$
Stabile et al. (2010) - Ruin Probabilities For Finite Horizon

The following theorem proves the asymptotic efficiency of the two simulation laws.

**Theorem 5** Under the assumptions of Theorem (2) we have that

\[
\lim_{u \to \infty} \frac{1}{u} \log E_{P_{\eta(z)}} \left[ \left( l^{P,P_{\eta(z)}}_{\tau_u} \right)^2 \mathbf{1}\{\tau_u \leq uz\} \right] = -2z\Lambda^\star \left( \frac{1}{z} + c \right)
\]

if \( z \in \left( 0, \frac{1}{N'(w)-c} \right) \) and

\[
\lim_{u \to \infty} \frac{1}{u} \log E_{P_w} \left[ \left( l^{P,P_w}_{\tau_u} \right)^2 \mathbf{1}\{\tau_u \leq uz\} \right] = -2w
\]

if \( z \in \left[ \frac{1}{N'(w)-c}, \infty \right) \).
Stabile et al. (2010) - Numerical Illustrations

In the last part, some simulations of the ultimate ruin probability $\Psi(u)$ using the importance sampling estimator

$$\hat{\Psi}(u)_{PW} = \frac{1}{n} \sum_{i=1}^{n} (l_{P,PW}^{T}(\tau_{u})(i))$$

are presented. For ease of notation and to emphasize the dependence on the number of replications $n$, let $\sigma_{n} := \hat{\Psi}(u)_{PW}$. Furthermore, let

$$\epsilon_{n} := \text{estimated relative error of } \sigma_{n}$$

$$\hat{n} := \inf\{n : \epsilon_{n} \leq 0.05\}$$

Assume that claims are exponentially distributed with mean $\frac{1}{\beta}$ and suppose that the fertility rate of the Hawkes process is $\mu(t) = \hat{\mu}\gamma e^{-\gamma t}$, where $\beta$ and $\gamma$ are positive constants.
Stabile et al. (2010) - Numerical Illustrations

To fulfil the assumptions of Theorem 2, the parameters $\lambda$, $\hat{\mu}$ and $c$ must be chosen to fulfil:

$$0 < \hat{\mu} < 1$$

$$\frac{\lambda}{\beta(1 - \hat{\mu})} < c < \frac{\lambda(1 - \hat{\mu})e^{\hat{\mu} - 1}}{\hat{\mu}\beta(e^{\hat{\mu} - 1} - \hat{\mu})}$$

The following values for the parameters are chosen:

$\beta = 2$, $\gamma = 2$, $\hat{\mu} = 0.2$, $\lambda = 2$, $c = 3$. 
Stabile et al. (2010) - Numerical Illustrations

The estimates for the ruin probability are compared with the Cramér-Lundberg upper bound $e^{-wu}$.

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Stabile et al. (2010) - Summary and Outlook

In conclusion:
This paper considers a risk model with Hawkes arrivals and derives the asymptotic behavior of infinite and finite horizon ruin probabilities and asymptotically efficient simulation laws using that compound Hawkes process fulfills a large deviation principle and assuming light-tailed claims.
This work was extended by Zhu (2013) who considers (subexponential) heavy tailed claims.
Stabile et al. (2010) - Summary and Outlook

Dassios and Zhao (2012) consider a risk process with the arrival of claims modelled by a dynamic contagion process, generalising the Hawkes process and the Cox process with shot noise intensity and thus including both self-excited and externally excited jumps.

Swishchuk (2017) and Cheng and Seol (2018) derive diffusion approximations and thus closed-form expressions for the ruin probabilities of risk models with Hawkes claims arrivals, Swishchuk (2017) even covers the more general case of Hawkes claims arrivals and bounded functions of Markov Chain claims ($Z_n$).
References


• Dassios, A. et al. (2012). Ruin by Dynamic Contagion Claims.


• Lundberg, F. (1903). Approximerad Framställning av Sannolikehetsfunktionen, Återförsäkering av Kollektivrisker.
References


The End

Thank You!

Q&A time!