Heterogeneous Beliefs, Speculation, and the Equity Premium *

Alexander David
Haskayne School of Business
University of Calgary
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ABSTRACT

Agents with heterogeneous beliefs about fundamental growth do not share risks perfectly but instead speculate with each other on the relative accuracy of their models’ predictions. They face the risk that market prices move more in line with the trading models of competing agents than with their own. Less risk averse agents speculate more aggressively and demand higher risk premiums. My calibrated model generates countercyclical consumption volatility, earnings forecast dispersion, and cross-sectional consumption dispersion. With a risk aversion coefficient less than one, agents’ speculation causes half the observed equity premium and lowers the riskless rate by 1%.

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The representative agent paradigm with identical agents fails to take into account speculative behavior among different agents in the economy. Under this paradigm, aggregate measures of fundamentals are sufficient to measure the risks faced by agents. Given the smoothness of these aggregate variables, this paradigm requires individuals to be implausibly highly risk averse to match the observed risk premium on stocks (this was first noted in the seminal paper of Mehra and Prescott 1985). Moreover, it implies that agents assign very low values to safe assets, that is, they require a high riskless rate to be induced to clear bond markets.

An alternative viewpoint is that individuals process new information and decide on their trades with competing models of economic fundamentals. In contrast to the identical agent case, such individuals do not always attempt to share risks with each other, but in periods of large disagreement of views take bets on the relative accuracy of their models’ predictions. In addition to the fundamental risk in assets, they face the risk that market prices move more in line with the trading models of other agents than with their own. Therefore speculative strategies by individuals introduce new risk factors that affect premiums on all assets — the beliefs of each trading type. Gains and losses from speculation are on net zero-sum and are thus not captured in the statistics of aggregate data, which therefore underestimate the amount of risk faced by agents.

I construct a general equilibrium exchange economy in which two types of agents have heterogeneous beliefs. These agents agree to disagree on the parameters of fundamental processes in the economy and update their beliefs about the state of the economy differently even though they observe the same data. While I exogenously specify the models of different agents, their trading profits and survival are endogenous: agents trading with models that do not fit the data will incur losses to agents trading with better models, and their share of the market will decline over time. To simplify the terminology, I will say that the two types of agents have different models of the fundamental process, even though in my analysis they simply disagree on its parameter values. In equilibrium, they agree on the prices of assets but disagree on the decomposition of asset returns.
into their expected return and shock components. Due to this disagreement, agents speculate with each other and as a result, as in models of incomplete markets, individuals’ consumptions are not perfectly correlated. At times of higher dispersion in beliefs and lower model disagreement the premium agents demand to hold stocks is larger.

While the effect of dispersion on the risk premium is intuitive, the role of model disagreement merits further clarification. The “disagreement value” as I will refer to it, in equilibrium is the ratio of the two agents’ state price densities. Since agents agree on all prices, it is a ratio of the relative likelihood that the current data on fundamentals was generated by the model of type 2 agents (for brevity I will simply refer to these agents collectively as agent 2) rather than agent 1. The level of this variable is shown to depend on the past performance of the two agents’ models. After a period in which the models perform comparably, the ratio is near its mean value (I say there is low model disagreement), agents’ consumptions become about equal, and each agent has a larger risk exposure to price movements due to beliefs of the other type. Conversely, after a period of dominance by, say, agent 2, the ratio becomes large, agent 2 consumes a larger share of output, and the price variability due to agent 1’s beliefs declines. Therefore, in periods of high model disagreement, there is a low exposure to risks of speculation.

I calibrate my model to fundamental data (aggregate earnings and consumption) and a series of dispersion of earnings forecasts obtained from the Survey of Professional Forecasters. I extend existing methodologies of the maximum likelihood procedure for unobservable regime-switching models (Hamilton 1989, Hamilton 1994) to a Generalized Method of Moments (GMM) method that can estimate heterogeneous parameters for two groups of agents. As part of the procedure, I estimate a set of beliefs at each date that agents of each type hold of fundamentals being in a strong growth phase. The procedure puts weight on the likelihood of each type of agent observing the fundamental data, as well as the dispersion in beliefs among the two groups.

(Insert Figure [1] about here)
There are two important properties of speculation and the risk it causes for individuals. First, the amount of risk faced by individuals is *endogenous*. Thus in contrast to the existing complete market models to understand the equity premium starting with Mehra and Prescott (1985), in my model the risk premium on stocks does not monotonically increase in the risk aversion of agents. As can be seen in the left panel of Figure 1 when agents’ coefficient of relative risk aversion (CRRA) is smaller than one, the risk premium and the CRRA are negatively related. The intuition for this finding is quite simple: less risk averse agents speculate more aggressively. As seen in the right panel, my model predicts that per capita consumption volatility increases as risk aversion declines. For low levels of risk aversion, the increase in amount of risk dominates the decrease in the market price of risk causing on net a higher risk premium.

Second, speculative activity in my calibrated model undergoes sharp bursts of intensity in periods of weak fundamentals since the two agents disagree most strongly on the transition probability out of a recession state. In such periods, per capita consumption volatility and stock volatility both expand to deliver a double impact on the risk premium through the market price of risk and the amount of risk to be priced. This positive time series covariation is important for my calibrated model to generate a large average risk premium with empirically plausible levels of average per capita consumption volatility. The bursts of speculative activity generate a countercyclical cross-sectional standard deviation of consumption growth across agents which is consistent with recent empirical evidence by Storesletten, Telmer, and Yaron (2004) on this statistic. I note that my calibrated model inherits this property from the dispersion series from surveys, which is strongly countercyclical.

I note that for each level of risk aversion plotted in Figure 1, the volatility of aggregate consumption is held constant at 1% a quarter, as is standard in the literature and the main binding constraint in the equity premium literature. However, recent evidence (Attansio, Banks, and Tanner 2002, Brav, Constantinides, and Geczy 2002) suggests that the per capita volatility of quarterly consumption
growth ranges between 6 and 12%. As seen, my model is capable of generating such levels of heterogeneity in consumption growth, and I explicitly calibrate to the mid-point of this interval. Finally, as the left panel of the figure shows, the average riskless rate in my model is fairly low for levels of the CRRA around 0.5, but climbs to implausible levels when the CRRA increases beyond one. This is the riskfree rate puzzle as first documented in Weil (1989). For extreme levels of the CRRA in excess of 50, the riskless rate falls again (not shown in the plot).

In their original article, Mehra and Prescott (1985) suggest the lack of perfect insurability due to incomplete markets as a promising direction for the resolution of the puzzle. This direction is followed by Mankiw (1986). However, subsequent articles suggest that if agents are allowed to trade in stocks and bonds, they are able to diversify most of their idiosyncratic risk, since asset prices in equilibrium do respond to changes in the aggregate income distribution (Telmer 1993, Lucas 1994, Heaton and Lucas 1996). Constantinides and Duffie (1996) are able to resolve most aspects of the puzzle with permanent idiosyncratic shocks in an economy in which individuals are content to not trade in equilibrium, and are thus unable to hedge the idiosyncratic shocks. My model has three significant differences: first, individuals do not receive idiosyncratic shocks, but instead, there is idiosyncratic variation in their beliefs. As in the case of incomplete markets, their marginal rates of substitution in equilibrium are not equated because of their inability or unwillingness to insure each other from belief movements. Second, individuals trade in equilibrium. Stock and bond prices do depend on the distribution of beliefs. However, despite the consumption smoothing attained by trading in financial assets, the difference in opinions remains, and agents continue to face an exposure to the changing beliefs of other agents. Therefore, despite trading, the model’s equity premium does not shrink. Finally, trading losses that lead to endogenous shocks to consumption in my model are not permanent, and hence do not cause a trend increase in the cross-sectional standard deviation of consumption growth across agents as in Constantinides and Duffie (1996).
Models of heterogeneous beliefs have become increasingly popular in recent years. As noted by early writers in this field (seminal papers in the field are Lintner 1969, Williams 1979, Varian 1989, Harris and Raviv 1993, Kandel and Pearson 1995), these models are able to generate patterns in trading volume because agents have differing opinions and agree to disagree after observing the same observed information. This is their main advantage over models based on asymmetric information, in which beliefs converge upon observing trades. None of these papers explicitly study the relationship between the equity premium and the time-variation in consumption moments of agents, which is the subject of this paper. Dumas, Kurshev, and Uppal (2005) refer to the speculative risk of unexpected movements in competitors’ beliefs modeled in this paper as “sentiment risk.”

More directly, my analysis extends the analysis of continuous time models of heterogeneous beliefs (Detemple and Murthy 1994, Zapatero 1998, Basak 2000, Basak and Croitoru 2000, Buraschi and Jiltsov 2002) to the case of recurrent jumps in the underlying drift of diffusion process. Crucially, the dispersion process in these models monotonically declines over time and asymptotes to zero. Therefore, the dispersion of beliefs across agents will have a temporary effect on the conditional risk premium, but will be unable to match the large risk premium in long samples of data. In contrast, in my analysis, agents have different underlying models of the data generating process as opposed to differing initial priors in the above papers and the dispersion process recurrently fluctuates and leads to a large equity premium over long horizons.

I compare my results with those of other authors using models in which agents have time-separable preferences and unrestricted access to capital markets. Reitz (1988) is able to generate a large equity premium with low risk aversion if agents price “peso problem” like events, such as a 70% drop in consumption. Longstaff and Piazzesi (2003) get some improvements — half the premium with a CRRA of five — with smaller jumps. Similarly, Bansal and Yaron (2004) price the risk from small but persistent changes in the expected growth rate of fundamentals and are able to justify half the observed premium with a CRRA of 7.5. In a paper related to this one, Brennan and
Xia (2001) model the learning process of homogeneous agents about the dividend process and find that with a CRRA of 10, equity volatility assumes empirically plausible levels and they are able to generate about half the equity premium. Ait-Sahalia, Parker, and Yogo (2004) use a value of 7 to generate the entire risk premium in a model with multiple goods. With the exception of Reitz (1988), none of the other papers can lower the riskless rate. In comparison in my calibrated model the Sharpe ratio attains a value in the range of about 9 – 15 %, a little less than half the value for the aggregate US stock market, stock volatility is generated at plausible levels of about 18%, and the equity premium is about 2.5-3% when agents have a coefficient of relative risk aversion (CRRA) in the 0.4 – 0.7 range. Moreover, the average riskless return is lower by almost 1% compared to a benchmark model with homogeneous beliefs and has a very low volatility as in the data.

The plan for the remainder of this paper is as follows: In Section I, I present the basic structure of the model, and in Section II I characterize the equilibrium and find approximate solutions for asset prices and portfolio choices. A calibration of the model is provided in Section III and the performance of the model in addressing the equity premium puzzle is discussed in Section IV. I conclude in Section V. Appendix A covers essential proofs, while Appendix B extends the analysis of my model to the case where stocks are in positive net supply. Two additional appendices, the first providing a detailed description of the projection method used to solve the PDE for asset prices, and the second of the calibration methodology will be made available to readers.

I Structure of the Model

In this section, I introduce the assumptions of my economic setting.

Assumption 1: Dividends, $q_t$, evolve according to the log-normal process

$$\frac{dq_t}{q_t} = \theta_t \, dt + \sigma_q \, d\tilde{W}_t,$$  \hspace{1cm} (1)
where $W_t = (W_{1t}, W_{2t})^T$ is a two-dimensional vector of independent Weiner processes; the $1 \times 2$ constant vector $\sigma_q$ is assumed known by all investors and is constant over time. The process for $\theta_t$ is described below.

**Assumption 2:** Total output in the economy, $x_t$, evolves according to the log-normal process

$$
\frac{dx_t}{x_t} = \kappa_t \, dt + \sigma_x \, dW_t,
$$

where the process followed by $\kappa_t$ is described below and $\sigma_x$ is a $1 \times 2$ constant vector known by investors.

It is convenient to stack together the “observation” processes (2), and (1): Let $y_t = (q; x_t)$\(^T\), so that

$$
\frac{dy_t}{y_t} = \nu_t \, dt + \Sigma \, dW_t,
$$

where $\frac{dy_t}{y_t}$ is to be interpreted as “element-by-element” division, $\nu_t = (\theta_t, \kappa_t)^T$, and $\Sigma = (\sigma_q^T, \sigma_x^T)^T$. I assume that $\Sigma$ is invertible.

**Assumption 3:** $\nu_t$ follows a 2—state, continuous-time finite state Markov chain with generator matrix $\Lambda$, that is, over the infinitesimal time interval of length $\lambda_{ij} \, dt = \text{prob} \{ \nu_{t+dt} = j | \nu_t = i \}$, for $i \neq j$, and $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$. The transition matrix over any finite interval of time, $s$, is $\exp(\Lambda s)$.

Assumptions 1-3 imply that real dividends and output follow a joint log-normal model with drifts that jump intermittently between two states. I provide some specification tests for the number of states in Appendix D. Following Cecchetti, Lam, and Mark (1990) and Brennan and Xia (2001), dividends are modeled as a part of the entire output of the economy. Therefore, the claim to dividends, the stock, will be in zero net supply.\(^4\) Aggregate (across agents) consumption in the economy will equal aggregate output. Agents can take ‘bets’ on the stock price as a vehicle for risk sharing.
Assumption 4: There are $M = 2$ classes of investors. All agents have time separable utility functions over infinitely dated stochastic consumption streams:

$$U(c) = E^{(m)} \int_0^\infty \exp(-\rho s) \cdot u(c_s) ds,$$

with time discount factor $\rho$, and felicity $u(c_t) = c_t^\gamma / \gamma$. The felicity function $u(.)$ displays a constant coefficient of relative risk aversion $1 - \gamma$, and satisfies the Inada conditions $\lim_{c \to 0} u'(c) = \infty$, and $\lim_{c \to \infty} u'(c) = 0$.

Assumption 5: Agents of type $m$ are collectively endowed with a constant fraction $0 < e^{(m)} < 1$ of output $x_t$ in the economy at period $t$. The endowment at time $t$ is $\epsilon^{(m)}_t = e^{(m)} x_t$, where $x_t$ follows the process in Assumption 2, $e^{(1)} + e^{(2)} = 1$.

Assumption 6: Individuals can trade in a short-term (instantaneous) riskless security, and two long term securities, stocks and consol bonds. Stocks pay a continuous dividend described by the process in Assumption 1. Bonds pay a continuous coupon of $c$ per instant. I denote the prices of stocks and bonds at time $t$ with the letters $P_t$ and $B_t$ respectively. Investors’ portfolio choices in these assets are $w_{P_t}$ and $w_{B_t}$ respectively and I impose constraints on admissible strategies that prohibit negative wealth at any future date (see Dybvig and Huang 1989). Both stocks and bonds are in zero net supply. I note that the number of long term assets equals the number of stochastic shocks driving the economy, so markets are dynamically complete.

The last assumption makes it possible to obtain fluctuating aggregate uncertainty and dispersion, which are the objects of the investigation of this paper.

Assumption 7: Investors do not observe the realizations of the drifts, $\nu_t$. An investor of class $m$ estimates that the vector of drift parameters, $\nu$, equals $\nu^{(m)}$, and the generator matrix, $\Lambda$, equals $\Lambda^{(m)}$. Investors of each type know the parameter values of all other types.
Investors learn about the drifts from observations of fundamentals. However, they agree to disagree about the evolution of states since they have different “models” of the state processes. Their perceptions are captured in different filtrations, \( \mathcal{F}^{(m)}_t \), and probability measures, \( \mathcal{P}^{(m)}_t \), of the states of fundamentals. In the standard heterogeneous beliefs framework (for example Basak 2000) agents disagree on the prior distribution of relevant state variables. My assumption is similar to that in Harris and Raviv (1993), in which investors have different likelihood functions of the relationship between observed signals and returns on assets, and each investor is absolutely convinced that his model is correct. It is also a form of the over-confidence model of Daniel, Hirshleifer, and Subrahmanyam (2001) in which each investor places an excessively large weight on his personal model. Given the observation of \( y_t \), investors form the posterior probability \( \pi^{(m)}_{1t} = \text{prob}(\nu_t = \nu^{(m)}_1 | \mathcal{F}^{(m)}_t) \) of fundamentals being in state 1 at time \( t \). I denote conditional means with bars, for example, \( \bar{\pi}_t^{(m)} = \sum_{i=1}^2 \pi_{it}^{(m)} \).

**Lemma 1** Given an initial condition \( 0 \leq \pi_0^{(m)} \leq 1 \), the probabilities \( \pi^{(m)}_{1t} \) follow the stochastic differential equations

\[
d\pi^{(m)}_{1t} = \mu^{(m)}_{1t} dt + \sigma^{(m)}_{1t} d\tilde{W}^{(m)}_t,
\]

where

\[
\mu^{(m)}_{1t} = (\lambda^{(m)}_{12} + \lambda^{(m)}_{21})[\pi^{(m)}_1 - \pi^{(m)}_{1t}], \quad (4)
\]

\[
\sigma^{(m)}_{1t} = \pi^{(m)}_{1t}(1 - \pi^{(m)}_{1t}) (\theta^{(m)}_1 - \theta^{(m)}_2, \kappa^{(m)}_1 - \kappa^{(m)}_2) : (\Sigma^\top)^{-1}, \quad \text{and} \quad (5)
\]

\[
d\tilde{W}^{(m)}_t = \Sigma^{-1} \left( \frac{dy_t}{y_t} - E^{(m)}_t \left[ \frac{dy_t}{y_t} \right] \right) = \Sigma^{-1} (\nu_t - \bar{\pi}^{(m)}) dt + d\tilde{W}_t. \quad (6)
\]

**Proof.** See Wonham (1964) or David (1993).

The first application of this result in financial economics as well as several properties of the filtering process are derived in David (1997). In particular, \( \pi^{(m)}_{1t} \) mean reverts to its unconditional mean, \( \pi^{(m)}_1 = \lambda^{(m)}_{21} / (\lambda^{(m)}_{12} + \lambda^{(m)}_{21}) \), with a speed proportional to \( (\lambda^{(m)}_{12} + \lambda^{(m)}_{21}) \), and the volatility
of an agent of type \( m \)'s updating process is the product of his uncertainty, \( \pi_1^{(m)}(1 - \pi_1^{(m)}) \) and the signal-to-noise ratio, \( (\theta_1^{(m)} - \theta_2^{(m)}, \kappa_1^{(m)} - \kappa_2^{(m)}) \cdot (\Sigma^\top)^{-1} \).

For later reference, I rewrite the fundamental process as
\[
\frac{dy_t}{y_t} = \left( \mu_t^{(m)} + d\bar{W}_t^{(m)} \right) dt + d\tilde{W}_t^{(m)},
\]
where \( d\tilde{W}_t^{(m)} \) is an “innovations” process under the filtration of agent \( m \). Under the separation principle it can be used for dynamic optimization (see David (1997) for a discussion). The difference between the two agents’ innovations processes is given by
\[
d\tilde{W}_t^{(2)} = d\tilde{W}_t^{(1)} + \sigma_{\pi t} dt, \tag{7}
\]
where \( \sigma_{\pi t} = \Sigma^{-1}(\mu_t^{(2)} - \mu_t^{(1)}) \). As can be seen, agents’ estimated switching probabilities between drift states, \( \lambda^{(m)}_{ij} \), can be substantially different and yet \( d\tilde{W}_t^{(2)} - d\tilde{W}_t^{(1)} \) is a term of the order \( O(dt) \).

Let \( \nu^*(m) = \pi_1^{s(m)} \nu_1^{(m)} + (1 - \pi_1^{s(m)}) \nu_2^{(m)} \) be the unconditional mean of fundamental growth assessed by investor \( m \). I do not restrict \( \nu^*(1) = \nu^*(2) \); that is, the unconditional mean estimates of the two agents may differ. Since the parameter differences affect only drift rates, I show in the following corollary that the probability measures of the two agents are equivalent over any finite interval.

**Corollary 1** The restriction of agents’ probability measures \( \mathcal{P}^{(1)} \) and \( \mathcal{P}^{(2)} \) to the filtration at time \( t \), \( \mathcal{F}_t, \mathcal{P}^{(1)}_t \) and \( \mathcal{P}^{(2)}_t \), are equivalent for all \( t \in [0, \infty) \). The Radon-Nikodym derivative of \( \mathcal{P}^{(2)}_t \) with respect to \( \mathcal{P}^{(1)}_t \) is given by
\[
\varrho_t = \varrho_0 \exp \left( -1/2 \int_0^t \sigma_{s\pi s} \sigma_{t\pi t} ds + \int_0^t \sigma_{s\pi s} d\bar{W}_t^{(1)} \right),
\]
which is a martingale with respect to \( \mathcal{P}^{(1)}_t \) on the time interval \( [0, t] \) for all \( t \).

Corollary 1 implies that the two probability measures are equivalent on the filtrations of agents at any finite time \( t \). The measures may be mutually singular over the the infinite horizon when the long-term mean drifts of the two agents are not equal. I will show in the next section that the mutual singularity will not preclude the agreement on security values by the two agents, nor does it imply the existence of arbitrage opportunities.
II Market Equilibrium

A rational expectations equilibrium is a set of utility-maximizing consumption choices for each agent and conjectured prices for all securities in each date and state for each agent so that total consumption equals total output in the economy, markets clear, and agents agree on prices in all dates and states. Due to the existence of two long-lived securities, markets are dynamically complete and ensure the existence of unique Arrow-Debreu (A-D) security prices for each agent under his own filtration. In equilibrium, agents will agree on these A-D prices as well.

I first examine individuals’ consumption choice problems. Each agent maximizes the utility function in Assumption 4 subject to the budget constraint

$$E^{(m)} \left[ \int_0^\infty c_s^{(m)} \xi_s^{(m)} ds \right] \leq E^{(m)} \left[ \int_0^\infty \xi_s^{(m)} \xi_s^{(m)} ds \right] = X_0^{(m)}, \quad (8)$$

where $\xi_t^{(m)}$, his state-price density (SPD) function for consumption at $t$, is determined endogenously in equilibrium, and $X_0^{(m)}$ is the value of his endowment as specified in Assumption 5 at period 0. The necessary conditions for optimality (see Karatzas, Lehoczky, and Shreve 1987, Cox and Huang 1989) are: $u'(c_t^{(m)}) = y_m \xi_t^{(m)}$ for each $m$. 6

Using the SPD, I can write the pricing kernel for an investor of type $m$ as

$$\frac{d\xi_t^{(m)}}{\xi_t^{(m)}} = -r_t dt - \phi_t^{(m)} d\bar{W}_t^{(m)}, \quad (9)$$

in which the real rate of interest, $r_t$, and market prices of risk, $\phi_t^{(m)}$, will be determined endogenously. Given the pricing kernel in eq. (9), the equilibrium price 7 of a traded security $i$ with a non-negative payout flow $\{\delta_{it}\}$ is determined by individuals of type $m$ as

$$\xi_t^{(m)} P_{it} = E^{(m)} \left[ \int_t^\infty \xi_s^{(m)} \delta_{is} ds \bigg\vert \mathcal{F}_t^{(m)} \right]. \quad (10)$$
For equilibrium to exist, agents must agree on the level of prices at each date and state. This requirement puts restrictions on the risk premiums on securities under the measures of the different agents and the objective measure, which I provide below. For security \( i \), I can write the dynamics of the price process under the objective measure as

\[
\frac{dP_T}{P_T} = \mu_i dt + \sigma_i d\tilde{W}_t,
\]

or in terms of the information filtration of agent \( m \),

\[
\frac{dP_T}{P_T} = \tilde{\mu}_T^{(m)} dt + \sigma_i d\tilde{W}_t^{(m)}. \tag{12}
\]

Using the definition of \( \tilde{W}_t^{(m)} \) in (6), agreement by all agents on the level of prices at each date implies that

\[
\mu_i - \tilde{\mu}_T^{(m)} = \sigma_i \Sigma^{-1}(\nu_t - \tilde{\nu}_t^{(m)}), \tag{13}
\]

for each \( m \), a relationship that I will closely examine in Section IV. In addition, the expected returns of the two different agents are related by

\[
\tilde{\mu}_T^{(1)} - \tilde{\mu}_T^{(2)} = \sigma_i \Sigma^{-1}(\tilde{\nu}_T^{(1)} - \tilde{\nu}_T^{(2)}). \tag{14}
\]

I now provide a condition under which agents will agree on prices.

**Proposition 1** Agents agree on the level of prices at all dates and states if and only if

\[
(\phi_t^{(1)} - \phi_t^{(2)})^\top = \Sigma^{-1}(\tilde{\nu}_t^{(1)} - \tilde{\nu}_t^{(2)}). \tag{15}
\]

The proof is in Appendix A.

It is relevant at this stage to point out why agents agree on prices despite having different probability measures over states of fundamental growth. Essentially, investors take bets on states of fundamental growth ‘trading away’ consumption from states which they think are less likely. Agents’ first order condition for optimization, \( u'(c_t^{(m)}) = \xi_t^{(m)} \), implies that dividends in states are priced so
the large marginal utility compensates for the small probability of the state. Differences in unconditional drift rates are compatible with these valuations: at long horizons, each agent consumes nearly the whole endowment in states where the realized mean is close to the agent’s believed long-term mean but far from what other agents believe, since for agents with constant relative risk aversion, no agent can have negative consumption.

A key state variable in my analysis is the disagreement value process: \( \eta_t = \frac{\xi_t^{(1)}}{\xi_t^{(2)}} \), which is the ratio of the SPDs of the two agents. Since the SPDs are the state prices per unit probabilities assessed by the two types of agents, and agents agree on all prices (including A-D state prices), \( \eta_t \) is the ratio of the likelihoods of observing the fundamentals at date \( t \) as realizations of the models of agent 2 to agent 1. Therefore, \( \eta_t \) will increase when the observed fundamental data at that date are more likely to arise from the model of agent 2 rather than that of agent 1. In periods when \( \eta_t \) is close to its mean value, I will say there is low disagreement since the fundamentals at that date were as likely to have been a realization of either agent’s model. Conversely, \( \eta_t \) gets far from its mean value when fundamental news supports one agent type’s model over the other. In the extreme cases, when news completely supports the model of type 1(2), \( \eta_t \rightarrow 0(\infty) \). Proposition \( \Pi \) implies that

\[
\frac{d\eta_t}{\eta_t} = (\tilde{\nu}_t^{(2)} - \tilde{\nu}_t^{(1)})^\top \Sigma^{-1\top} d\tilde{W}_t^{(1)} = \sigma_\eta d\tilde{W}_t^{(1)}. \tag{16}
\]

This formulation enables me to study the evolution of the disagreement process given the history of beliefs of each agent.

As can be seen, \( \eta_t \) increases under two conditions: (i) when agent 1 has a positive surprise \( (d\tilde{W}_t^{(1)} > 0) \) and agent 2 is more optimistic than agent 1, or (ii) when agent 1 has a negative surprise and agent 2 is more pessimistic than agent 1. For example, in the calibrated equilibrium discussed in the next section, I find that (i) will hold in 57% of the sample. In these cases, positive surprises to fundamentals lend more support to the model of agent 2. In the remaining 43% of
the sample, positive surprises to fundamentals lead to decreases in $\eta$, and lend more support to the model of agent 1. The intuition of the sign of these effects is quite straightforward: for example, if agent 1 receives a positive surprise, and agent 2 is more pessimistic than agent 1, then he would receive an even larger positive surprise, and the likelihood that the data were a realization of agent 1’s model would increase ($\eta$ would decrease). The characterization explains why the correlation between agents’ consumption growths and stock returns is time-varying, and in fact switches sign over time.

The following proposition characterizes equilibrium consumptions, the riskless rate, and the market prices of risk of the two agents. As a prelude, I will require the following lemma, which characterizes the consumption processes of the two agents.

**Lemma 2** To be consistent with utility maximization, the consumption process of an individual of type $m$ follows the diffusion process $d c_t^{(m)} = \mu_t^{(m)} dt + \sigma_t^{(m)} d\tilde{W}_t^{(m)}$, with volatility and drift coefficients

$$
\begin{align*}
\sigma_t^{(m)} &= \frac{1}{a_t^{(m)}} \phi_t^{(m)} \\
\mu_t^{(m)} &= \frac{1}{a_t^{(m)}} r_t + \frac{1}{2} b_t^{(m)} \phi_t^{(m)} (\phi_t^{(m)})^\top 
\end{align*}
$$

where, $a_t^{(m)} = -u''_m(c_t^{(m)})/u'_m(c_t^{(m)}) = (1 - \gamma)/c_t^{(m)}$, and $b_t^{(m)} = -u''_m(c_t^{(m)})/u'_m(c_t^{(m)}) = (2 - \gamma)/c_t^{(m)}$.

**Proof:** In Appendix A.

The lemma shows in particular that the volatilities of individuals’ consumption growths are time-varying and equal the product of the inverse of the CRRA and the market prices of risk. Since the norm of the market prices of risk of agent $m$ at any given time $t$ will equal the conditional maximal Sharpe ratio (as perceived by agent $m$) of all assets in the economy, the volatilities of individuals’ consumption growths will summarize the information about conditional Sharpe ratios attainable by my model. As I will show though, neither the volatility of aggregate consumption
growth nor the per capita consumption volatility will be sufficient statistics for conditional Sharpe ratios. I look at \( \phi_t^{(m)} \) carefully below.

**Proposition 2** *In equilibrium,*

(i) The individual consumption flow rates are

\[
\begin{align*}
  c_t^{(1)} &= \frac{x_t}{1 + k \eta_t^{1-\gamma}}; \\
  c_t^{(2)} &= \frac{k \eta_t^{1-\gamma} x_t}{1 + k \eta_t^{1-\gamma}}.
\end{align*}
\]

where \( k = (y_1/y_2)^{\frac{1}{1-\gamma}} \).

(ii) The riskless rate in the economy is given by

\[
\begin{align*}
  r_t &= \rho - \frac{1}{2} (2 - \gamma) (1 - \gamma) \sigma_x \sigma_x^\top + \frac{1 - \gamma}{1 + k \eta_t^{1-\gamma}} \left( \tilde{\kappa}_t^{(1)} + k \eta_t^{1-\gamma} \tilde{\kappa}_t^{(2)} \right) \\
   &\quad - \frac{\gamma k \eta_t^{1-\gamma}}{2 (1 - \gamma) \left( 1 + k \eta_t^{1-\gamma} \right)^2} \left[ \left( \tilde{\theta}_t^{(1)} - \tilde{\theta}_t^{(2)} \right) \sigma_x - \left( \tilde{\kappa}_t^{(1)} - \tilde{\kappa}_t^{(2)} \right) \sigma_q \right]^2.
\end{align*}
\]

(iii) Finally, the market prices of risk of the two types of agents are

\[
\begin{align*}
  \phi_q^{(1)} &= (1 - \gamma) \sigma_{x,1} + \frac{k \eta_t^{1-\gamma}}{1 + k \eta_t^{1-\gamma}} \left( \tilde{\theta}_t^{(1)} - \tilde{\theta}_t^{(2)} \right) \sigma_{x,2} + \left( \tilde{\kappa}_t^{(2)} - \tilde{\kappa}_t^{(1)} \right) \sigma_{q,2}, \\
  \phi_q^{(2)} &= (1 - \gamma) \sigma_{x,1} + \frac{1}{1 + k \eta_t^{1-\gamma}} \left( \tilde{\theta}_t^{(2)} - \tilde{\theta}_t^{(1)} \right) \sigma_{x,2} + \left( \tilde{\kappa}_t^{(1)} - \tilde{\kappa}_t^{(2)} \right) \sigma_{q,2}, \\
  \phi_x^{(1)} &= (1 - \gamma) \sigma_{x,2} + \frac{k \eta_t^{1-\gamma}}{1 + k \eta_t^{1-\gamma}} \left( \tilde{\theta}_t^{(1)} - \tilde{\theta}_t^{(2)} \right) \sigma_{x,1} + \left( \tilde{\kappa}_t^{(2)} - \tilde{\kappa}_t^{(1)} \right) \sigma_{q,1}, \\
  \phi_x^{(2)} &= (1 - \gamma) \sigma_{x,2} + \frac{1}{1 + k \eta_t^{1-\gamma}} \left( \tilde{\theta}_t^{(2)} - \tilde{\theta}_t^{(1)} \right) \sigma_{x,1} + \left( \tilde{\kappa}_t^{(1)} - \tilde{\kappa}_t^{(2)} \right) \sigma_{q,1}.
\end{align*}
\]

*Proof.* In Appendix A.

It is important to note that the riskless rate and the market prices of risk characterized above depend on the constant \( k = (y_1/y_2)^{\frac{1}{1-\gamma}} \). Therefore, the SPD functions of the agents (eq. 5) are also dependent on \( k \). I will determine \( k \) by ensuring that the budget constraints of the agents (eq. 8) are satisfied with equality with the assumed SPDs. Since this step will involve the solution of a
PDE, I defer its discussion to section [III A]. I provide a discussion of the qualitative features of the equilibrium below.

I first make some comments on the riskless rate in eq. (20). The first two terms in this riskless rate are standard, reflecting the time preference and the precautionary demand arising from the noise in the consumption process — with higher consumption volatility, agents’ demand for riskless assets increases as they desire safer portfolios to offset risk, lowering the equilibrium real rate. The precautionary demand increases in the prudence of agents, captured by the term \((1 - \gamma) \cdot (2 - \gamma)\). The third term is the usual wealth effect on consumption: when the expected growth rate of consumption increases, agents are less willing to save for the future, leading to a higher equilibrium real rate. In my case, the expected growth rates of the two agents are weighted by their respective shares of total consumption.

(Insert Figure 2 about here)

The last term in the interest rate expression represents the ‘hedging’ demand term and is the product of two parts, which implies that it is necessary for each part to be large for an impact on the rate. The first part, \(k \eta_t \frac{1}{1 - \gamma} \left( 1 + k \eta_t \frac{1}{1 - \gamma} \right)^2\), is a concave function of \(\eta\) with a maximum at \(\eta = 1/k(1 - \gamma) = \eta_2/\eta_1\). From (i), it is the product of the shares of consumption of the two types of agents. When the relative performance of the two types has been similar, \((\eta \approx \bar{\eta})\), and I say there is low model disagreement. At such times, the share in the market of each type of agent is near a half, and each group of agents can potentially impact market prices. Therefore, agents face the risk that prices move with the beliefs of each type, or conversely, they perceive higher speculative opportunities during these times. Similarly, the second part represents speculative opportunities that arise from the dispersion in agents’ beliefs — in periods of higher dispersion, the difference in expected growth rates increases. This effect is shown in the left panel of Figure 2 in which I hold \(\eta\) constant and allow agents’ beliefs of the expansion states to diverge. As I depart farther from the diagonal line, the dispersion in expected growth rates increases so that the riskless rate falls.
In my model calibration, I find that the two agents’ beliefs are highly correlated, but in periods of increased dispersion of beliefs, the rate drops significantly, and will help achieve an overall low average riskless rate. In either case, low disagreement or high dispersion, agents’ savings response depends on their CRRA. For investors with CRRA larger than one, an improvement in opportunities makes them want to consume more currently due to a dominating wealth effect, causing a higher market clearing interest rate. Investors with CRRA less than one want to save and invest more currently due to a dominating substitution effect, leading to a lowering of the riskless rate. To obtain a low riskless rate I will require that investors are of the latter type, that is with CRRA less than one. Notably in this case, the impact of speculation is to drop the riskless rate below that of a benchmark economy in which agents have homogeneous beliefs.

Also as can be seen from eq. (20), the short rate in an economy with heterogeneous beliefs depends not only on current beliefs of agents through the terms $\kappa_t^{(m)}$ and $\theta_t^{(m)}$, but also on lagged beliefs through the disagreement value $\eta_t$. From (16), it is evident that $\eta_t$ can be written as

$$\eta_t = \exp \left[ \int_0^t -\frac{1}{2} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)})^\top (\Sigma \Sigma^\top)^{-1} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)}) ds + \int_0^t \Sigma^{-1} (\bar{\nu}_s^{(2)} - \bar{\nu}_s^{(1)}) d\tilde{W}_s^{(1)} \right],$$

(25)

that is, $\eta_t$ is an integral of weighted averages of past dispersions in expected drifts of earnings and consumption. In other words, the short rate process displays path dependence with respect to past differences in opinions of the different agents. When agents have homogeneous beliefs, $\eta_t = 1$ at each time, and the last term in (20) vanishes, and so does the path dependence. The path dependence of the short rate will be useful for obtaining a low volatility of interest rates in my model, a feature of the data.

Turning to the market prices of risk in (iii), I note these are comprised of two terms, the risk in aggregate fundamentals, $(1 - \gamma)\sigma_x$, and a term related to the speculative risk. For a given agent, the second term is the product of two components: the share of the other agent’s consumption of total output and a speculative component, which is proportional to the amount that an agent’s
expected growth of earnings exceeds the other agent’s \((\tilde{\theta}_t^{(1)} - \tilde{\theta}_t^{(2)})\) for agent 1). Both terms have intuitive meanings: the first component implies that when an agent consumes a smaller share of output, he faces greater exposure to prices moving in the direction of the other agent’s beliefs. The second component implies that when an agent is more optimistic relative to the other, he faces potentially a larger correction of prices moving in line with the other agent’s beliefs, and his price of risk increases.\(^{10}\) Indeed the second term is negative when an agent is more pessimistic than the other agent. Finally, I note that as \(\gamma\) increases (agents are less risk averse) their consumption shares become more volatile, thereby giving larger weights to the dispersion terms in the speculative risk component. In Section IV I show that for a region of low risk aversion, this effect can dominate the first effect (risk in aggregate fundamentals), causing Sharpe ratios to increase for lower CRRA.

It is interesting to note that agents’ market prices of dividend risk are non-zero despite stocks (a claim to future dividends) being in zero net supply. The intuition behind this result is that fluctuations in dividend growth lead to divergence in opinions about future growth rates and speculative possibilities, which in turn lead to fluctuations in individual consumptions. Therefore, each agent demands a risk premium to bear these shocks. With zero dispersion in beliefs, this channel disappears.

A key feature of my model is its ability to generate high and persistent stock market volatility that arises endogenously as agents learn about fundamentals. When fundamental growth changes rapidly, agents’ confidence in their current estimates declines, and their beliefs fluctuate rapidly with news. As shown in David (1997), such a learning-based volatility process satisfies many of the stylized facts in the GARCH literature. In this model, the speculation between agents has a further impact on volatility as displayed in Figure 2 (middle panel). Volatility increases in the uncertainty of each type, and in periods when both agents have maximum uncertainty (expansion probability of 0.5) is 50% higher than in the case where a representative agent has maximum uncertainty.
The equity risk premium for each agent is an inner product of the market prices of risk of that agent in Proposition 2 and equity volatility in (C6). The risk premium inherits many of the features of the price of risk discussed below Proposition 2 and I will not repeat these discussions. Notably different though, in periods of high certainty, is the decline in risk premiums due to lower volatility in these periods. In the right panel of Figure 2 I look at the effects of dispersion on the premium of agent 2. In periods when agent 2 is more optimistic of an expansion state, his risk premia are positive. In such periods, the premium of agent 1 is negative. As we will see, agent 2 takes long positions in stocks in most such periods, while agent 1 takes positions of the opposite sign.

A natural question that arises is the relationship between the conditional risk premium under the objective measure in the model and agents’ beliefs. I summarize this relationship in the following corollary.

**Corollary 2**

\[ \mu_i - r = (1 - \gamma) \sigma_x \sigma_i^T + \left[ \sum_{m=1}^{2} \frac{\alpha_i^{(m)}}{x_t} \Sigma^{-1} (\nu_t - \nu_t^{(m)}) \right] \sigma_i^T. \]

The premium under the objective measure is thus the sum of the premium in the benchmark economy with homogeneous beliefs and the consumption share weighted estimation errors of the two agents. It follows at once that the conditional premium is \( (1 - \gamma) \sigma_x \sigma_i^T \) if both agents are conditionally unbiased. The following case will be particularly relevant in analyzing the premium in my calibrated economy in the next section: \( \Sigma^{-1}(\nu_t - \nu_t^{(1)}) = (\delta, 0) \) and \( \Sigma^{-1}(\nu_t - \nu_t^{(2)}) = (0, 0) \). In this case, agent 1 is pessimistic about earnings growth and agent 2 is unbiased, in which case the premium is \( (1 - \gamma) \sigma_x \sigma_i^T + \frac{\alpha_i^{(1)}}{x_t} \delta \sigma_i,1 \). The average premium under the objective measure in a long sample will remain large only if the consumption share of agent 1 remains large despite trading using a biased estimate.\(^{11}\)
A Asset Prices and Portfolio Choices

Since there are two shock processes in the economy, agents require at least two multi-period securities in addition to instantaneous bonds to complete the market (in the traditional sense of market completeness). I implement the equilibrium with stocks paying the dividend process in Assumption 1, and consol (perpetuity) bonds paying a constant coupon flow \( c \). I briefly describe the pricing of these securities below.

The prices are functions of beliefs of each type of agent, and the disagreement value process. I use standard no-arbitrage analysis (see, for example Cochrane 2001) to value these securities. Since agents agree on prices of all assets, I formulate the PDE under the filtration of agents of type 1. The stock price \( P(\pi(1), \pi(2), \eta, q) \) is obtained by solving

\[
E^{(m)} \left[ \frac{dP}{P} \right] + \left( \frac{q}{P} - r(\pi(1), \pi(2), \eta) \right) dt = -E^{(m)} \left[ \frac{dP}{P} \frac{d\xi^{(m)}}{\xi^{(m)}} \right].
\]

(27)

I show in Appendix C that \( P(\pi(1), \pi(2), \eta, q) = p(\pi(1), \pi(2), \eta) \cdot q \), where \( p(\pi(1), \pi(2), \eta) \) follows an elliptic PDE with natural boundary conditions.

While closed form solutions for the PDE as well as the one for individuals’ wealth processes (to be introduced) are not available, I am able to provide polynomial approximations to these PDEs using projection methods described for example in Judd (1992) and Judd (1999).

I can similarly solve for the wealth of agents of type 1, \( X^{(1)}_t \), when they have made optimal portfolio and consumption decisions. Given their consumption choices in (19), \( X^{(1)}_t \) must satisfy

\[
X^{(1)}_t = \frac{1}{\xi^{(1)}_t} E^{(1)} \left[ \int_t^\infty c^{(1)}_s \xi^{(1)}_s ds \right].
\]

(28)

Linearity of optimal consumption in output, \( x \), implies that wealth is homogeneous of degree one in output so that \( X^{(1)}(\pi(1), \pi(2), \eta, x) = f^{(1)}(\pi(1), \pi(2), \eta) \cdot x \). The PDE for \( f^{(1)}(\cdot, \cdot) \) is provided in Appendix C and is solved using projection methods. Using the method of Cox and Huang (1989) the portfolio choices of agent 1 are:
where the volatilities can be obtained from the polynomial solutions of the prices. Since both bonds and stocks are in zero net supply, the portfolio choices of agents of type 2 are simply \( w^{(2)}_t = -w^{(1)}_t \).

I end the characterization of the equilibrium by proving its existence and determining the constant \( k = (y_1/y_2)^{1/\gamma} \), which is used in all the pricing formulas above. At the equilibrium level of \( k \), the budget constraint of each individual in (8) at time 0 is satisfied with equality. I assume the initial beliefs of each agent to be at their unconditional values \( \pi^{*(m)} \) given below Lemma \( \Pi \) and \( \eta_0 = 1 \). The value of agent 1’s endowment, the right hand side of (8), can be formulated as the solution of (28) with the flow rate of consumption \( c^{(1)}_t \) replaced by the endowment flow \( e^{(1)}_x \) given in Assumption 5. Let us call this value \( V^{(1)}(\pi^{(1)}, \pi^{(2)}, \eta; k) \). I show that an equilibrium \( k \) is the implicit solution to the equation

\[
X^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k) = V^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k),
\]

(30)

**Proposition 3** There exists an equilibrium, and for \( 0 \leq \gamma < 1 \), the equilibrium is unique.

I find the equilibrium by numerically solving for the roots of (30). While uniqueness is only established for \( 0 \leq \gamma < 1 \), in my computations I found that the equilibrium was unique for the full range of reported \( \gamma s \).

### III Calibration

In this section, I describe my methodology for estimating parameters for the fundamental processes used by the two sets of agents. The method finds the parameters which jointly maximize the likelihood of each set of agents observing the fundamental processes, as well as matching the dispersion of these agents’ forecasts to those available from surveys.
A Data Series

The fundamental series that I use are the real earnings of S&P 500 companies obtained from Standard and Poor’s, and the real consumption for non-durables and services obtained from the Federal Reserve Board. The regime-switching model with heterogeneous beliefs is fitted to these fundamentals and a time series of dispersion in forecasts of earnings.

I construct the measure of cross-sectional dispersion using data from the Survey of Professional Forecasters, available at the Federal Reserve Bank of Philadelphia. Reliable data are available from around 1970. Specifically, at each time $t$, let $FD_i(t, \tau)$ be the forecast of nominal profit growth $\tau$ quarters ahead by forecaster $i$, and $FI_i(t, \tau)$ be the forecast of the price level $\tau$ quarters ahead. I then define $FRD_i(t, \tau) = FD_i(t, \tau)/FI_i(t, \tau)$, as a measure of forecasted real earnings growth. If $n_t$ is the number of individuals at time $t$, then the time $\tau$-quarters ahead dispersion of real earnings growth at time $t$ as

$$d(t, \tau) = \sqrt{\frac{1}{n_t - 1} \sum_{i=1}^{n_t} \left( \frac{FRD_i(t, \tau)}{FRD(t)} \right)^2 - \frac{1}{n_t} \sum_{i=1}^{n_t} \left( \frac{FRD_i(t, \tau)}{FRD(t)} \right)}.$$  (31)

The time series of the four-quarter ahead dispersion so obtained is in the top panel of Figure 4.

B Calibrated Parameter Values and Implied Beliefs

(Insert Table I about here)

The calibration methodology is described in detail in Appendix D. The parameter values used by the agents of each type are shown in Table I. The parameter values are quite compelling. The earnings drift estimates of agent 1 sandwich those of agent 2, suggesting that the former will have more volatile expectations. For this reason I will often refer to agent 1 as the more “volatile” investor, and agent 2 as the more “stable” investor. Agent 2 estimates that the economy shifts out of recessions more rapidly than agent 1; however, he estimates that earnings grows less rapidly.
in expansions. Agent 1 has an unconditional bias of -3%, about 0.375 of a standard deviation from the mean of annualized earnings growth, while Agent 2 has an almost unbiased unconditional expectation.

I show in the top panels of Figure 3 that agents’ beliefs are highly correlated, although in line with my comments above, agent 1’s beliefs overshoot agent 2’s in each direction. The bottom panel shows that this is also true of the agents’ expected growth rate of earnings. In upturns (downturns), agent 1 (2) is relatively more optimistic. In addition, the expectations of the agents are more volatile during downturns, and tend to be more dissimilar. The reason this happens is that both agents have higher transition probabilities out of the recession state, and their estimates are more dissimilar. Overall, the dispersion in the two agents’ expected earnings growth is asymmetric and countercyclical.

As mentioned in the introduction, the two models have very similar fits for historical earnings growth. I report the regressions of actual growth on expected growth for each of the two types of agents in Table I (bottom panel), and find the $R^2$s of the two agents’ models to be 64.7 and 67.3% respectively. Both intercepts are insignificantly different from zero, and their beta coefficients are highly statistically significant and very close to one. While agent 2’s model is slightly better in fitting current growth, in forecasting, agent 1’s model is more accurate at horizons of 1 and 2 quarters ahead (results not shown in table).

I create a model-based series of dispersions in forecasts by taking the standard deviation of forecasted growth of the two agents for any given horizon. The construction is analogous to the one from survey data in (31). The actual and model-based dispersions are shown in the top panel of Figure 4. As seen, both data and model-based series tend to increase in and around recessions of the U.S. economy. The two series are strongly positively correlated. A regression of the historical
dispersion measure on its model-based counterpart yields an $R^2$ of nearly 20% (Table I, bottom panel). The beta coefficient of the regression is about 0.71, and is not significantly different from 1; however, the model-based dispersion is on average lower, leading to a positive intercept term. Nonetheless, the overall chi-squared statistic for the model, which penalizes fitting errors of both agents and the dispersion error, is low with a $p$-value of more than 0.1.

The calibrated disagreement value process is shown in the bottom panel of Figure 4. Some comments on the dynamics of this process are made following Lemma 5. In addition, its path dependence (see (25)) implies that it is slow moving. This should make the disagreement value smoother than expected growth rates which is evident by comparing it with the series in the bottom panel of Figure 3. As seen, it has fewer local peaks and troughs than agents’ beliefs, and plays a role in my model in generating smoother consumption series. Since this process is martingale with a starting value of one, in a long sample, it should exceed one for half the sample periods. In my sample, the process exceeds one in about 63% of the sample. Another relevant point is that even though dispersion is strongly countercyclical, the disagreement value process, being slow moving, is not. In fact, the levels of disagreement value during past recessions have differed quite significantly.

Equilibrium consumption levels of the two types of agents are completely determined by the disagreement value process and realized output in the economy as seen in eq. (19). I examine the calibrated consumption levels of the two types of agents in the top panel of Figure 4 (third panel). Agent 2’s consumption increases in $\eta$, therefore his consumption growth rate is strongly positively correlated to positive earnings shocks (and as I will show, to stock returns) in periods when agent 2 is relatively more optimistic. Conversely, in periods when agent 1 is more optimistic, the consumption growth of agent 1 (2) will have a positive (negative) correlation with earnings shocks. The figures show that the consumption of agent 1 increases faster in and around most NBER-dated recessions and other periods of low growth in earnings. Agent 2’s consumptions follow the opposite pattern.
Both series have positive trends, and the two are negatively correlated. It is also worth noticing that starting from the same level, the calibrated consumption paths crossed about twelve times in the thirty-year sample, although by the end of the sample, the consumption of agent 1 outpaced that of agent 2. In addition, there are fairly long swings of the domination of a type 1, followed by rapid declines, that bring their average growth rates about level. As I show in the next section, the volatilities of the two agents’ consumption growths vary significantly over time, and on average the volatility of agent 1 is higher. However, since his mean growth rates is also higher, I find that the two agents’ welfare is almost identical, and the constant \( k = (y_1/y_2)^{1/(1-\gamma)} \) determined in eq. (30) is slightly below one in my calibrated model. I will shed further light on consumption patterns when discussing portfolio choices and risk premia of the two types of agents in the next section.

IV Equity Premium and Riskless Rate Results

I use the parameters of fundamental processes calibrated in Section III to study the equity premium and related statistics implied by the model. In addition, I use different parameters for the preferences of agents. I split my analysis into three parts: Subsections IV A and IV B discuss the properties of unconditional and conditional moments of variables relevant to the puzzles, respectively, and in Appendix B, I examine the effects of relaxing my assumption on stocks being in zero net supply.

A Unconditional Moments

I use the pricing functions to generate price series and returns from the calibration exercise. For the calibrated results, I use the calibrated belief and disagreement value processes shown in Figures 3 and 4. For example, to obtain a real stock price series, I use the product of the realized proxy for dividends (50% of earnings), and the P-D ratio of \( p(\pi_{1t}^{(1)}, \pi_{1t}^{(2)}, \eta_t) \), whose value is obtained from the solution of eq. (C2) in Appendix C. Similarly, other variables are generated. Calibrated portfolio
weights held in stocks by agent 1 are shown in the bottom panel of Figure 4 (fourth panel). The weight varies between -0.2 and 0.2, as the relative optimism of agent 1 changes. As seen in the figure, agent 1 holds short positions more often, consistent with his relative pessimism seen from the bottom panel of Figure 3. The weights are generally small since consumption volatility is lower than stock price volatility.

(Insert Table II about here)

The results from the calibration exercise are shown in Table II. Column (3) of the table provides the average equity volatility generated by the model. In each case average annualized stock return volatility is between 18 and 20%, and increases in $\gamma$ (I discuss this effect below). This exceeds the volatility of the assumed dividend process in the model of around 7%, due to the volatility caused by the beliefs and the disagreement value. Comparing lines 2 and 12, I see that the effect of increasing the time discount from two to three increase stock volatility. The intuition is that with higher time discount, greater weight is given to current news on fundamentals.

The average model implied riskless rate is reported in col. (4). As mentioned in the discussion of Proposition 2(ii), the rate is determined by three different effects. With low calibrated volatility of aggregate consumption, the precautionary savings effect is small for a very large range of investor risk aversion. In the range where $0 < \gamma < 1$, the wealth effect is positive, while the hedging demand effect is negative. Both the wealth and hedging demand effects imply lower rates as $\gamma$ increases. For example, for the sample period the average riskless rate is 2.75% and 2.17% for $\gamma$ of 0.5 and 0.6 respectively (lines 1 and 2). I note that while the model’s average rate is above the empirically estimated 1% average real riskless return, it is 75-90 basis points below the benchmark economy in which agents have homogeneous beliefs. Moreover, the standard deviation of this real rate is extremely low, between 1 and 2% at an annual rate. In the calibration, I have used a time-discount factor, $\rho$, of 2 higher time discount is more reasonable. Using instead $\rho = 0.03$ raises the average rate by 1% and does not change its standard deviation (lines 2 and 12). With a higher
The average riskless rate drops further as agents’ savings demand increases in periods of low disagreement and high dispersion. However, with a higher \( \gamma \) the volatility of the short rate as well as individual’s consumption volatilities increase beyond observed levels (results are not reported). For \( \gamma = 0 \) (log-utility) the hedging effect is zero and the riskless rate hits 4.8%. For \( \gamma < 0 \), both effects are positive and increase as \( \gamma \) is lowered, pushing the riskless rate even higher. For extremely high CRRA in excess of 50, the precautionary savings effect finally dominates and the riskless rate falls again (not shown in Table).

The equity premium under the objective measure calculated as the average annualized excess returns from the model generated price series is in col. (6). For the case \( \gamma = 0.5 \), the equity premium is about 2.4%, compared to an historical equity premium of 6.1% in this particular sample. Somewhat remarkably, when I increase \( \gamma \) to 0.6, the equity premium rises to 2.95%. The intuition for this result is provided in the discussion of Proposition 2(iii). Essentially, the speculative component of the market prices of risk of both agent types increases in absolute value as \( \gamma \) increases. Less risk-averse agents take more speculative positions and their relative shares of the economy fluctuate more. For \( \gamma > 0 \), the speculative component of the equity premium outweighs the premium from aggregate risk in fundamentals, which declines with lower risk aversion. The cost of obtaining a higher risk premium is higher individual consumption volatilities which I discuss below. For \( \gamma = 0.5 \), the Sharpe ratio generated (column 8) is about 13%, less than half its historical value. Nonetheless, at the low level of risk aversion assumed, this is a substantial improvement on current models. As \( \gamma \) is lowered below zero, agents speculate less, their consumptions shares become less diverse, and the speculative component of the market price of risk declines. As \( \gamma \) is lowered below -9, the premium from the aggregate risk in fundamentals eventually dominates and the risk premium starts increasing again, reaching 2.6% at \( \gamma = -29 \). This non-monotonicity can also be viewed in Figure[1]
Using the individual consumption functions in Proposition 2\(^1\) and the calibrated disagreement value process, I generate processes for individual consumption and calculate their sample volatilities. Of the two investors in my model, agent 1 has a higher volatility of consumption, in line with the larger volatility of his belief process. As I increase \(\gamma\), the volatilities of both agents’ consumption growths increase. For the case \(\gamma = 0.5\), the per-capita volatility is nearly 10\%, within the range estimated in recent papers cited in the introduction. The higher per-capita volatilities are essential for generating higher Sharpe ratios in my model relative to the homogeneous agent benchmark. However, as I show in the next subsection, this level of volatility is not in itself sufficient for generating my results. Finally, when \(\gamma\) is lowered below zero, individual consumption volatilities rapidly decline as agents reduce their speculation.

I can also compute the average ex-ante equity premium expected by agents in my model. These are shown in cols. (12) and (13) for the two types of agents. The expected equity premium for each agent is calculated using the market prices of risk in Proposition 2\(^2\) and the volatilities of asset prices at each date, using the calibrated belief and disagreement processes. As anticipated, I see that the equity premium of agent 1 underestimates the long term historical equity premium in the model by 5 to 6\%, while that of agent 2 is very close to its historical average. In the next subsection when examining conditional equity premia, I show that the ‘bias’ of agent 1 is consistent with eq. (13). I do note, however, that in periods when agent 1 is more pessimistic he is on average short in equities, and so his strategy earns positive expected returns.

One perhaps surprising aspect of my results is that agent 1, who in the sample underestimates earnings growth and trades based on his model, survives, and in fact has a slightly higher average growth rate of consumption than agent 2. This result may seem at odds with recent results by Blume and Easley (2004) and Yan (2005), who show that ‘irrational’ agents will not survive in the long term. In their analysis, the rational agent exactly knows the data generating process (DGP), while the irrational agent has a misspecified model, such as a biased estimate of the drift rate. The
latter paper reports that when the bias of a type is substantially larger than that of agent 1 in this paper, it may take hundreds of years for his wealth share of the economy to decline noticeably. Nevertheless, I note that there are key differences in my analysis in that both agents face estimation risk and neither agent knows the exact DGP. Agent 2 is ‘more rational’ only in the sense that the econometrician’s estimate of his unconditional mean equals the sample mean. The unconditional mean of the data generating process may be different from its sample mean. Moreover, both agents’ models may be slightly misspecified relative to the DGP, as every model is. While I have specified both agents’ models without specific constraints on their rationality, I find that each model explains a similar fraction of variation of the realized earnings growth, and when trading with their models, both agents survive.

The last two columns give the equity premiums and riskless rate when beliefs are homogeneous. With a non-stochastic drift rate of aggregate output, investors’ beliefs and consumption are uncorrelated and hence the component of volatility arising from belief variation does not command a market risk premium. As in David and Veronesi (2002), the equity premium equals \((1 - \gamma) \cdot \sigma_q \sigma_x^\top\). These columns highlight that the equity premium without the speculation modeled here is of the order of 0.0005\%, while the riskless rate is between 3 and 4\% (one to two percentage points higher than the time-discount factor) when the CRRA is smaller than one. Therefore in this range, the dispersion of beliefs in my model is the driving force, and none of the other features of the calibration exercise can account for my results. For very low \(\gamma\), the aggregate risk of fundamentals dominates and the risk premium for the model resembles that of the homogeneous agent economy.

A.1 Back-of-the-Envelope Approximation of Sample Moments

While calculation of the unconditional moments of asset returns and consumption as described above is cumbersome, I am able to provide some approximate calculations that are easily replicated and show that the premium in my model is close to half its empirically observed level, which is
more than 50 times higher than in a benchmark economy with the same level of risk aversion and homogeneous beliefs. The exercise does miss some crucial properties of conditional moments of my model which I will discuss in Section IV B.

I make use of the following sample moments: First, as pointed out above, agent 1 underestimates earnings growth by 3% and agent 2 has an essentially unbiased estimate. Second, the average sample moment of the consumption share of agent 1 is 0.55. Third, average stock volatility in the sample is 19%.

Using these sample moments, I approximate the moments of the equity premium puzzle. First, Corollary 2 implies that the equity premium with a CRRA of 0.5 would average $0.5 \cdot 0.02 \cdot 0.19 + 0.55 \cdot \frac{0.03}{0.083} \cdot 0.19 = 0.039$, which is 1% higher than the premium reported above. Second, using (20), the approximate sample average of the riskless rate is

$$\bar{r} \approx 0.02 - \frac{1}{2} (2 - \gamma) (1 - \gamma) \cdot 0.0004 + (1 - \gamma) \cdot 0.03 - \frac{\gamma}{2(1 - \gamma)} \cdot 0.55 \cdot 0.45 \cdot \frac{0.03^2}{0.083^2},$$

where $\rho = 0.02$, $\kappa^{(m)} = 0.028$, $\sigma_{x,1} = 0$, $\sigma_x^+ = 0.004$, $\sigma_{q,1} = 0.083$, and $k \simeq 1$. For $\gamma = 0.5$, I obtain an $\bar{r} = 0.016$, which is 1% below that from the exact calculation. Similarly from (21) and (22),

$$\bar{\phi}_{q}^{(1)} = 0.5 \cdot 0.02 + 0.45 \cdot \frac{0.03}{0.083} = 0.172,$$

and

$$\bar{\phi}_{q}^{(2)} = 0.5 \cdot 0.02 + 0.55 \cdot \frac{b}{0.083} = 0.208.$$

By Lemma 2 for $\gamma = 0.5$, the quarterly consumption volatilities of the two agents equal these market prices of risk. I note that these volatilities are about twice as high as those reported from the exact calculation in Table II. In fact, I show in subsection III that all the moments of my model vary significantly over time and the lower unconditional consumption volatility is consistent with the reported moments of asset prices.
A.2 Bootstrapped Distributions of Asset Pricing Moments

The asset pricing models above are obtained from one 30-year sample realization of the data, and a natural question to ask is if this was a ‘typical’ sample path. In particular, I would like to determine standard errors of all statistics and to ascertain whether the survival of the pessimistic agent in the sample was fortuitous. Monte-Carlo simulations of the model are not informative for the analysis since the relative performance of the two agents’ models must be compared with the true data generating process, which is unknown to the econometrician. I therefore resort to a standard bootstrap methodology in which “innovations” to each model are resampled with replacement and all statistics are recalculated for the case $\rho = 0.02$ and $\gamma = 0.5$. The results from 10,000 repetitions (and a basic description) of this exercise are reported in Figure 5.

(Insert Figure 5 about here)

The left panel shows the distribution of $\bar{\theta}^{(1)} - \bar{\theta}^{(2)}$, which is the relative bias of agent 1. The mean of the bias is -3.1% compared to the -2.72% in my sample. The distribution is heavily negatively skewed with a minimum bias of -12%. However, 58% of the observations are above the mean value. The middle panel shows the distribution of the equity premium under the objective measure. The mean of this distribution is 2.2%, close to the mean of my sample, and a standard deviation of 0.63%. This distribution is negatively skewed, but not as much as that of the relative bias. Finally, the distribution of the riskless rate in the right panel has a mean of 2.9% with a standard deviation of 0.2%, and it is heavily negatively skewed.

Further intuition for the relative difference in skewness of these distributions can be obtained from the shapes of the pricing functions in the top panels of Figure 2. First, as noted above, with the unconditional biases of the two types of agents, the premium of agent 2 is likely to be close to the premium under the objective measure. The right panel shows that in samples when the relative bias of agent 1 is negative (positive), the objective premium will be positive (negative) and
is close to linear in this relative bias. However, when the bias is very large in either direction, the premium flattens out due to the reduction in volatility (middle panel) in periods when agents are very confident of either state. Therefore, the premium is likely to take on values both above and below its mean, but the skewness is limited due to this last observation. The left panel shows that the riskless rate has a hump shape over the relative bias: for both negative and positive values of the relative bias, the riskless rate drops. This causes the distribution of the riskless rate to be more negatively skewed. I finally note that the minimum and maximum values for the disagreement value, \( \eta_t \), in the 10,000 resamplings of 30-year samples were 0.43 and 3.13, which are clearly bounded away from 0 and infinity, thus implying that each agent survives with probability one.

\section*{B Conditional Moments}

Since agents in the model do not face any trading frictions, Lemma 2 implies that agents’ market prices of risk and their conditional consumption volatilities are closely related. Therefore, for a given \( \gamma \), statistics of individual consumption growth rates should summarize the information in agents’ prices of risk. In this subsection, I take a closer look at the moments of individual consumptions that are generated by the state variables in my model, and reconcile the equity premia of agents reported in the previous subsection with conventional Euler equations.

The conditional risk premium of agent \( m \) for any given asset is the conditional covariance between his market price of risk and the asset volatility. Eq. (17) of Lemma 2 further shows that the market prices of risk of agent \( m \) equals the CRRA times the volatility of the agent’s conditional consumption growth. Therefore, his risk premium is

\[
\mu_P^{(m)} + \delta_t - r_t = \rho_t^{(m)} \cdot (1 - \gamma) \cdot \| \sigma_{ct}^{(m)} \| \cdot \| \sigma_{Pt} \|, \tag{32}
\]

where \( \rho_t^{(m)} \) is the conditional correlation between his consumption growth and stock returns. If the quantities on the right hand side are time-varying (as is implied by Lemma 2) then care has to be
taken to extend the implications of this pricing equation to its unconditional form (see Hansen and Richard 1987, Campbell and Cochrane 2000).

Several authors use the unconditional form of (32) to place bounds on Sharpe ratios and the equity premium. If consumption and stock price volatilities are assumed constant at their expected values, then indeed the unconditional equity premium of investor \( m \) must be bounded by \((1 - \gamma) \cdot E[||\sigma_{ct}^{(m)}||] \cdot E[||\sigma_{Pt}||]\). For example, looking at line 3 of Table III the case where \( \rho = 0.02 \) and \( \gamma = 0.5 \), the equity premium of agent 2 would be bounded by \(0.5 \cdot (2 \cdot 0.0971) \cdot 0.1857 = 0.018\), clearly smaller than the ex-ante expected premium computed for agent 2 in col. (13) as 2.4% (note that the consumption volatility in the table is stated in quarterly units and must be doubled to find its implications for the annualized premium). The bound is large because it assumes a correlation between the consumption growth of agent 2 and equity returns of one. If I were further to assume that the correlation was constant at its unconditional value reported in col. (15) of 0.1445, then the premium implied by the unconditional moments would be much smaller at 0.26%. This is of course just the correlation puzzle as stated in Cochrane (2001), that the correlation between aggregate consumption and stock returns is no more than 20%, thus further deepening the equity premium puzzle. In the remainder of this section, I will point out key features of the time variation of the moments of individual consumption growths and stock returns that will justify the premiums reported in Table III.

I show in the remainder of this section that the conditional distribution of most calibrated variables will be markedly different depending on the relative optimism of the two agents. Therefore, I partition all the series into two parts with the former group having dates in which \( \bar{\theta}^{(1)}_t > \bar{\theta}^{(2)}_t \). In this group of observations, the volatile investor (agent 1) is the conditional optimist. Table III shows the conditional sample moments of various variables in my calibration exercise, and it is useful also to look at the bottom panel of Figure 4, which plots time series of the conditional expectations of
dividend drift of the two types of agents. About 43% of the observations in the sample fall into the first group. The reason I partition my data into this form is that the sign of the conditional risk premium of agent 1 (2) will be positive (negative) for observations in the first group, and the signs are reversed for observations in the alternative group.

Using the calibrated beliefs and disagreement value process, I generate the ex-ante conditional correlations, \( \rho_t^{(m)} = (\phi_t^{(m)} \cdot \sigma_P^T) / (||\phi_t^{(m)}|| \cdot ||\sigma_P||) \), and generate their averages in cols. (1) and (2). As seen, the correlations of the simulated consumption growth of agent 1 and 2 are 0.94 and -0.91 in the first group, and -0.96 and 0.95 in the second group respectively. The correlations differ from one (in absolute value) due to the presence of the small exogenous volatility of aggregate output terms in the market prices of risk. Since the correlations switch sign in the two sub-samples, the unconditional correlations of the two agents are close to -0.14 and 0.14 respectively, similar to estimates in other papers. The high conditional correlations are critical in yielding conditional risk premia close to the product of individual consumption volatility and stock volatility for the two types of agents.

As seen in columns (3) and (4), the average growth rate of consumption of agent 1 (2) is positive (negative) in the first group of observations, that is, consumption of each group of agents grows faster in periods when they are relatively more optimistic. Columns (5) and (6) show that the volatilities of consumption growth of both groups of agents are much lower in the top group of observations: the average volatilities are about 2% in the first group and 13% in the second group. Intuition from this observation can be obtained from Figure 3 where the bottom panel shows that the difference in expected growth rates of the two agents is small when \( \tilde{\theta}_t^{(1)} > \tilde{\theta}_t^{(2)} \). Col. (7) shows the same for the volatility of stock returns in the two sub-samples. Overall, for both types of agents, market prices of risk (consumption volatilities) are higher in absolute value in the second group of observations, in periods when the volatility of stock returns is also higher. Due to this comovement,
risk premiums in the second group of observations rise (in absolute value) more than proportionally than the increase in consumption volatilities.

Within each group of observations, correlations between agents’ consumption growths and stock returns are quite stable. Hence, taking expectations of eq. (A1) conditional on being in either group implies that

\[
E \left[ \bar{\mu}_{P_t} + \delta_t - r_t \mid \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right] 
\approx E[\rho^{(m)} \mid \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}] \cdot E \left[ ||\phi_t^{(m)}|| \cdot ||\sigma_P|| \mid \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right],
\]

\[
\approx E[\rho^{(m)} \mid \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)}] \cdot \{ E \left[ ||\phi_t^{(m)}|| \cdot \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right] \} \cdot E \left[ ||\sigma_P|| \mid \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right] + \text{Cov} \left[ ||\phi_t^{(m)}||, ||\sigma_P|| \mid \bar{\theta}_t^{(1)} > \bar{\theta}_t^{(2)} \right].
\]

The first component provides the risk premium due to higher average volatilities of consumption growth and stock prices for a given group. The second component measures the risk premium due to changes in dispersion of beliefs within the group of observations, which causes comovement of the two volatilities. Below, I look at the sample conditional moments in this equation to decompose the risk premium.

The two components of the ex-ante risk premiums are given in cols. (8) and (10) for agent 1 and (9) and (11) for agent 2 respectively. I note that by Lemma [2] the norm of the market prices of risk must equal the product of the agents’ CRRA and the norm of their consumption volatilities. Indeed, by comparing the norms of the consumption volatilities in (5) and (6), I find that the first component of each agent (cols. (8) and (9)) is very close to that implied by Lemma [2]. Due to low volatilities in the first group of observations, these components are very small, on the order of 0.3%. In the second group of observations this component is much larger at -3.4 and 2.5% for the two agents. The second components, shown in cols. (10) and (11), capture the portion of the risk premiums of the two agents arising from the covariance of the market prices of risk and stock volatility within each of the subgroups. These components are also much larger in absolute value.
in the second group of observations, and are nearly as large as the first components. Overall, the conditioning shows that about half the generated premium arises from higher average consumption volatilities in periods of higher average stock market volatility. The other half is generated from the covariance between these quantities within each subgroup. Due to the sign of the correlations of the two types of agents, the premia of the volatile (type 1) agents are positive in the first group of observations, when they are conditional optimists, and negative in the second group, and negative overall. The reverse holds for stable (type 2) agents. The sum of the components, as shown in cols. (12) and (13) is again small in the first group of observations, and equals -5.3 % and 5.2% for the second group. The unconditional expectations of these premia equal -2.8% and 2.59% as in Table II I come back to this issue after discussing the conditional dispersions of agents’ expected growth rates.

Using the calibrated price process, I next report the ex-post realized excess returns of stocks in col. (14). Somewhat surprisingly, the excess returns are nearly 4% in the first group of observations and much smaller at about 1% in the second group. The unconditional average of the excess returns is about 2.4 %, as reported in Table II As I have noted earlier, given the small positive bias of agent 2 in estimating earnings growth, his ex-ante risk premium should be fairly close to the risk premium under the objective measure. For the full sample as reported in the previous sub-section, I indeed found these two premia to be quite similar. Therefore, we can infer that the equity premium under the objective measure is close to -0.8% and 5.2% in the two groups of data as reported in col. (13). The relative size of these premia varies inversely with realized excess returns on stocks reported in col. (14), which means that stock prices fall in periods of higher ex-ante forward-looking risk premiums under the objective measure. Indeed, the excess returns over the following one to four quarters are large and positive in the second group, and are negative in the first group of observations (results not shown in table).
I next discuss the impact of dispersion in beliefs on the riskless rate and the cross-sectional dispersion in consumption growth in the two groups of data. As seen in col. (15), the riskless rate is quite high at nearly 3.56% in the first group, and 2.13% in the second. I noted earlier that since the two types of agents disagree more on the transition probability from the weak to strong dividend state (Table 1), their beliefs are more volatile in the second set of observations, which I noted earlier were periods of weaker earning growth. Indeed, as seen in col. (16), the difference in expected growth rates is much larger in the second group of observations, in periods when volatile (type 1) agents are relatively more pessimistic. As seen in cols. (17) and (18) both the historical dispersion and the model-fitted dispersion are about 25% higher in the second group relative to the first group of observations. As seen in the left panel of Figure 2, the riskless rate in the economy falls in periods of higher dispersion due to the higher speculative opportunities in such periods (see the discussion following Proposition 2). Therefore, periods of weak growth in fundamentals are also accompanied by lower (real) rates.

I now address the issue of whether the difference in ex-ante risk premia of the two types of agents is consistent with their expected growth rates of fundamentals. The relationship between these variables can be approximated from eq. (14). Focusing on the second group of observations, the average difference in expected dividend growth rates is about -5%, and the volatility of stock returns is 23.7%, implying a difference in equity premia of around -14% (roughly $-1/0.083 \cdot 0.237 \cdot 0.05$), larger in absolute value than the -10.5% difference in the ex-ante premia reported in cols. (12) and (13) from the exact calculation. In the first group of observations, the difference in equity premia is of positive sign and smaller in magnitude, consistent with the smaller difference in estimated fundamentals growth rates.

The higher volatilities of consumption and dispersion in beliefs also lead to more dispersion in growth rates of consumption across agents. As seen in col. (19), the unconditional mean of the cross-sectional standard deviation (cs-sd) is 0.077 at a quarterly rate and it is about twice as high
in the second group of observations, which are periods of low realized returns and high volatility. The negative relationship between the cs-sd in consumption growth and stock returns is consistent with the analysis in Constantinides and Duffie (1996), and in addition is endogenously generated in equilibrium. However, unlike their model, the endogenous shocks to consumption in my model are not permanent, and hence do not cause a trend increase in the cs-sd with the age of agents. Estimates of the cs-sd depend critically on the filtering method used in empirical studies. Jacobs and Wang (2004) provide an estimate of 0.1 based on age- and education-based cohorts in the Consumer Expenditure Survey (CEX) data, while Storesletten, Telmer, and Yaron (2004) report that it varies between 0.06 and 0.105 (both estimates at a quarterly rate) among cohorts of individuals in the Panel Study of Income Dynamics data. This latter study finds that the cs-sd on average increases by 75% as the macroeconomy moves from peak to trough. My model-based average cs-sd lies within the range of these estimates, and even though my conditioning criterion is somewhat different, displays a countercyclical variation as in Storesletten, Telmer, and Yaron (2004).

V Conclusions

I show that in a model in which agents have heterogeneous beliefs about the state of fundamental growth, the risk premium increases with lower risk aversion (as long the CRRA is less than one) because the exposure to speculative risk is endogenous, and increases as less risk averse agents undertake more aggressive trading strategies. The model implies that the equity premium is higher, the riskless rate is lower, and consumption growth is more volatile and dispersed in periods when the dispersion among agents’ expectations of future growth is high, and when the disagreement between their models is low. My calibrated model shows that these conditions are more likely to prevail during periods of weak fundamental growth. These stylized facts have some support in recent empirical research on individual level consumption (Storesletten, Telmer, and Yaron 2004).
The degree of freedom I take relative to existing work on the equity premium is the higher level of per capita consumption volatility. The higher volatility in itself justifies only a tenth of the premium in my model, but, its positive covariance with stock volatility, and high conditional correlations between individuals’ consumption growth and stock returns, which switch signs depending on the relative optimism of agents, together raise the model’s unconditional equity risk premium to nearly half its historical level. The former property implies that higher risk (stock volatility) is also priced higher, while the latter helps address the consumption correlation puzzle by generating high conditional correlations, but unconditional correlations near empirical estimates. While exact properties of individual consumption processes are hard to measure and verify, my results support a growing literature that finds useful information for asset pricing in statistics of cross-sectional consumption growth (see, e.g., Brav, Constantinides, and Geczy 2002, Jacobs and Wang 2004). I view my results as complementary to this line of research, providing further empirical predictions to be tested at the individual level.

Notes

1 I borrow a colorful description of this kind of risk from Abreu and Brunnermeier (2003):

   For example, when Stanley Druckenmiller, who managed George Soros’s $8.2 million Quantum Fund, was asked why he didn’t get out of internet stocks earlier even though he knew that technology stocks were overvalued, he replied that he thought the party wasn’t going to end so quickly. In his words “We thought it was the eight inning, and it was the ninth.”

While dispersion of beliefs is a key ingredient of the “bubble” equilibrium in Abreu and Brunnermeier (2003), they do not analyze its impact on the risk premium.
In this sense, even though the number of traded long-lived securities equals the number of shocks in the economy, so that markets are complete, they are effectively incomplete.


This results because I have a pure exchange economy with no explicit production process or fruit-bearing tree as in Lucas (1978). The Euler equation for asset prices is the same as that in an economy where the stock is in positive net supply, although the market prices of risk will in general differ from the latter case. I note that in Mehra and Prescott (1985) the ratio of aggregate dividends to consumption is identically one in each period, while in my model it is identically zero. As evident from the analysis in Cecchetti, Lam, and Mark (1990) and Brennan and Xia (2001), the equity premium puzzle is robust to the assumption of stocks being in zero net supply. I show later that retaining all my assumptions, but instead assuming that agents have homogeneous beliefs, my model equity premium is very small as well. Empirically, stocks in the U.S. are in ‘small’ positive net supply: Cecchetti, Lam, and Mark (1990) note that the dividend-consumption ratio is of the order of 2 to 5%. In Appendix B, I show by generalizing my model that the market prices of risk and riskless rates with such small supply of stock will be very similar to the case modeled here.

Lemma 1 in Huang and Pagès (1992) shows that the measures are mutually singular if $\int_0^\infty \sigma_{\eta_s} = \infty$, almost surely as is true if $\hat{\nu}^{(1)} \neq \hat{\nu}^{(2)}$. As a simple comparison, the objective
and risk-neutral measures in the infinite horizon problems in Samuelson (1965) and Merton (1990) are mutually singular.

6 The condition can also be written as $c_t^{(m)} = I_m(y_m e_t^{(m)})$, where $I_m(z)$ is the inverse of $u'_m(e^{(m)})$, and $y_m$ is the Lagrange multiplier with respect to the budget constraint. By Assumption 4, the marginal felicity function is monotonically declining and satisfies the Inada conditions implying a unique solution for $I_m(.)$. The existence of an optimal solution for the complete variational problem including consumption and portfolio choices can be established by verifying a boundedness condition on the felicity function, and Lipschitz and growth conditions on the SPD functions with respect to each state variable (for the finite and infinite horizon cases respectively, see Cox and Huang 1991, Huang and Pagès 1992). For $0 \leq \gamma < 1$ it is straightforward to verify these conditions from the explicit functional forms for $r$ and $\phi^{(m)}$ in Proposition 2 below (see Footnote 9).

7 I assume that the transversality condition $\lim_{t \to \infty} E^{(m)}[\exp(- \int_0^t r_s ds) P_{it}] = 0$ holds for each agent, so that only fundamental valuations are compatible with equilibrium.

8 This effect has been well known since the papers of Hakansson (1971) and Merton (1973) for the case of general state variables where investors have homogeneous beliefs. My incremental contribution is to study the effect of changes in the opportunity set simultaneously for two types of agents, creating buying opportunities for one type of agent and selling opportunities for the other type.

9 In addition, the derivative of the function $\left( k \eta_t^{\frac{1}{1-\gamma}} \right) / \left( 1 + k \eta_t^{\frac{1}{1-\gamma}} \right)^p$, for $p = 1, 2$ with respect to $\eta$ is bounded on $(0, \infty)$ for $0 \leq \gamma < 1$, and hence the riskless rate and market prices of risk satisfy the sufficient Lipschitz conditions for the existence of investors’ variational problem in footnote 6.
While agents trade on their own models as in models of investor over-confidence (for example Daniel, Hirshleifer, and Subrahmanyam 2001), their market price of dividend risk increases when their model predictions are more optimistic relative to traders of the other type.

Using the same steps I can show that the statement of the corollary holds when there is an arbitrary number of types. Once again the premium under the objective measure will remain large if the consumption shares of the pessimists remain large enough after trading.

Following Longstaff and Piazzesi (2003) I assume that ‘true’ dividends are one half of earnings, an assumption that matches the average historical dividend yield, but has twice its volatility. I might add that the average historical equity premium calculated for my sample with this approximation is very close to the average premium calculated with historical dividends. This assumption on the payout series enables us to match the historical volatility of stock returns, while the pure dividend payout assumption does not.
Table I: 2-State Heterogeneity Model Calibration

**Top Panel:** GMM estimates of the following (discretized) model for real consumption, $x_t$, and real earnings, $q_t$:

$$
x_{t+1} = x_t \cdot e^{(\kappa^{(m)}_t - \frac{1}{2} \sigma_x^2 t)} \Delta t + \sigma_x \varepsilon_{t+1};
$$

$$q_{t+1} = q_t \cdot e^{(\kappa^{(m)}_t - \frac{1}{2} \sigma_q^2 t)} \Delta t + \sigma_q \varepsilon_{t+1},$$

where $\sigma_q = (\sigma_{q1}, \sigma_{q2})$, $\sigma_x = (0, \sigma_{x,2})$ and where $\left(\theta^{(m)}_t, \kappa^{(m)}_t\right)$ jointly follow a two-state regime switching model. I note that the variance-covariance matrix between the two fundamental shocks is identified by setting $\sigma_{x1} = 0$. I estimate the quarterly transition probability matrix whose estimates and standard errors are shown. The implied generator is $\Lambda^{(m)} = \sum_{i=1}^{\infty} (-1)^{i+1} \left((P^{(m)}(0.25))^4 - I\right)^i$, whose value I estimate using a series approximation of length 10 (see Israel, Rosenthal, and Wei 2001). The GMM errors include the scores of the likelihood function of each type of agent and the difference in model-implied and historical dispersion in forecasts of Professional Forecasters as described in Appendix 3. The $\chi^2(4)$ statistic for the specification test of the model is 7.6341, which has a $p$-value of 0.1059. **Bottom Panel:**

Standard errors of parameter estimates are in parentheses. Units of measurement are quarterly and in percentage points. $T$-statistics are in parentheses. All $t$-statistics are adjusted for heteroskedasticity and autocorrelation using the methodology of Newey and West (1987). Figure [3](top panel) shows the belief processes of the two agents. The top panel of Figure 4 shows the actual and model implied 4-quarter ahead dispersions of earnings growth, which are in the 3rd regression.

---

**Series Used:** Real Earnings, Real Consumption, and Dispersion of Earnings Growth Forecasts

**Time Span (Quarterly):** 1971-2001

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<th>Drifts:</th>
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<th>Agent 2:</th>
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<td>$\theta^{(1)}_1$</td>
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<td>-0.2305 (0.0192)</td>
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<td>0.0795 (0.0258)</td>
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<td>$\sigma_{q,1}$</td>
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**Model Fits:**

$\Delta \log(q_t) = \alpha + \beta \cdot (\theta^{(m)}_1 \pi^{(m)}_1(t_t) + \theta^{(m)}_2 \pi^{(m)}_1(t_t)) + \epsilon(t), \ m = 1, 2$

<table>
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</tr>
</tbody>
</table>

**Dispersion:**

$d(t, 4) = \alpha + \beta \cdot d(\pi^{(1)}(t, 4), \pi^{(2)}(t, 4)) + \epsilon(t)$

<table>
<thead>
<tr>
<th>Agent 1:</th>
<th>Agent 2:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>4.1301 (12.1873)</td>
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<tr>
<td>$\hat{\beta}$</td>
<td>0.7190</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.7190</td>
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</table>
Table II: Calibrated Equity Premium and Related Statistics (1971 to 2001)

The unconditional means of the following variables (annualized units, unless stated) are generated from the model: col. (3), $||\sigma_P||$, volatility of equity returns, col. (4), $\mu_R$, riskless rate, col. (5) $\sigma_R$, volatility of-riskless rate, col. (6), $\mu_P - \mu_R$, equity premium (objective measure), col. (7), $(\mu_P - \mu_R)/||\sigma_P||$, Sharpe ratio, col. (8) and col. (9), $\sigma_{c_j}$, j = 1, 2, volatilities (quarterly rate) of individual consumption growth, col. (10), $\sigma_c$, per capita volatility (quarterly rate) of consumption growth, cols. (11) and (12), $\mu_{E_j} - \mu_R$, j = 1, 2, equity premium under measure of two different agents, cols. (13) and (14), $\rho^{(m)}$, correlations between consumption growth and stock returns for the two agent types, and cols. (15) and (16) $\rho^{(h)} - \rho^{(h)}$, equity premium and riskless return under homogeneous beliefs. Equity Premiums and all other statistics are calculated using the solutions to the PDEs in Section II A, using the parameter values in Table I. The belief process, $\{\pi_t^{(m)}\}$, is generated using the discretized version of the SDE in Lemma 1 as shown in (D1) and (D1) of Appendix D. The disagreement value process is generated using the discretized version of (25). Calibrated belief and disagreement value processes are shown in Figures 3 and 4. The price-dividend ratio at period $t$ is $p(\pi_t^{(1)}, \pi_t^{(2)}, \eta_t)$ and is obtained from the solution of PDE (C2) in Appendix C. The calibrated price at time $t$ is given by $p(\pi_t^{(1)}, \pi_t^{(2)}, \eta_t) \cdot q_t/2$, where $q_t$ are S&P 500 earnings per share. Other variables are similarly calculated. The riskless rate is calculated from (20) as shown in Figure 2. Agents’ consumptions are in (19) and are shown in the third panel of Figure 4. The ex-ante expected premium of each agent is obtained from eq. (A1). The risk premium for the homogeneous investor case is $(1 - \gamma)\sigma \sigma_d^{(h)}$, and the riskless rate is in (20) with the last term set to zero.

<table>
<thead>
<tr>
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<tbody>
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<td>0.02</td>
<td>0.6</td>
<td>0.192</td>
<td>0.021</td>
<td>0.017</td>
<td>0.029</td>
<td>0.153</td>
<td>0.138</td>
<td>0.124</td>
<td>0.131</td>
<td>-0.035</td>
<td>0.032</td>
<td>-0.140</td>
<td>0.145</td>
<td>$3 \times 10^{-4}$</td>
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<tr>
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<td>0.5</td>
<td>0.186</td>
<td>0.028</td>
<td>0.011</td>
<td>0.024</td>
<td>0.130</td>
<td>0.102</td>
<td>0.097</td>
<td>0.099</td>
<td>-0.028</td>
<td>0.026</td>
<td>-0.139</td>
<td>0.144</td>
<td>$4 \times 10^{-4}$</td>
<td>0.034</td>
</tr>
<tr>
<td>$</td>
<td></td>
<td>\sigma_P</td>
<td></td>
<td>$</td>
<td>0.02</td>
<td>0.4</td>
<td>0.183</td>
<td>0.032</td>
<td>0.008</td>
<td>0.020</td>
<td>0.109</td>
<td>0.087</td>
<td>0.081</td>
<td>0.084</td>
<td>-0.024</td>
<td>0.022</td>
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<tr>
<td>$\mu_R$</td>
<td>0.02</td>
<td>0.3</td>
<td>0.181</td>
<td>0.036</td>
<td>0.005</td>
<td>0.018</td>
<td>0.094</td>
<td>0.074</td>
<td>0.068</td>
<td>0.071</td>
<td>-0.021</td>
<td>0.018</td>
<td>-0.139</td>
<td>0.144</td>
<td>$5 \times 10^{-4}$</td>
<td>0.039</td>
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<tr>
<td>$\sigma_R$</td>
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<td>0.165</td>
<td>0.048</td>
<td>0.000</td>
<td>0.016</td>
<td>0.109</td>
<td>0.051</td>
<td>0.049</td>
<td>0.050</td>
<td>-0.019</td>
<td>0.018</td>
<td>-0.111</td>
<td>0.124</td>
<td>$8 \times 10^{-4}$</td>
<td>0.048</td>
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<tr>
<td>$\mu_P - \mu_R$</td>
<td>0.02</td>
<td>-4.0</td>
<td>0.144</td>
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<td>0.022</td>
<td>0.014</td>
<td>0.118</td>
<td>0.024</td>
<td>0.023</td>
<td>0.023</td>
<td>-0.015</td>
<td>0.017</td>
<td>-0.109</td>
<td>0.114</td>
<td>0.004</td>
<td>0.154</td>
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<tr>
<td>$\sigma_{c_1}$</td>
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<td>0.133</td>
<td>0.287</td>
<td>0.024</td>
<td>0.013</td>
<td>0.120</td>
<td>0.012</td>
<td>0.011</td>
<td>0.012</td>
<td>-0.008</td>
<td>0.017</td>
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<td>0.108</td>
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<td>0.160</td>
<td>0.011</td>
<td>0.010</td>
<td>0.010</td>
<td>0.003</td>
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<td>0.393</td>
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<tr>
<td>$\sigma_c$</td>
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<td>0.507</td>
<td>0.026</td>
<td>0.019</td>
<td>0.184</td>
<td>0.010</td>
<td>0.010</td>
<td>0.010</td>
<td>0.010</td>
<td>0.023</td>
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<td>0.106</td>
<td>0.017</td>
<td>0.498</td>
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<td>$\mu_{P}^{(1)} - \mu_{R}$</td>
<td>0.02</td>
<td>-24.0</td>
<td>0.114</td>
<td>0.602</td>
<td>0.027</td>
<td>0.023</td>
<td>0.201</td>
<td>0.010</td>
<td>0.010</td>
<td>0.010</td>
<td>0.015</td>
<td>0.025</td>
<td>-0.100</td>
<td>0.104</td>
<td>0.021</td>
<td>0.593</td>
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<td>$\mu_{P}^{(2)} - \mu_{R}$</td>
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<td>-29.0</td>
<td>0.111</td>
<td>0.688</td>
<td>0.027</td>
<td>0.026</td>
<td>0.234</td>
<td>0.010</td>
<td>0.010</td>
<td>0.010</td>
<td>0.018</td>
<td>0.026</td>
<td>-0.100</td>
<td>0.103</td>
<td>0.025</td>
<td>0.679</td>
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<tr>
<td>$\rho^{(1)}$</td>
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<td>0.5</td>
<td>0.188</td>
<td>0.037</td>
<td>0.011</td>
<td>0.024</td>
<td>0.129</td>
<td>0.102</td>
<td>0.097</td>
<td>0.099</td>
<td>-0.028</td>
<td>0.026</td>
<td>-0.139</td>
<td>0.144</td>
<td>$4 \times 10^{-4}$</td>
<td>0.043</td>
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The following (annualized, unless stated) conditional moments are generated from the model for the case when $\rho = 0.02$ and $\gamma = 0.5$ corresponding to line 3 of Table III. Cols. (1) and (2), $\rho^{(m)}$, $m=1,2$ are the conditional correlations of ex-ante consumption growth with stock returns of the two types of agents respectively, which at time $t$ is calculated as $(\phi_t^{(m)} \cdot \sigma_P^T) / (||\phi_t^{(m)}|| \cdot ||\sigma_P||)$, where $\phi_t^{(m)}$ are the market prices of risk in Proposition 2(iii). Cols. (3) and (4), $\mu_{c_t}^{(m)}$, are individual ex-post mean consumption growth rates (quarterly), and cols. (5) and (6), $||\sigma_{c_t}^{(m)}||$, are individual ex-post consumption growth rate volatilities (quarterly). These are calculated using the agents’ consumption functions in eq. (A1), and the consumption paths are shown in the third panel of Figure 4. The calibrated disagreement value process is in Figure 4. Col. (7) is the volatility of equity returns obtained from the solutions to the PDEs in Section II A as shown in eq. (6) of Appendix C using the parameter values in Table I. For its generation, in addition to the disagreement $\mu_P - \mu_R$, is the riskless return calculated as in eq. (20). Col. (16), $E[\phi_t^{(1)} \cdot \phi_t^{(2)}]$, is the difference in expectations of earnings growth of the two types of agents, for the two types of agents, col. (17), $\hat{d}(t, 4)$, is its model fitted (scaled) counterpart (the two series are shown in Figure 4). Col. (19), $\sigma_{cs}$, is the cross-sectional standard deviation of ex-post consumption growth rates (quarterly) across agents.

Table III: Conditional Moments of Model-Generated Variables

| Cases | Prob. | $\rho^{(1)}$ | $\rho^{(2)}$ | $\mu_{c_t}^{(1)}$ | $\mu_{c_t}^{(2)}$ | $||\sigma_{c_t}^{(1)}||$ | $||\sigma_{c_t}^{(2)}||$ | $||\sigma_P||$ |
|-------|-------|-------------|-------------|------------------|------------------|----------------|----------------|----------------|
| $\hat{\theta}_t^{(1)} > \hat{\theta}_t^{(2)}$ | 0.4308 | 0.9431 | -0.9127 | 0.0255 | -0.0098 | 0.0416 | 0.0769 | 0.1184 |
| $\hat{\theta}_t^{(1)} < \hat{\theta}_t^{(2)}$ | 0.5691 | -0.9596 | 0.9448 | -0.0038 | 0.0152 | 0.1482 | 0.1124 | 0.2366 |
| Averages | -0.1398 | 0.1445 | 0.0088 | 0.0044 | 0.1023 | 0.0971 | 0.1857 |

| Cases | Prob. | $\rho^{(1)} E[||\phi_t^{(1)}||] : E[||\sigma_P||]$ | $\rho^{(2)} E[||\phi_t^{(2)}||] : E[||\sigma_P||]$ | $\rho^{(1)} Cov[||\phi_t^{(1)}||, ||\sigma_P||]$ | $\rho^{(2)} Cov[||\phi_t^{(2)}||, ||\sigma_P||]$ | $\mu_{E_t}^{(1)} - \mu_R$ | $\mu_{E_t}^{(2)} - \mu_R$ |
|-------|-------|---------------------------------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|----------------|----------------|
| $\hat{\theta}_t^{(1)} > \hat{\theta}_t^{(2)}$ | 0.4308 | 0.0046 | -0.0083 | -0.0003 | 0.0003 | 0.0043 | -0.0086 |
| $\hat{\theta}_t^{(1)} < \hat{\theta}_t^{(2)}$ | 0.5691 | -0.0336 | 0.0251 | -0.0195 | 0.0269 | -0.0531 | 0.0520 |
| Averages | -0.0171 | 0.0107 | -0.0112 | 0.0152 | -0.0284 | 0.0259 |

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<th>Cases</th>
<th>Prob.</th>
<th>$\mu_P - \mu_R$</th>
<th>$\mu_R$</th>
<th>$E[\hat{\theta}_t^{(1)} - \hat{\theta}_t^{(2)}]$</th>
<th>$d(t, 4)$</th>
<th>$\tilde{d}(t, 4)$</th>
<th>$\sigma_{cs}$</th>
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</thead>
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<td>0.4308</td>
<td>0.0422</td>
<td>0.0356</td>
<td>0.0048</td>
<td>6.3881</td>
<td>6.0375</td>
<td>0.0492</td>
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<tr>
<td>$\hat{\theta}_t^{(1)} &lt; \hat{\theta}_t^{(2)}$</td>
<td>0.5691</td>
<td>0.0104</td>
<td>0.0213</td>
<td>-0.0514</td>
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<td>7.8357</td>
<td>0.0984</td>
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<tr>
<td>Averages</td>
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<td>-0.0272</td>
<td>7.0603</td>
<td>7.0604</td>
<td>0.0771</td>
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Figure 1: The Effects of Risk Aversion on the Average Riskless Rate, Equity Premium, Per Capita Consumption Volatility, and Stock Volatility

The left panel plots the riskfree rate and the equity premium for alternative levels of $\gamma$, where agents’ constant coefficient of relative risk aversion is $1 - \gamma$. The right panel plots per-capita consumption volatility and stock volatility for alternative levels of $\gamma$. Details of the calculation are provided in the footnote to Table II.

Figure 2: The Riskless Rate, Stock Return Volatility, and Agent 2’s Equity Premium

The riskless rate and stock return volatility are in eqs. (20) and eq. (C6). The risk premium of Agent 2 is $-\phi^{(2)} \sigma_p^T$ where the market prices of risk are in eqs. (22) and (24). In the figures I set the disagreement value to one, and Prob 1. and Prob 2. are the two agents’ beliefs of earnings currently being in an expansion state. The parameters of the fundamental processes are shown in Table II. In addition, I use $\rho = 0.02$ and $\gamma = 0.5$. 
The top panel has the time series of filtered beliefs about real earnings growth of the two types of agents. Filtered beliefs of the two agents are obtained from the discretized version of the belief processes in Lemma 1 as shown in eqs. (D1) and (D2) of Appendix 3 using the calibrated parameters for each type of agent shown in Table 1. The bottom panel has the actual and expected earnings growth of the two types of agents using these filtered beliefs.
The first panel has the model implied dispersion and the dispersion of four-quarter ahead earnings growth from the Survey of Professional Forecasts calculated as shown in (31). Model series are analogously calculated. The second panel shows the disagreement value process, \( \{ \eta_t \} \), implied by the belief series shown in Figure 3 and eq. (16). The third panel shows the consumption levels of each agent as calculated from (19) using the disagreement value process, \( \{ \eta_t \} \), in the second panel. The fourth panel shows the proportion of wealth in the stock held by investors of type 1 (pessimistic investors) calculated as \( w_t^{(1)} = \sigma_{X_t}^{(1)} \cdot (\sigma_{B_t}^T, \sigma_{P_t}^T)^{-1} \). The volatilities of stocks, bonds, and wealth are calculated as described in Section II A and at time \( t \) are calculated conditional on agents’ beliefs and disagreement value. Parameter values for fundamentals are as shown in Table III and in addition, \( \rho = 0.02 \) and \( \gamma = 0.5 \).
Distributions of the three statistics are obtained by resampling the series of fitted \textit{innovations}, \{\hat{W}(t)\} of type $m$ agents, with replacement $J = 10,000$ times (see, e.g. Runkle 1987). The innovations at time $t$ are $\hat{W}^{(m)}(t) = (\Sigma^T)^{-1}[\Delta \log(y)(t) - (\hat{\nu}^{(m)}(t|t-\Delta t) - \frac{1}{2} \text{diag}(\Sigma \Sigma^T)) \Delta t]$, where $\hat{\nu}^{(m)}(t|t-\Delta t)$ is Type $m$’s expected value of the drift conditional on information available up to time $t - \Delta t$. Paths of fundamentals on the $j$th resampling are created using: $\log(y)(t, j) = \log(y)(t-\Delta t, j) + \hat{\nu}^{(m)}(t|t-\Delta t, j) - \frac{1}{2} \text{diag}(\Sigma \Sigma^T) \Delta t + \hat{W}^{(m)}(t, j)$. Updating for each type of agent’s beliefs and the disagreement value on the following discrete interval is as described in Appendix 3. Parameter values for fundamentals are as shown in Table III and in addition, $\rho = 0.02$ and $\gamma = 0.5$. Statistics along each path are computed as described in the footnote to Table III.
Appendix A

Proof of Corollary 1 To establish that $\varrho_t$ (a zero drift process) is a martingale on $[0,t]$ under $\mathcal{P}^{(1)}$, it is sufficient to show that the Novikov condition holds, that is, $E^{(1)} \exp \left[ 1/2 \int_0^t (\tilde{\nu}_s^{(2)} - \tilde{\nu}_s^{(1)})^\top (\Sigma \Sigma^\top)^{-1} (\tilde{\nu}_s^{(2)} - \tilde{\nu}_s^{(1)}) ds \right] < \infty$ (see e.g., Proposition 2.24 in Nielsen 1999). But,

$$E^{(1)} \exp \left[ 1/2 \int_0^t (\tilde{\nu}_s^{(2)} - \tilde{\nu}_s^{(1)})^\top (\Sigma \Sigma^\top)^{-1} (\tilde{\nu}_s^{(2)} - \tilde{\nu}_s^{(1)}) ds \right] < \exp \left[ 1/2 (\tilde{\nu} - \lambda)^\top (\Sigma \Sigma^\top)^{-1} (\tilde{\nu} - \lambda) \cdot t \right] < \infty,$$

where $\tilde{\nu} = \max_i \nu_i^\top (\Sigma \Sigma^\top)^{-1} \nu_i$, and $\lambda = \min_i \nu_i^\top (\Sigma \Sigma^\top)^{-1} \nu_i$. Since $\varrho_t$ is positive and finite for all $t$, the measure $\mathcal{P}_t^{(2)}(A) = E^{(1)}[1_A \cdot \varrho_t]$ is equivalent to $\mathcal{P}_t^{(1)}$. An application of Girsanov’s Theorem to the relation between the two innovation processes in (7) then implies that the Radon-Nikodym derivative of $\mathcal{P}_t^{(2)}$ with respect to $\mathcal{P}_t^{(1)}$ is $\varrho_t$. ■

In proving Proposition 1 I will find the following result useful.

Lemma 3 If $(\phi_t^{(1)} - \phi_t^{(2)})^\top = \Sigma^{-1} (\tilde{\nu}_t^{(1)} - \tilde{\nu}_t^{(2)})$, then $\eta_t = \xi_t^{(1)}/\xi_t^{(2)}$ follows the process

$$\frac{d\eta_t}{\eta_t} = (\phi_t^{(2)} - \phi_t^{(1)}) d\tilde{W}_t^{(1)}.$$

Proof of Lemma 3 Let $\eta_t = g(\xi_t^{(1)}, \xi_t^{(2)})$. Then, its partial derivatives are $g_{\xi_t^{(1)}} = 1/\xi_t^{(1)}$; $g_{\xi_t^{(2)}} = -\xi_t^{(1)}/\xi_t^{(2)}$; $g_{\xi_t^{(1)}} g_{\xi_t^{(2)}}(\xi_t^{(1)}) = 0$; $g_{\xi_t^{(1)}} g_{\xi_t^{(2)}} = -1/(\xi_t^{(2)})^2$; $g_{\xi_t^{(2)}} g_{\xi_t^{(2)}} = 2\xi_t^{(1)}/(\xi_t^{(2)})^3$. Using the dynamics of the real kernels in (5), an application of Ito’s lemma implies that

$$d\eta_t = \frac{1}{\xi_t^{(2)}} \left( -\tau_t \xi_t^{(1)} dt - \phi_t^{(1)} \xi_t^{(1)} d\tilde{W}_t^{(1)} \right)$$

$$- \frac{\xi_t^{(1)}}{(\xi_t^{(2)})^2} \left( -\tau_t \xi_t^{(2)} dt - \phi_t^{(2)} \xi_t^{(2)} d\tilde{W}_t^{(2)} \right) + \frac{1}{2} \frac{2\xi_t^{(1)}}{(\xi_t^{(2)})^3} \phi_t^{(2)} \phi_t^{(2)}^\top dt - \frac{1}{(\xi_t^{(2)})^2} \xi_t^{(1)} \phi_t^{(2)}^\top dt.$$
Therefore,

\[
\frac{d\eta_t}{\eta_t} = (-r_t dt - \phi_t^{(1)} \, d\tilde{W}_t^{(1)}) - (-r_t dt - \phi_t^{(2)} \, d\tilde{W}_t^{(2)}) + \left(\phi_t^{(2)} \phi_t^{(2)\top} - \phi_t^{(1)} \phi_t^{(1)\top}\right) dt.
\]

Now using (7) and the stated condition and collecting terms completes the proof. \(\blacksquare\)

**Proof of Proposition** Suppose agents agree on prices at all dates. By the definition of the market prices of risk, for an asset with current payout flow rate of \(\delta_{it}\) and volatility \(\sigma_{it}\), the instantaneous risk premium for agent \(m\) is

\[
\hat{\mu}_{it}^{(m)} + \delta_{it} - r_t = \sigma_{it} \phi_t^{(m)\top}, \tag{A1}
\]

which implies that

\[
\hat{\mu}_{it}^{(1)} - \hat{\mu}_{it}^{(2)} = \sigma_{it} (\phi_t^{(1)} - \phi_t^{(2)})\top. \tag{A2}
\]

Since (14) and (A2) hold for every asset \(i\) at each time \(t\), (15) must hold.

Now suppose that (15) holds. By Lemma 8 \(\frac{d\eta_t}{\eta_t} = \sigma_{\eta_t} d\tilde{W}_t^{(1)} = \frac{d\eta_t}{\eta_t}\). By Corollary 1, \(\eta_t\) is the Radon-Nikodym derivative of \(P_t^{(2)}\) with respect to \(P_t^{(1)}\). Fix an arbitrary time horizon \(T\). Then,

\[
E_t^{(2)} \left[ \int_t^T \xi_s^{(2)} \, \delta_s \, ds \right] = E_t^{(1)} \left[ \int_t^T \xi_s^{(2)} \, \frac{\eta_t}{\eta_t} \, \delta_s \, ds \right] = E_t^{(1)} \left[ \int_t^T \xi_s^{(1)} \, \delta_s \, ds \right] \tag{A3}
\]

where the first equality follows from the definition of a Radon-Nikodym derivative and the second from the definition of \(\eta_t\). Therefore agents agree on the level of the expected discounted value of fundamentals up to a fixed horizon \(T\). Now, since \(\delta_t\) and \(\xi_t^{(m)}\) are positive, both discounted values are positive and increase in \(T\). Letting \(T \to \infty\), by the monotone convergence theorem then implies that the agents agree on the discounted value of fundamentals. Under their respective transversality conditions, they agree on the level on prices, as claimed. \(\blacksquare\)
Proof of Lemma 2. By individual m’s first order condition for optimal consumption, $c_t^{(m)} = I_m(y_m \xi_t^{(m)})$, where $y_m$ is the Lagrange multiplier for agent m. A straightforward application of Ito’s lemma implies that $dc_m = \partial I_m(y_m \xi_t^{(m)})/\partial \xi^{(m)} d\xi^{(m)} + 1/2 \partial^2 I_m(y_m \xi_t^{(m)})/\partial \xi^{(m)} \partial \xi^{(m)} (d\xi^{(m)})^2$.

Since the optimum condition can also be written as $u_m'(c_t^{(m)}) = y_m \xi_t^{(m)}$, I can also write $c_t^{(m)} = I_m(u_m'(c_t^{(m)}))$. Differentiating both sides of the equality implies that $1 = I'_m(y_m \xi_t^{(m)}) u''(c_t^{(m)})$.

By the chain rule, $\partial I_m(y_m \xi_t^{(m)})/\partial \xi^{(m)} = I'_m(y_m \xi_t^{(m)}) y_m = y_m/u_m''(c_t^{(m)})$. Using the characterization of individual m’s state price density in 12, I obtain $\sigma_{c_t}^{(m)} = -y_m \xi_t^{(m)}/u_m''(c_t^{(m)}) \phi^{(m)} = -u_m'(c_t^{(m)})/u_m''(c_t^{(m)}) \phi^{(m)}$, which equals the statement of the volatility. Similarly, differentiating consumption twice, I obtain $0 = I''_m(u_m'(c_t^{(m)})) u''(c_t^{(m)})^2 + I'_m(u_m'(c_t^{(m)})) u'''(c_t^{(m)})$, hence $I''_m(y_m \xi_t^{(m)}) = -u''(c_t^{(m)})/u_m''(c_t^{(m)})^3$, and $\partial^2 I_m(y_m \xi_t^{(m)})/\partial \xi^{(m)} = -u_m''(c_t^{(m)})/u_m''(c_t^{(m)})^3 y_m^2$.

Therefore, $\mu_c^{(m)} = y_m \xi_t^{(m)}/u_m''(c_t^{(m)})(-r_t) + 1/2 u_m''(c_t^{(m)})/u_m''(c_t^{(m)})^3 y_m^2 \xi_t^{(m)}$, which equals the drift term in the statement. ■

Proof of Proposition 2.

To facilitate the analysis of equilibrium, I follow the approach of Cuoco and Hè (1994) to solve for the equilibrium in the effectively incomplete markets model by formulating stochastic weights for the representative agent. The solution method was extended to models with heterogeneous beliefs by Basak (2000).13 For given weights $\lambda_{1t}$ and $\lambda_{2t}$ for the two agents, the representative agent’s utility function solves:

$$U(c_t; \lambda_{1t}, \lambda_{2t}) = e^{-\rho t} \max_{c_{1t}+c_{2t}=c_t} \lambda_{1t} \frac{c_{1t}^{\gamma}}{\gamma} + \lambda_{2t} \frac{c_{2t}^{\gamma}}{\gamma}.$$  

Solving this problem gives the equivalent form:

$$U(c_t; \lambda_{1t}, \lambda_{2t}) = e^{-\rho t} \lambda_{1t} \left(1 + \left(\frac{\lambda_{2t}}{\lambda_{1t}}\right)^{1-\gamma}\right)^{1-\gamma}.$$  \hspace{1cm} (A4)
Following the analysis in Basak (2000), I formulate the equilibrium with the weights \( \lambda_{1t} = 1/y_1 \) and \( \lambda_{2t} = \eta_t/y_2 \), where \( y_1 \) and \( y_2 \) are the Lagrange multipliers associated with the budget constraints of the two agents at time 0 in eq. (8). It is evident that with these weights, consumption allocations coincide with those of competitive equilibrium: that is they satisfy \( u'(c^{(1)}_t)/u'(c^{(2)}_t) = (y_1\xi^{(1)}_t)/(y_2\xi^{(2)}_t) \), the ratio of individuals’ optimality conditions, and by construction, the goods market clears.

I define the inverse function of the representative agent’s marginal utility of aggregate consumption as \( U_c(q_t; \lambda_{1t}, \lambda_{2t})^{-1} \equiv I(z_t; \lambda_{1t}, \lambda_{2t}) \equiv I_1(z_t; y_1) + I_2(z_t; y_2) \). Now, by the special choice of the weights, \( \lambda_{1t} = 1/y_1 \) and \( \lambda_{2t} = \eta_t/y_2 \), where \( \eta_t = \xi^{(1)}_t/\xi^{(2)}_t \), \( I(z_t; y_1, y_2, \eta_t) = I_1(y_1 z_t) + I_2(y_2/\eta_t z_t) \). Therefore, \( I(z_t; y_1, y_2, \eta_t) = I_1(y_1 \xi^{(1)}_t) + I_2(y_2 \xi^{(2)}_t) = c_{1t} + c_{2t} = x_t \).

Furthermore, since \( U_c(\cdot)^{-1} = I(\cdot), I(\xi^{(1)}_t) = x_t \), or \( U_c(x_t) = \xi^{(1)}_t \). In addition, \( U_c(\cdot)/\eta_t = \xi^{(2)}_t \).

Now using the form of the representative agent’s utility function in (A4) gives the individual consumption processes in (i).

Using the characterization of agent 1’s state price density, \( U_c(x) = \xi^{(1)}_t \), for the weights \( \lambda_{1t} = 1/y_1 \) and \( \lambda_{2t} = \eta_t/y_2 \), I obtain

\[
\xi^{(1)}_t = e^{-\rho t} x_t^{\gamma - 1} \frac{1}{y_1} \left[ 1 + \left( \frac{y_1 \eta_t}{y_2} \right)^{1/(1-\gamma)} \right]^{1-\gamma}.
\] (A5)

Since the drift of \( d\xi^{(1)}_t/\xi^{(1)}_t \) equals minus the short rate, an application of Ito’s lemma along with the equations for the processes \( x_t \) and \( \eta_t \) in (2) and (16) implies the riskless rate in (ii).

Market clearing for the consumption good implies that

\[
\sigma^{(1)}_{c,qt} + \sigma^{(2)}_{c,qt} = \sigma_{x,1} x_t,
\] (A6)

\[
\sigma^{(1)}_{c,xt} + \sigma^{(2)}_{c,xt} = \sigma_{x,2} x_t.
\] (A7)
Using the volatilities of consumption from Lemma 2 implies that the equilibrium conditions are:

\[
\frac{1}{a_t^{(1)}} \phi_{q_t}^{(1)} + \frac{1}{a_t^{(2)}} \phi_{q_t}^{(2)} = \sigma_{x,1}x_t \quad \text{and} \quad (A8)
\]

\[
\frac{1}{a_t^{(1)}} \phi_{x_t}^{(1)} + \frac{1}{a_t^{(2)}} \phi_{x_t}^{(2)} = \sigma_{x,2}x_t \quad \text{(A9)}
\]

Now equations (15), (A8), and (A9) contain four equations in the four market prices of risk, that lead to the unique solution

\[
\phi_{q_t}^{(1)} = \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \frac{\sigma_{x,2} (\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)}) + \sigma_{q,2} (\bar{\kappa}_t^{(2)} - \bar{\kappa}_t^{(1)})}{|\Sigma|} + \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \sigma_{x,1}x_t, \quad (A10)
\]

\[
\phi_{q_t}^{(2)} = \frac{a_t^{(2)}}{a_t^{(1)} + a_t^{(2)}} \frac{\sigma_{x,2} (\bar{\theta}_t^{(2)} - \bar{\theta}_t^{(1)}) + \sigma_{q,2} (\bar{\kappa}_t^{(1)} - \bar{\kappa}_t^{(2)})}{|\Sigma|} + \frac{a_t^{(2)}}{a_t^{(1)} + a_t^{(2)}} \sigma_{x,1}x_t, \quad (A11)
\]

\[
\phi_{x_t}^{(1)} = \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \frac{\sigma_{x,1} (\bar{\theta}_t^{(2)} - \bar{\theta}_t^{(1)}) + \sigma_{q,1} (\bar{\kappa}_t^{(1)} - \bar{\kappa}_t^{(2)})}{|\Sigma|} + \frac{a_t^{(1)}}{a_t^{(1)} + a_t^{(2)}} \sigma_{x,2}x_t, \quad (A12)
\]

\[
\phi_{x_t}^{(2)} = \frac{a_t^{(2)}}{a_t^{(1)} + a_t^{(2)}} \frac{\sigma_{x,1} (\bar{\theta}_t^{(1)} - \bar{\theta}_t^{(2)}) + \sigma_{q,1} (\bar{\kappa}_t^{(2)} - \bar{\kappa}_t^{(1)})}{|\Sigma|} + \frac{a_t^{(2)}}{a_t^{(1)} + a_t^{(2)}} \sigma_{x,2}x_t, \quad (A13)
\]

Notice that the market prices of risk depend on beliefs of investors of each type through the conditional means of each of the state variables, as well as their risk-aversions and consumption levels, through the coefficients \(a_m\). For the case of CRRA preferences, \(a_t^{(m)} = -u''[c^{(m)}]/u'[c^{(m)}] = (1 - \gamma)/c_t^{(m)}\). Substituting these into eqs. (A10) – (A13) implies (iii). □

**Proof of Corollary 2** Equations (19) and (21) – (24), imply that \(\phi_t^{(1)} \frac{\xi^{(1)}_t}{x_t} + \phi_t^{(2)} \frac{\xi^{(2)}_t}{x_t} = (1 - \gamma)\sigma_x\). Multiplying (24) by \(\xi^{(m)}_t\) for each m, adding across agents, and using the above equality completes the proof. □

**Proof of Proposition 3** Proposition 2 provides consumption processes and SPDs for each type as functions of \(k\) that are consistent with utility maximization, market clearing, and the structure of
the state variables in the economy. Proposition (agreement of values) implies that for all $k$,

$$
\begin{align*}
\sum_{m=1}^{2} X^{(m)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k) &= E^{(1)}[\int_0^\infty \xi_s^{(1)} c_s^{(1)} ds] + E^{(2)}[\int_0^\infty \xi_s^{(2)} c_s^{(2)} ds] \\
&= E^{(1)}[\int_0^\infty \xi_s^{(1)} c_s^{(1)} ds] + E^{(1)}[\int_0^\infty \xi_s^{(1)} c_s^{(2)} ds] \\
&= E^{(1)}[\int_t^\infty \xi_s^{(1)} x_s ds] = E^{(1)}[\int_t^\infty \xi_s^{(1)} (e^{(1)} + e^{(2)}) x_s ds] \\
&= E^{(1)}[\int_t^\infty \xi_s^{(1)} x_s ds] + E^{(2)}[\int_t^\infty \xi_s^{(2)} e^{(2)} x_s ds] \\
&= \sum_{m=1}^{2} V^{(m)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k),
\end{align*}
$$

where the second equality holds because agents agree on the value of agent 2’s consumption flow, the third and fourth from consumption flows in (19) and Assumption 5 respectively, and the fifth because the agents agree on the value of agent 2’s endowment stream. With this equality, I have that when $X^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k) = V^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k)$, then $X^{(2)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k) = V^{(2)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k)$ as well. Next notice that $X^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k) - V^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k)$ is a continuous function of $k$, is positive at $k = 0$ and negative as $k \to \infty$ since by (19), $c_s^{(1)}(k = 0) - e^{(1)} x_s > 0, \forall s$ and $c_s^{(1)}(k \to \infty) - e^{(1)} x_s < 0, \forall s$. Therefore, there exists a $k^* > 0$ such that $X^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k^*) = V^{(1)}(\pi^{*(1)}, \pi^{*(2)}, \eta_0; k^*)$, and hence equilibrium exists.

The derivative $\frac{\partial(X^{(1)}-V^{(1)})}{\partial k} = E^{(1)}[\int_0^\infty A_s \cdot B_s \cdot (C_s - D_s) ds]$, in which $A_s = (k \eta_s^{-\gamma}) + 1)^{-\gamma}, \quad B_s = (k \eta_0^{-\gamma} + 1)\gamma^{-2}, \quad C_s = \eta_s^{1-\gamma}(e^{(m)} (\gamma - 1)[k \eta_s^{-\gamma} + 1] - \gamma), \quad D_s = ([e^{(m)} \cdot (\gamma - 1) + 1] k \eta_0^{-\gamma} + (e^{(m)} - 1)(\gamma - 1))\eta_0^{-\gamma}$. It is straightforward to see that $A_s > 0, B_s > 0, C_s < 0, and D_s > 0$ for all $s$ when $0 \leq \gamma < 1$, and hence the derivative is strictly negative, which implies that $k^*$ is unique. ■

Appendix B: Stocks in Positive Net Supply
In the model discussed so far, stocks have been in zero net supply. As claimed in footnote 4, with the alternative assumption of stocks being in ‘small’ positive supply the equilibrium premium and riskless rate are very similar. Now total consumption in any period equals $x_t + q_t$, the sum of dividends and output in the economy produced from resources not financed by public equity.

The additional complication is the introduction of another state variable, $q_t/(x_t + q_t)$, the share of dividends of total output. I am once again able to solve for the riskless rate and market prices of risk in this alternative economy. Similar to (20), I find the riskless rate to be

$$
r_t = \rho - \frac{1}{2} (2 - \gamma) (1 - \gamma) \left( \frac{x_t}{x_t + q_t} \sigma_x + \frac{q_t}{x_t + q_t} \sigma_q \right)^2 + \frac{q_t}{x_t + q_t} \frac{1 - \gamma}{1 + k \eta_t} \left( \theta_t^{(1)} + k \eta_t^{\frac{1}{1-\gamma}} \theta_t^{(2)} \right) + \frac{x_t}{x_t + q_t} \frac{1 - \gamma}{1 + k \eta_t} \left( \kappa_t^{(1)} + k \eta_t^{\frac{1}{1-\gamma}} \kappa_t^{(2)} \right) - \frac{\gamma k \eta_t^{\frac{1}{1-\gamma}} \left( (\theta_t^{(1)} - \theta_t^{(2)}) \sigma_x - (\kappa_t^{(1)} - \kappa_t^{(2)}) \sigma_q \right)^2}{2 (1 - \gamma) \left( 1 + k \eta_t^{\frac{1}{1-\gamma}} \right)^2 |\Sigma|^2}.
$$

As before, the terms have similar interpretation, where the first term, for precautionary savings, must now incorporate the weighted average of volatility of the two processes, and the wealth effect term similarly incorporates the expected growth of dividend growth as well. As $q_t/(x_t + q_t) \to 0$, I get back (20). Notably, the speculative risk component (last term), is identical in the two expressions.

Similarly solving for the market prices of risk (I only provide the equation for $\phi_q^{(1)}$)

$$
\phi_q^{(1)} = (1 - \gamma) \left( \frac{q_t}{q_t + x_t} \sigma_{q,1} + \frac{x_t}{q_t + x_t} \sigma_{x,1} \right) + \frac{k \eta_t^{\frac{1}{1-\gamma}}} {1 + k \eta_t^{\frac{1}{1-\gamma}}} \left( \theta_t^{(1)} - \theta_t^{(2)} \right) \sigma_{x,2} + \left( \kappa_t^{(2)} - \kappa_t^{(1)} \right) \sigma_{q,2},
$$

which is identical to (21) with the sole exception that the market price of aggregate risk contains the weighted average of dividend and consumption volatilities. Once again, the price of the speculative risk component is the same.
Solving for asset prices would be similar, but I would have to incorporate the additional state variable, which would be tedious but is still possible using projection methods. Nonetheless, since the equity premium puzzle can be restated as the difficulty of attaining a low riskless return and high Sharpe ratio, it is straightforward to assess if the change in assumption will significantly affect my results. Calibrating my economy to aggregate dividends and output I find that the ratio $q_t/(q_t + x_t)$ is on average about 2.6%, and for the case $\gamma = 0.5$, leads to an average riskless rate which is higher only by four basis points. The values of the four market prices of risk also differ by similar amounts relative to the zero net supply case. Therefore, the alternative assumption do not affect Sharpe ratios under the measures of the different agents by large amounts.

References


Blume, Lawrence, and David Easley, 2004, If you’re so smart, why aren’t you rich? belief selection in complete and incomplete markets, Discussion paper Department of Economics, Cornell University. 28


Cuoco, Domenico, and Hua Hè, 1994, Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets, Mimeo The Wharton School, University of Pennsylvania.


Shefrin, Hersh, 2001, On kernels and sentiment, mimeo Santa Clara University.


