Multi-Factor Lévy-Based Models

in

Financial and Energy Markets

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Outline of Presentation

1. Literature Review
2. Lévy and α-Stable Processes
3. Gaussian- and Lévy-Based Models
4. Change of Time Method: Gaussian- and Lévy-Based
5. Pricing of Financial and Energy Derivatives
6. Conclusion

Literature Review on Time-Changing

Bochner (1949): introduced the notion of change of time (time-changed Brownian motion)

Clark (1973): introduced Bochner's change of time into financial economics

Feller (1966): introduced subordinated process $X(T(t))$ with Markov process $X(t)$ and $T(t)$ as a process with independent increments ($T(t)$ was called 'randomized operational time')

Johnson (1979): introduced time-changed stochastic volatility model (SVM) in continuous time

Literature Review on Time-Changing

Johnson & Shanno (1987): studied pricing of options using time-changed SVM

Ikeda & Watanabe (1981): introduced and studied change of time for the solution of SDEs

Barndorff-Nielsen, Nicolato & Shephard (2003): studied relatiomship between subordination and SVM using change of tuime ($T(t)$-'chronometer')

Carr, Geman, Madan, Yor (2003): used subordinated processes to construct SV for Lévy processes ($T(t)$-'business time')

Literature Review on Time-Changing: Embedding Problem

The change of time method is closely associated with the *embedding problem*: to embed a process $X(t)$ in Brownian motion is to find a Wiener process process $W(t)$ and an increasing family of stopping times $T(t)$ such that $W(T(t))$ has the same joint distribution as $X(t)$.

Skorokhod (1965): first treated the embedding problem, showing that the sum of any sequence of independent r.v. with mean zero and finite variation could be embedded in Brownian motion using stopping times

Literature Review on Time-Changing: Lévy Processes

Dambis (1965), Dubins & Schwartz (1965): independently showed that every continuous martingale could be embedded in Brownian motion

Knight (1971): discovered multivariate extension of Dambis (1965), Dubins & Schwartz (1965) result

Huff (1969): showed that every process of pathwise bounded variation could be embedded in Brownian motion

Literature Review on Time-Changing: Lévy Processes

Monroe (1972): proved that every right continuous martingale could be embedded in a Brownian motion

Monroe (1978): proved that a process can be embedded in Brownian motion if and only if this process is a local semimartingale

Literature Review on Time-Changing: Lévy Processes

Meyer (1971), Papangelou (1972): independently discovered Knight's (1971) result for point processes

Rosiński & Woyczyński (1986): considered time changes for integrals over a stable Lévy processes

Kallenberg (1992): considered time change representations for stable integrals

Literature Review on Time-Changing: Lévy Processes

Lévy processes can also be used as a time change for other Lévy processes (*subordinators*)

Madan & Seneta (1990): introduced Variance Gamma (VG) process (Brownian motion with drift time changed by a gamma process)

Geman, Madan & Yor (2001): considered time changes for Lévy processes ('business time')

Literature Review on Time-Changing: Lévy Processes

Carr, Geman, Madan & Yor (2003): used change of time to introduce stochastic volatility into a Lévy model to achieve leverage effect and a long-term skew

Kallsen & Shiryaev (2001): showed that Rosiński-Woyczyński-Kallenberg statement can not be extended to any other Lévy processes but symmetric α-stable

Swishchuk (2004, 2007): applied change of time method for options and swaps pricing for Gaussian models

One-Factor and Multi-Factor Gaussian Models

One-Factor Gaussian Models

1. *The Geometric Brownian Motion Model*.
$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$.

2. *The Continuous-Time GARCH Model*.
$dS(t) = \mu(b - S(t))dt + \sigma S(t)dW(t)$.

Another models such as OU, Vasićek, CIR, Hull-White, HJM, should be mentioned as well.

One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Multi-Factor Gaussian SIRMs

Multi-factor models driven by Brownian motions can be obtained using various combinations of above-mentioned processes. We give one example of two-factor SIRM:

$$\begin{cases} dS(t) &= \mu(b(t) - S(t))dt + \sigma S(t) dW^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t) dW^2(t), \end{cases}$$

where W^1, W^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Change of Time Method for SDE driven by Brownian motion

Definition 1. A *time change* is a right-continuous increasing $[0, +\infty]$-valued process $(T_t)_{t \in R_+}$ such that T_t is a stopping time for any $t \in R_+$. By $\hat{\mathcal{F}}_t := \mathcal{F}_{T_t}$ we define the time-changed filtration $(\hat{\mathcal{F}}_t)_{t \in R_+}$. The *inverse time change* $(\hat{T}_t)_{t \in R_+}$ is defined as $\hat{T}_t := \inf\{s \in R_+ : T_s > t\}$. (See Ikeda and Watanabe (1983)).

We consider the following SDE driven by a *Brownian motion*:

$$dX(t) = a(t, X(t))dW(t),$$

where $W(t)$ is a Brownian motion and $a(t, X)$ is a continuous and measurable by t and X function on $[0, +\infty) \times R$.

Change of Time Method for SDE driven by Brownian motion (cntd)

Theorem. (*Ikeda and Watanabe* (1981), Chapter IV, Theorem 4.3) Let $\widehat{W}(t)$ be an one-dimensional $\mathcal{F}_t$-Wiener process with $\widehat{W}(0) = 0$, given on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and let $X(0)$ be an $\mathcal{F}_0$-adopted random variable.

Change of Time Method for SDE driven by Brownian motion (cntd)

Define a continuous process $V = V(t)$ by the equality

$$V(t) = X(0) + \widehat{W}(t).$$

Let T_t be the change of time process:

$$T_t = \int_0^t a^{-2}(T_s, X(0) + \widehat{W}(s))ds.$$

If

$$X(t) := V(\widehat{T}_t) = X(0) + \widehat{W}(\widehat{T}_t)$$

and $\widehat{\mathcal{F}}_t := \mathcal{F}_{\widehat{T}_t}$, then there exists $\widehat{\mathcal{F}}_t$-adopted Wiener process $W = W(t)$ such that $(X(t), W(t))$ is a solution of $dX(t) = a(t, X(t))dW(t)$ on probability space $(\Omega, \mathcal{F}, \widehat{\mathcal{F}}_t, P)$, where $\widehat{T}_t$ is the inverse to T_t time change.

Solutions to the One-Factor and Multi-Factor Gaussian Models

Solution of One-Factor Gaussian SIRMs Using CTM

We use change of time method (see Ikeda and Watanabe (1981)) to get the solutions to the following below equations (see Swishchuk (2007)). $W(t)$ below is an standard Brownian motion, and $\widehat{W}$ is a $(\hat{T}_t)_{t\in R_+}$-adapted standard Brownian motion on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t\in R_+}, P)$.

Solutions to the One-Factor and Multi-Factor Gaussian Models (cntd)

Solution of One-Factor Gaussian SIRMs Using CTM

1. *Geometric Brownian Motion*. $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$. Solution $S(t) = e^{\mu t}[S(0) + \hat{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t [S(0) + \hat{W}(\hat{T}_s)]^2 ds$.

2. *Continuous-Time GARCH Process*. $dS(t) = \mu(b - S(t))dt + \sigma S(t)dW(t)$. Solution $S(t) = e^{-\mu t}(S(0) - b + \hat{W}(\hat{T}_t)) + b$, where $\hat{T}_t = \sigma^2 \int_0^t [S(0) - b + \hat{W}(\hat{T}_s) + e^{\mu s} b)^2 ds$.

Analogous results may be written for many other well-known models: OU, Vasićek, CIR, Hull-White, HJM, etc.

Solutions to the One-Factor and Multi-Factor Gaussian Models (cntd)

Solution of Multi-Factor Gaussian Models Using CTM

Solution of *multi-factor models driven by Brownian motions* can be obtained using various combinations of solutions of the above-mentioned processes, see subsection 5.1, and CTM. We give one example of two-factor Continuous-Time GARCH model driven by Brownian motions:

$$\begin{cases} dS(t) &= r(t)S(t))dt + \sigma S(t)dW^1(t) \\ dr(t) &= a(m - r(t))dt + \sigma_2 r(t)dW^2(t), \end{cases}$$

where W^1, W^2 may be correlated, $m \in R, \sigma, a > 0$.

Solutions to the One-Factor and Multi-Factor Gaussian Models (cntd)

Solution of Multi-Factor Gaussian Models Using CTM

Solution, using CTM for the first and the second equations:

$$\begin{aligned} S(t) &= e^{\int_0^t r_s ds}[S_0 + \widehat{W}^1(\widehat{T}_t^1)] \\ &= e^{\int_0^t e^{-as}[r_0 - m + \widehat{W}^2(\widehat{T}_s^2)]ds}[S_0 + \widehat{W}^1(\widehat{T}_t^1)], \end{aligned}$$

where $\widehat{T}^i$ and $\widehat{W}^i$ are defined in 1. $(i = 1)$ and 2. $(i = 2)$, respectively.

Lévy Models

Lévy Processes

Definition 2. By *Lévy process* we define a stochastically continuous process with *stationary and independent increments*, Sato (1999), Applebaum (2003), Schoutens (2003).

Examples of Lévy Processes:

- linear function $L(t) = \gamma t$
- Brownian motion with drift
- Poisson process
- compound Poisson process

Lévy Models

Lévy-Khintchine Formula for Lévy Processes $L(t)$

$$\begin{aligned} E(e^{i(u,L(t))}) &= \exp\{t[i(u,\gamma) - \tfrac{1}{2}(u,Au) \\ &+ \textstyle\int_{R^d-\{0\}}[e^{i(u,y)} - 1 - i(u,y)\mathbf{1}_{B_1(0)}]\nu(dy)]\} \end{aligned}$$

(γ, A, ν)-*Lévy-Khintchine triplet*

Lévy Models

Interpretation of Lévy-Khintchine triplet

- γ stands for linear function, drift
- Diffusion matrix A stands for Brownian motion
- Lévy measure ν stands for jumps

Lévy Models

Lévy-Itô Decomposition

If L is a Lévy process, then there exists $\gamma \in R^d$, a Brownian motion B_A with covariance matrix A and an independent Poisson random measure N on $R^+ \times (R^d - \{0\})$ such that, for each $t \geq 0$,

$$L(t) = \gamma t + B_A(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx).$$

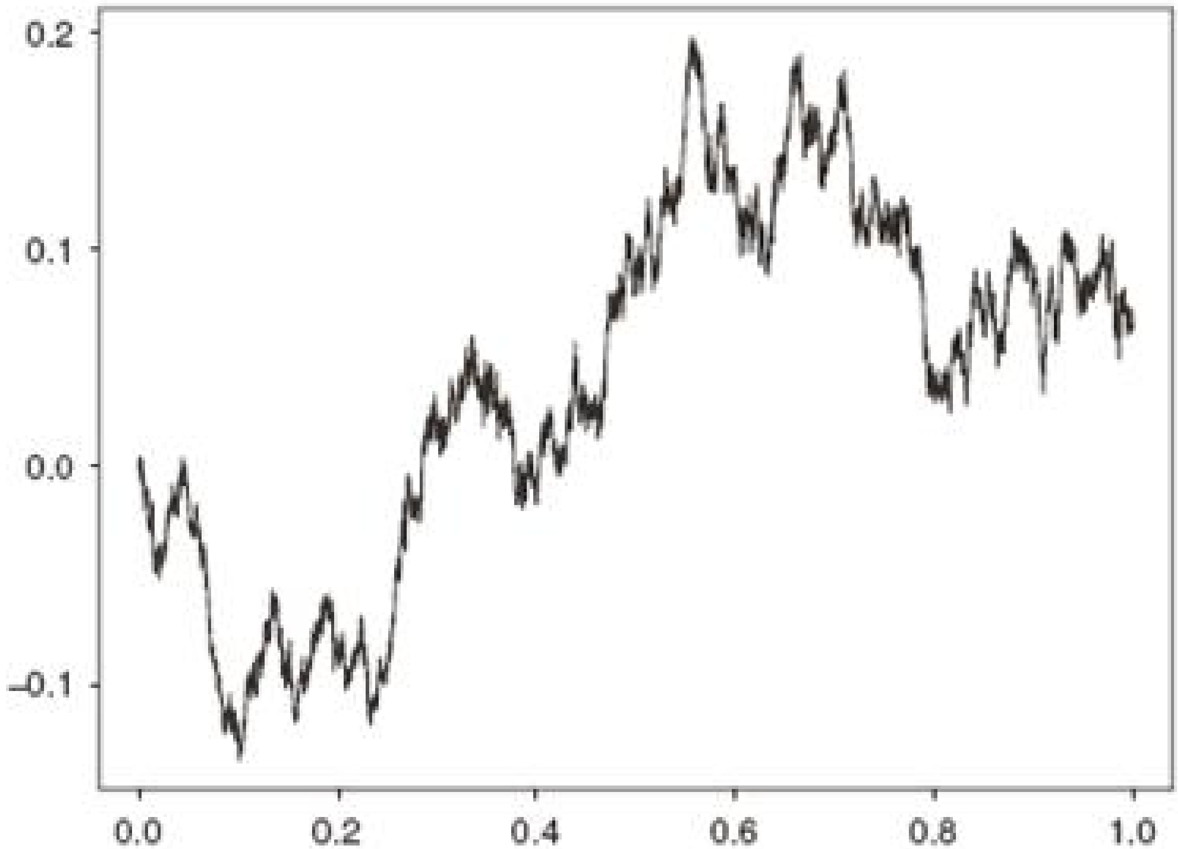

Figure 3: A sample path of a Brownian motion; $\Psi(\theta) = \theta^2/2$.

Fig. 3. Lévy Processes: Wiener Process (Brownian motion)

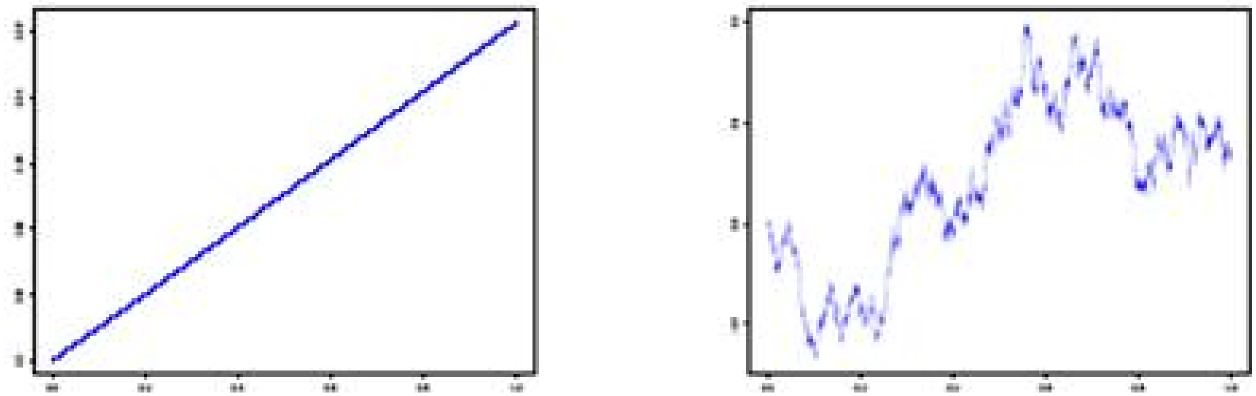

FIGURE 2.4. Examples of Lévy processes: linear drift (left) and Brownian motion

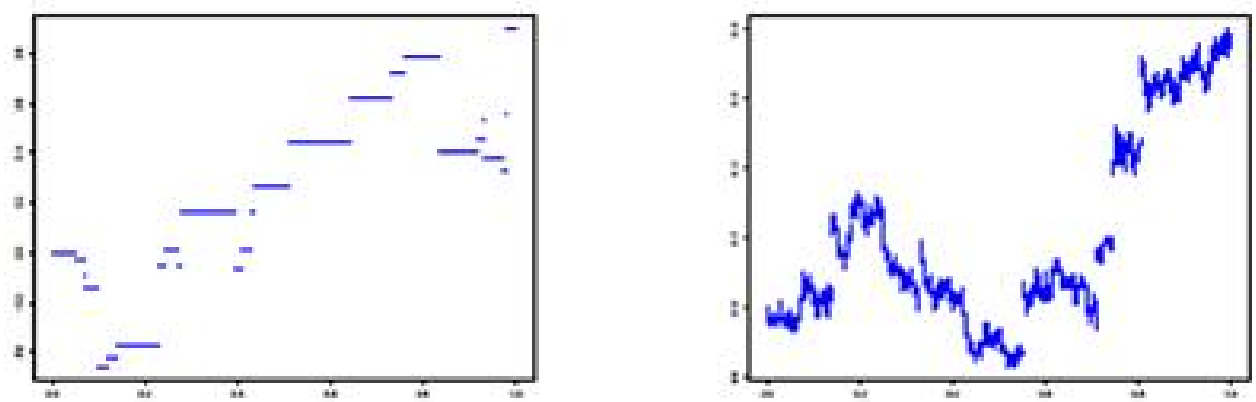

FIGURE 2.5. Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion

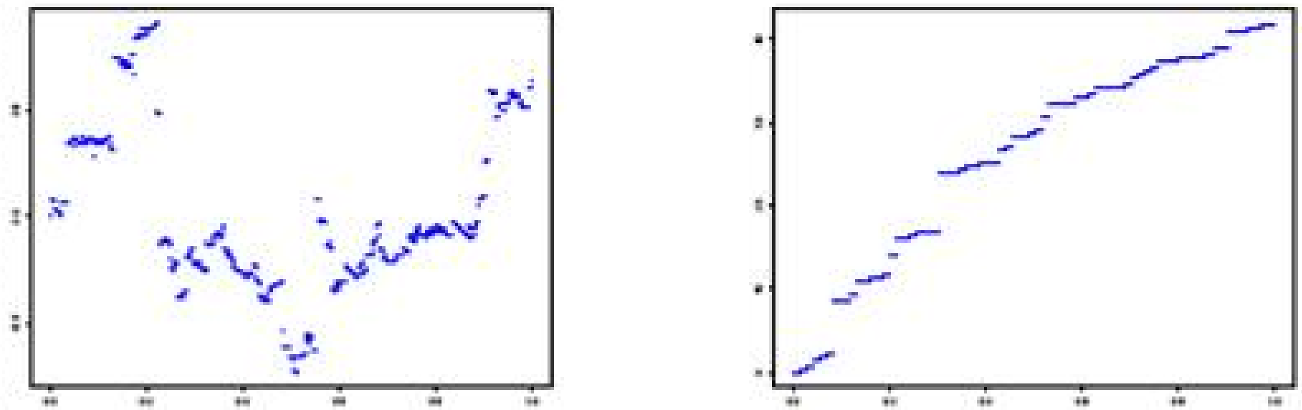

FIGURE 7.10. Simulated path of a normal inverse Gaussian (left) and an inverse Gaussian process

Fig. 3. Lévy Processes II

Lévy Models

Lévy Processes in Finance

- Brownian motion with drift (only continuous Lévy process)
- Merton model=Brownian motion+drift+Gaussian jumps
- Kou model=Brownian motion+drift+exponential jumps
- VG, IG, NIG, Generalized hyperbolic processes
- *α-stable Lévy processes*

Lévy Models

Symmetric α-Stable ($S\alpha S$) Distribution (cntd)

Characteristic function:

$$\phi(u) = e^{(i\delta u - \sigma|u|^{\alpha})},$$

where α is the *characteristic exponent* ($0 < \alpha \leq 2$), $\delta \in (-\infty, +\infty)$ is the *location* parameter, and $\sigma > 0$ is the *dispersion*.

Lévy Models

Symmetric α-Stable ($S\alpha S$) Distribution (cntd)

For values of $\alpha \in (1, 2]$ location parameter δ corresponds to the mean of the α-stable distribution, while for $0 < \alpha \leq 1$, δ corresponds to its median.

The dispersion parameter σ corresponds to the spread of the distribution around its location parameter δ.

The characteristic exponent α determines the shape of the distribution.

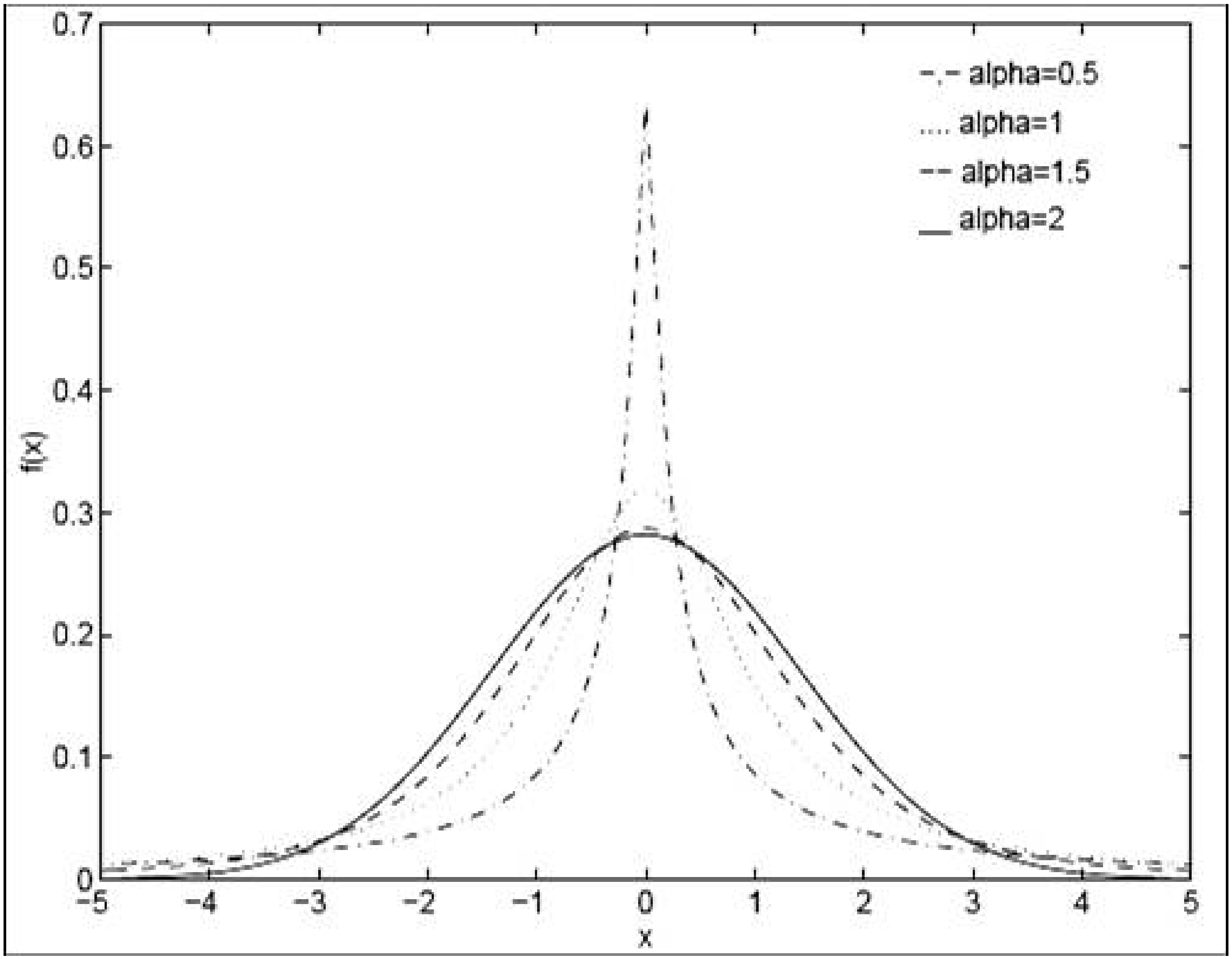

Fig. 1. Standard $S\alpha S$ densities

Lévy Models

Symmetric α-Stable ($S\alpha S$) Distribution (cntd)

A stable distribution is called *standard* if $\delta = 0$ and $\sigma = 1$.

If a random variable L is stable with parameters α, δ, σ, then $(L - \delta)/\sigma^{1/\alpha}$ is sdandard with characteristic exponent α.

By letting α take the values 1/2, 1 and 2, we get three important special cases: the *Lévy ($\alpha = 1/2$)*, *Cauchy ($\alpha = 1$)* and the *Gaussian ($\alpha = 2$)* distributions:

$$\begin{aligned} f_{1/2}(\gamma, \delta; x) &= \left(\frac{t}{2\sqrt{\pi}}\right) x^{-3/2} e^{-t^2/(4x)} \\ f_1(\gamma, \delta; x) &= \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x-\delta)^2}, \\ f_2(\gamma, \delta; x) &= \frac{1}{\sqrt{4\pi\gamma}} \exp\left[-\frac{(x-\delta)^2}{4\gamma}\right]. \end{aligned}$$

Lévy Models

Symmetric α-Stable ($S\alpha S$) Distribution (cntd)

Unfortunately, no closed form expression exist for general α-stable distribution other than the Lévy, the Cauchy and the Gaussian.

However, power series expansions can be derived for density $f_\alpha(\delta, \sigma; x)$.

Its tails (algebraic tails) decay at a lower rate than the Gaussian density tails (exponential tails).

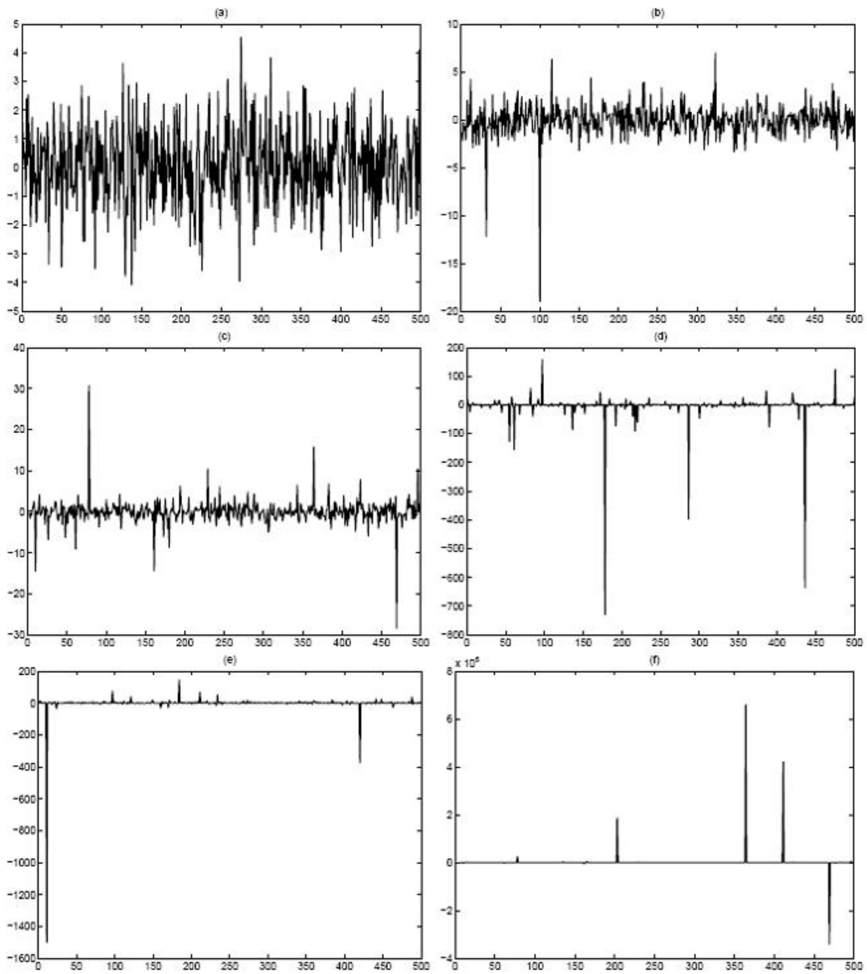

Fig. 2. $S\alpha S$ time series. a) $\alpha = 2$, b) $\alpha = 1.95$, c) $\alpha = 1.5$, d) $\alpha = 1.0$, e) $\alpha = 0.85$, f) $\alpha = 0.45$.

Lévy Models

Symmetric α-Stable ($S\alpha S$) Distribution (cntd)

The smaller the characteristic exponent α is, the heavier the tails of the α-stable density.

This implies that random variables following α-stable distribution with small characteristic exponent are *highly impulsive*, and it is this heavy-tail characteristic that makes this density appropriate for modeling noise which is impulsive in nature, for example, electricity prices or volatility.

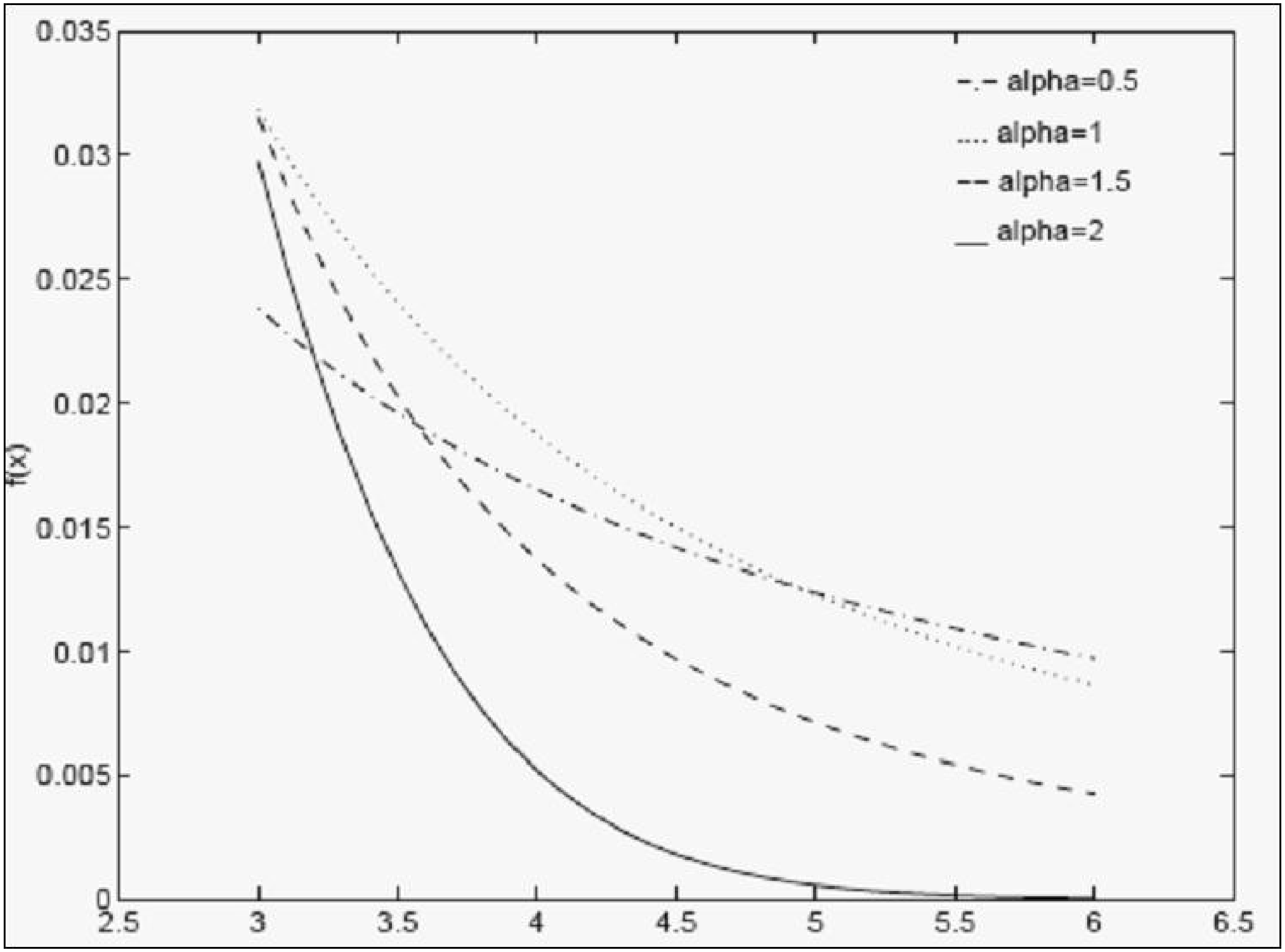

Fig. 3. Tails of the densities in Figure 1

Lévy Models

Symmetric α-Stable ($S\alpha S$) Distribution (cntd)

Only moments of order less than α exist for the non-Gaussian family of α-stable distribution.

The fractional lower order moments with zero location parameter and dispersion σ are given by

$$\begin{aligned} E|X|^p &= D(p,\alpha)\sigma^{p/\alpha}, \quad for \quad 0 < p < \alpha, \\ D(p,\alpha) &= \frac{2^p \Gamma(\frac{p+1}{2})\Gamma(1-\frac{p}{\alpha})}{\alpha\sqrt{\pi}\Gamma(1-\frac{p}{2})}, \end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function (Sato (2005)).

Lévy Models

Symmetric α-Stable ($S\alpha S$) Distribution (cntd)

Since the $S\alpha S$ r.v. has 'infinite variance', the covariation of two jointly $S\alpha S$ real r.v. with dispersions γ_x and γ_y defined by

$$[X, Y]_\alpha = \frac{E[X|Y|^{p-2}Y]}{E[|Y|^p]}\gamma_y$$

has often been used instead of the covariance (and correlation), where $\gamma_y = [Y, Y]_\alpha$ is the dispersion of r.v. Y.

Lévy Models

α-stable Lévy Processes

Definiton 3. Let $\alpha \in (0,2]$. An *α-stable Lévy process* L such that L_1 (or equivalently any L_t) has a strictly α-stable distribution (i.e., $L_1 \equiv S_\alpha(\sigma,\beta,\delta)$) for some $\alpha \in (0,2] \setminus \{1\}$, $\sigma \in R_+$, $\beta \in [-1,1]$, $\delta = 0$ or $\alpha = 1$, $\sigma \in R_+$, $\beta = 0$, $\delta \in R$). We call L a *symmetric α-stable Lévy process* if the distribution of L_1 is even symmetric α-stable (i.e., $L_1 \equiv S_\alpha(\sigma,0,0)$ for some $\alpha \in (0,2]$, $\sigma \in R_+$.) A process L is called $(T_t)_{t\in R_+}$-adapted if L is constant on $[T_{t-},T_t]$ for any $t \in R_+$. (See Sato (2005)).

Lévy Models

α-stable Lévy Processes (cntd)

- the only self-similar Lévy processes: $L(at) \stackrel{Law}{=} a^{1/\alpha} L(t), a \geq 0$

- either Brownian motion or pure jump

- characteristic exponent, Lévy-Khintchine triplet known in closed form

- 4 parameters

- infinite variance (except for Brownian motion)

Lévy Models

α-stable Lévy Processes (cntd)

• α-stable Lévy Processes are semimartingales ($\int_0^t f_s dL_s$ can be defined)

• α-stable Lévy Processes are pure discontinuous Markov processes with generator

$$Af(x) = \int_{R-\{0\}} [f(x+y) - f(x) - yf'(y)\mathbf{1}_{|y|<1}(y)] \frac{K_\alpha}{|y|^{1+\alpha}} dy$$

Lévy Models

α-stable Lévy Processes (cntd)

$E|L(t)|^p$ is finite or infinite according as $0 < p < \alpha$ or $p > \alpha$, respectively.

In particular, for an α-stable process $EL(t) = \delta t$ $(1 < \alpha < 2)$ (Sato (2005)).

Lévy Models

SDE driven by α-stable Lévy Processes

$$dX_t = b(X_{t-})dt + \sigma(X_{t-})dL(t)$$

Janicki, Michna & Weron (1996): there exists unique solution for continuous b, σ and α-stable Lévy process $S_\alpha((t-s)^{1/\alpha}, \beta, \delta), \beta \in [-1, +1]$.

Zanzotto (1997): solutions of one-dimensional SDEs driven by stable Lévy motion

Cartea & Howison (2006): option pricing with Lévy-stable processes generated by Lévy-stable integrated variance

Lévy Models

One-Factor Lévy Models

$L(t)$ below is a symmetric α-stable Lévy process. We define below various processes via SDE driven by α-stable Lévy process.

1. *Geometric α-stable Lévy Motion*.
$dS(t) = \mu S(t-)dt + \sigma S(t-)dL(t)$.

2. *Ornstein-Uhlenbeck Process Driven by α-stable Lévy Motion*.
$dS(t) = -\mu S(t-)dt + \sigma dL(t)$.

Lévy Models

One-Factor Lévy Models

3. *Vasićek Process Driven by α-stable Lévy Motion*.
$dS(t) = \mu(b - S(t-))dt + \sigma dL(t)$.

4. *Continuous-Time GARCH Process Driven by α-stable Lévy process*.
$dS(t) = \mu(b - S(t-))dt + \sigma S(t-)dL(t)$.

5. *Cox-Ingersoll-Ross Process Driven by α-stable Lévy Motion*.
$dS(t) = k(\theta - S(t-))dt + \gamma\sqrt{S(t-)}dL(t)$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy Models

6. *Ho and Lee Process Driven by α-stable Lévy Motion*.
$dS(t) = \theta(t-)dt + \sigma dL(t)$.

7. *Hull and White Process Driven by α-stable Lévy Motion*.
$dS(t) = (a(t-) - b(t-)S(t-))dt + \sigma(t)dL(t)$

Lévy Models

One-Factor Lévy Models

8. *Heath, Jarrow and Morton Process Driven by α-stable Lévy Motion*. Define the forward interest rate $f(t, s)$, for $t \leq s$, that represents the instanteneous interest rate at time s as ‘anticipated‘ by the market at time t.

Lévy Models

One-Factor Lévy Models

8. *Heath, Jarrow and Morton Process Driven by α-stable Lévy Motion (cntd)*.
The process $f(t,u)_{0\le t\le u}$ satisfies an equation

$$f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dL(v),$$

where the processes a and b are continuous.

Eberlein & Raible (1999): Lévy-based term structure models

Lévy Models

Multi-Factor Lévy Models

Multi-factor models driven by α-stable Lévy motions can be obtained using various combinations of above-mentioned processes. We give one example of two-factor continuous-time GARCH model driven by α-stable Lévy motions.

$$\begin{cases} dS(t) &= r(t-)S(t-))dt + \sigma S(t-)dL^1(t) \\ dr(t) &= a(m - r(t-))dt + \sigma_2 r(t-)dL^2(t), \end{cases}$$

where L^1, L^2 may be correlated, $m \in R, \sigma_i, a > 0, i = 1, 2.$

Lévy Models

Multi-Factor Lévy Models

Also, we can consider various combinations of models, presented above, i.e., mixed models containing Brownian and Lévy motions. For example,

$$\begin{cases} dS(t) &=& \mu(b(t-) - S(t-))dt + \sigma S(t-)dL(t) \\ db(t) &=& \xi b(t)dt + \eta b(t)dW(t), \end{cases}$$

where Brownian motion $W(t)$ and Lévy process $L(t)$ may be correlated.

Change of Time Method for SDE Driven by Lévy Motion

We denote by $L^{\alpha}_{a.s.}$ the family of all real measurable $\mathcal{F}_t$-adapted processes a on $\Omega \times [0, +\infty)$ such that for every $T > 0$, $\int_0^T |a(t,\omega)|^{\alpha} dt < +\infty$ a.s. We consider the following *SDE driven by a Lévy motion*:

$$dX(t) = a(t, X(t-))dL(t).$$

Change of Time Method for SDE Driven by Lévy Motion

Theorem. *(Rosinski and Woyczynski (1986)*, Theorem 3.1., p.277). Let $a \in L^{\alpha}_{a.s.}$ be such that $T(u) := \int_0^u |a|^{\alpha} dt \to +\infty$ a.s. as $u \to +\infty$. If $\hat{T}(t) := \inf\{u : T(u) > t\}$ and $\hat{\mathcal{F}}_t = \mathcal{F}_{\hat{T}(t)}$, then the time-changed ctochastic integral $\hat{L}(t) = \int_0^{\hat{T}(t)} adL(t)$ is an $\hat{\mathcal{F}}_t - \alpha$-stable Lévy process, where $L(t)$ is $\mathcal{F}_t$-adapted and $\mathcal{F}_t$-α-stable Lévy process. Consequently, a.s. for each $t > 0$ $\int_0^t adL = \hat{L}(T(t))$, i.e., the stochastic integral with respect to a α-stable Lévy process is nothing but another α-stable Lévy process with randomly changed time scale.

Solutions of One-Factor Lévy-based Models using CTM

Below we give the solutions to some one-factor Lévy Models decribed by SDE driven by α-stable Lévy process. $L(t)$ below is a symmetric α-stable Lévy process, and $\hat{L}$ is a $(\hat{T}_t)_{t\in R_+}$-adapted symmetric α-stable Levy process on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t\in R_+}, P))$.

1. *Geometric α-stable Lévy Motion*. $dS(t) = \mu S(t-)dt + \sigma S(t-)dL(t)$. *Solution* $S(t) = e^{\mu t}[S(0) + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t [S(0) + \hat{L}(\hat{T}_s)]^\alpha ds$.

2. *Ornstein-Uhlenbeck Process Driven by α-stable Lévy Motion*. $dS(t) = -\mu S(t-)dt + \sigma dL(t)$. *Solution* $S(t) = e^{-\mu t}[S(0) + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t (e^{\mu s}[S(0) + \hat{L}(\hat{T}_s)])^\alpha ds$.

Change of Time Method for SDE Driven by Levy Motion

3. *Vasićek Process Driven by α-stable Lévy Motion*. $dS(t) = \mu(b - S(t-))dt + \sigma dL(t)$. *Solution* $S(t) = e^{-\mu t}[S(0) - b + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^{\alpha} \int_0^t (e^{\mu s}[S(0) - b + \hat{L}(\hat{T}_s)] + b)^{\alpha} ds$.

4. *Continuous-Time GARCH Process Driven by α-stable Lévy process*. $dS(t) = \mu(b - S(t-))dt + \sigma S(t-)dL(t)$. *Solution* $S(t) = e^{-\mu t}(S(0) - b + \hat{L}(\hat{T}_t)) + b$, where $\hat{T}_t = \sigma^{\alpha} \int_0^t [S(0) - b + \hat{L}(\hat{T}_s) + e^{\mu s} b]^{\alpha} ds$.

5. *Cox-Ingersoll-Ross Process Driven by α-stable Lévy Motion*. $dS(t) = k(\theta^2 - S(t-))dt + \gamma\sqrt{S(t-)}dL(t)$. *Solution* $S^2(t) = e^{-kt}[S_0^2 - \theta^2 + \hat{L}(\hat{T}_t)] + \theta^2$, where $\hat{T}_t = \gamma^{\alpha} \int_0^t [e^{k\hat{T}_s}(S_0^2 - \theta^2 + \hat{L}(\hat{T}_s)) + \theta^2 e^{2k\hat{T}_s}]^{\alpha/2} ds$.

Change of Time Method for SDE Driven by Levy Motion

6. *Ho and Lee Process Driven by α-stable Lévy Motion*. $dS(t) = \theta(t-)dt + \sigma dL(t)$. *Solution* $S(t) = S(0) + \hat{L}(\sigma^\alpha t) + \int_0^t \theta(s)ds.$

7. *Hull and White Process Driven by α-stable Lévy Motion*. $dS(t) = (a(t-) - b(t-)S(t-))dt + \sigma(t-)dL(t)$.
Solution $S(t) = \exp[-\int_0^t b(s)ds][S(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T}_t)],$
where $\hat{T}_t = \int_0^t \sigma^\alpha(s)[S(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T}_s) + \exp[\int_0^s b(u)du]\frac{a(s)}{b(s)}]^\alpha ds.$

8. *Heath, Jarrow and Morton Process Driven by α-stable Lévy Motion*. $f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dL(v)$. *Solution* $f(t,u) = f(0,u) + \hat{L}(\hat{T}_t) + \int_0^t a(v,u)dv,$ where $\hat{T}_t = \int_0^t b^\alpha(f(0,u) + \hat{L}(\hat{T}_s) + \int_0^s a(v,u)dv)ds.$

Change of Time Method for SDE Driven by Lévy Motion

Solution of Multi-Factor Lévy SIRMs Using CTM

Solution of multi-factor models driven by α-stable Lévy motions can be obtained using various combinations of solutions of the above-mentioned processes and CTM. We give one example of two-factor continuous-time GARCH model driven by α-stable Lévy motions:

$$\begin{cases} dS(t) &= r(t-)S(t-))dt + \sigma S(t-)dL^1(t) \\ dr(t) &= a(m - r(t-))dt + \sigma_2 r(t-)dL^2(t), \end{cases}$$

where L^1, L^2 may be correlated, $m \in R, \sigma_i, a > 0, i = 1, 2.$

Change of Time Method for SDE Driven by Lévy Motion

Solution of Multi-Factor Lévy Models Using CTM

Solution, using CTM for the first and the second equations:

$$\begin{aligned} S(t) &= e^{\int_0^t r_s ds}[S_0 + \hat{L}^1(\hat{T}_t^1)] \\ &= e^{\int_0^t e^{-as}[r_0 - m + \hat{L}^2(\hat{T}_s^2)]ds}[S_0 + \hat{L}^1(\hat{T}_t^1)], \end{aligned}$$

where $\hat{T}^i$ are defined in 1. ($i = 1$) and 4. ($i = 2$), respectevely.

Applications in Financial and Energy Markets

Variance Swaps for Lévy-Based Heston Model

Assume that underlying asset S_t in the risk-neutral world and variance follow the following model:

$$\begin{cases} \frac{dS_t}{S_t} &= r_t dt + \sigma_t dw_t \\ d\sigma_t^2 &= k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dL_t, \end{cases}$$

where r_t is deterministic interest rate, σ_0 and θ are short and long volatility, $k > 0$ is a reversion speed, $\gamma > 0$ is a volatility (of volatility) parameter, w_t and L_t are independent standard Wiener and α-stable Lévy processes ($\alpha \in (0, 2]$).

Applications in Financial and Energy Markets

Variance Swaps for Lévy-Based Heston Model

Solution:

$$\sigma^2(t) = e^{-kt}[\sigma_0^2 - \theta^2 + \hat{L}(\hat{T}_t)] + \theta^2,$$

where $\hat{T}_t = \gamma^\alpha \int_0^t [e^{k\hat{T}_s}(\sigma_0^2 - \theta^2 + \hat{L}(\hat{T}_s)) + \theta^2 e^{2k\hat{T}_s}]^{\alpha/2} ds.$

Applications in Financial and Energy Markets

Variance Swaps for Lévy-Based Heston Model

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$N(\sigma_R^2(S) - K_{var}),$$

where $\sigma_R^2(S)$ is the realized stock variance (quoted in annual terms) over the life of the contract,

$$\sigma_R^2(S) := \frac{1}{T}\int_0^T \sigma^2(s)ds,$$

K_{var} is the delivery price for variance, and N is the notional amount.

Applications in Financial and Energy Markets

Variance Swaps for Lévy-Based Heston Model

Valuing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract P on future realized variance with strike price K_{var} is the expected present value of the future payoff in the risk-neutral world:

$$P_{var} = E\{e^{-rT}(\sigma_R^2(S) - K_{var})\},$$

where r is the risk-free discount rate corresponding to the expiration date T, and E denotes the expectation.

Applications in Financial and Energy Markets

Variance Swaps for Lévy-Based Heston Model

Realized Variance:

$$\sigma_R^2(S) := \frac{1}{T}\int_0^T \sigma^2(s)ds = \frac{1}{T}\int_0^T \{e^{-ks}[\sigma_0^2 - \theta^2 + \hat{L}(\hat{T}_s)] + \theta^2\}ds,$$

Value of Variance Swap:

$$\begin{aligned} P_{var} &= E\{e^{-rT}(\sigma_R^2(S) - K_{var})\} \\ &= E\{e^{-rT}(\tfrac{1}{T}\textstyle\int_0^T \{e^{-ks}[\sigma_0^2 - \theta^2 + \hat{L}(\hat{T}_s)] + \theta^2\}ds - K_{var}) \end{aligned}$$

Applications in Financial and Energy Markets

Variance Swaps for Lévy-Based Heston Model

Thus, for calculating variance swaps we need to know only $E\{\sigma_R^2(S)\}$, namely, mean value of the underlying variance, or $E[\widehat{L}(\widehat{T}_s)]$.

Only moments of order less than α exist for the non-Gaussian family of α-stable distribution. We suppose that $1 < \alpha < 2$ to find $E[\widehat{L}(\widehat{T}_s)]$.

Applications in Financial and Energy Markets

Variance Swaps for Lévy-Based Heston Model

Value of Variance Swap for Lévy-Based Heston Model:

$$P_{var} = e^{-rT}[\frac{1-e^{-kT}}{kT}(\sigma_0^2-\theta^2)+\theta^2-K_{var}].$$

Applications in Financial and Energy Markets

Volatility Swaps for Lévy-Based Heston Model?

A stock *volatility swap* is a forward contract on the annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R(S) - K_{vol}),$$

where $\sigma_R(S)$ is the realized stock volatility (quoted in annual terms) over the life of contract,

$$\sigma_R(S) := \sqrt{\frac{1}{T}\int_0^T \sigma_s^2 ds},$$

σ_t is a stochastic stock volatility, K_{vol} is the annualized volatility delivery price, and N is the notional amount

Applications in Financial and Energy Markets

Volatility Swaps for Lévy-Based Heston Model?

To calculate volatility swaps we need more. From Brockhaus-Long (2000) approximation (which is used the second order Taylor expansion for function $\sqrt{x}$) we have:

$$E\{\sqrt{\sigma_R^2(S)}\} \approx \sqrt{E\{V\}} - \frac{Var\{V\}}{8E\{V\}^{3/2}},$$

where $V := \sigma_R^2(S)$ and $\frac{Var\{V\}}{8E\{V\}^{3/2}}$ is the convexity adjustment.

Thus, to calculate the value of volatility swaps

$$P_{vol} = \{e^{-rT}(E\{\sigma_R(S)\} - K_{vol})\}$$

we need both $E\{V\}$ and $Var\{V\}$.

Applications in Financial and Energy Markets

Volatility Swaps for Lévy-Based Heston Model?

For $S\alpha S$ processes only the moments of order $p < \alpha$ exist, $\alpha \in (0, 2]$.

Since the $S\alpha S$ r.v. has 'infinite variance', the covariation of two jointly $S\alpha S$ real r.v. with dispersions γ_x and γ_y defined by

$$[X, Y]_\alpha = \frac{E[X|Y|^{p-2}Y]}{E[|Y|^p]}\gamma_y$$

has often been used instead of the covariance (and correlation), where $\gamma_y = [Y, Y]_\alpha$ is the dispersion of r.v. Y.

One of possible way to get volatility swaps for Lévy-based Heston model is to use covariation.

Applications in Financial and Energy Markets

Gaussian-Based SABR or LMM models

SABR model (see Hagan, Kumar, Lesniewski and Woodward (2002)) and the Libor Market Model (LMM) (Brace, Gatarek and Musiela (BGM, 1996)) have become industry standards for pricing plain-vanilla and complex interest rate products, respectively.

Gaussian-based SABR model, a stochastic volatility model in which the forward value satisfies:

$$\begin{cases} dF_t &= \sigma_t F_t^{\beta} dW_t^1 \\ d\sigma_t &= \nu \sigma_t dW_t^2, \end{cases}$$

Applications in Financial and Energy Markets

Lévy-Based SABR

Lévy-based SABR model, a stochastic volatility model in which the forward value satisfies:

$$\begin{cases} dF_t &= \sigma_t F_t^{\beta} dW_t \\ d\sigma_t &= \nu \sigma_t dL_t, \end{cases}$$

Applications in Financial and Energy Markets

Lévy-Based SABR: solution using change of time

$$F_t = F_0 + \widehat{W}(\widehat{T}_t^1),$$

$$T_t^1 = \int_0^t \sigma_{T_s^1}^{-2} (F_0 + \widehat{W}(s))^{-2\beta} ds,$$

$$\sigma_t = \sigma_0 + \widehat{L}(\widehat{T}_t^2),$$

$$T_t^2 = \nu^{-\alpha} \int_0^t (\sigma_0 + \widehat{L}(s))^{-\alpha} ds.$$

Applications in Financial and Energy Markets

Energy Forwards and Futures

Random variables following α-stable distribution with small characteristic exponent are *highly impulsive*, and it is this heavy-tail characteristic that makes this density appropriate for modeling noise which is impulsive in nature, for example, energy prices such as electricity.

Applications in Financial and Energy Markets

Energy Forwards and Futures: Lévy-Based Schwartz-Smith Model

$$\begin{cases} \ln(S_t) &= \kappa_t + \xi_t \\ d\kappa_t &= (-k\kappa_t - \lambda_\kappa)dt + \sigma_\kappa dL_\kappa \\ d\xi_t &= (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dW_\xi, \end{cases}$$

where S_t current spot price, κ_t is the short-term deviation in prices, ξ_t is the equilibrium price level.

Applications in Financial and Energy Markets

Energy Forwards and Futures: Lévy-Based Schwartz-Smith Model

Let $F_{t,T}$ denotes the market price for a futures contract with maturity T, then:

$$\ln(F_{t,T}) = e^{-k(T-t)}\kappa_t + \xi_t + A(T-t),$$

where $A(T-t)$ is a deterministic function with explicit expression.‘ We note that κ_t, using change of time for α-stable processes can be presented in the following form:

$$\kappa_t = e^{-kt}[\kappa_0 + \frac{\lambda_\kappa}{k} + \hat{L}_\kappa(\hat{T}_t)],$$

$$\hat{T}_t = \sigma_\kappa^\alpha \int_0^t (e^{-ks}[\kappa_0 + \frac{\lambda_\kappa}{k} + \hat{L}_\kappa(\hat{T}_s)] - \frac{\lambda_\kappa}{k})^\alpha ds$$

Applications in Financial and Energy Markets

Energy Forwards and Futures: Lévy-Based Schwartz-Smith Model

In this way, the market price for a futures contract with maturity T has the following look:

$$\begin{aligned} \ln(F_{t,T}) &= e^{-kT}[\kappa_0 + \frac{\lambda_\kappa}{k} + \widehat{L}_\kappa(\widehat{T}_t)] \\ &+ \xi_0 + (\mu_\xi - \lambda_\xi)t + \sigma_\xi W_\xi + A(T-t), \end{aligned}$$

where Lévy process $\widehat{L}_\kappa$ and Wiener process W_ξ may be correlated.

If $\alpha \in (1, 2]$, then we can calculate the value of Lévy-based futures contract.

Applications in Financial and Energy Markets

Energy Forwards and Futures: Lévy-Based Schwartz Model

$$\begin{cases} d\ln(S_t) &= (r_t - \delta_t)S_t dt + S_t \sigma_1 dW_1 \\ d\delta_t &= k(a - \delta_t)dt + \sigma_2 dL \\ dr_t &= a(m - r_t)dt + \sigma_3 dW_2, \end{cases}$$

where Wiener processes W_1, W_2 and α-stable Lévy process L may be correlated. δ_t and r_t are instantaneous convenience yield and interest rate, respectively.

Applications in Financial and Energy Markets

Energy Forwards and Futures: Lévy-Based Schwartz Model

We note that:

$$\begin{aligned}\delta_t &= e^{kt}(\delta_0 - a + \hat{L}(\hat{T}_t)),\\ \hat{T}_t &= \sigma_2^\alpha \int_0^t (e^{ks}[\delta_0 - a + \hat{L}(\hat{T}_s)] + a)^\alpha ds\end{aligned}$$

and

$$\begin{aligned}r_t &= e^{at}(r_0 - m + \hat{W}_2(\hat{T}_t)),\\ \hat{T}_t &= \sigma_3^2 \int_0^t (e^{as}[r_0 - m + \hat{W}_2(\hat{T}_s)] + m)^2 ds.\end{aligned}$$

Applications in Financial and Energy Markets

Energy Forwards and Futures: Lévy-Based Schwartz Model

Solution for $\ln[S_t]$:

$$\ln[S_t] = e^{\int_0^t [e^{as}(r_0 - m + \hat{W}_2(\hat{T}_s^2)) - e^{ks}(\delta_0 - a + \hat{L}(\hat{T}_s))]ds}[\ln S_0 + \hat{W}_1(\hat{T}_t^1)].$$

Applications in Financial and Energy Markets

Energy Forwards and Futures: Lévy-Based Schwartz Model

In this way, the futures contracts has the following form:

$$\begin{aligned}
\ln(F_{t,T}) &= \frac{1-e^{-k(T-t)}}{k}\delta_t + \frac{1-e^{-a(T-t)}}{a}r_t + \ln(S_t) + C(T-t) \\
&= \frac{1-e^{-k(T-t)}}{k}[e^{kt}(\delta_0 - a + \hat{L}(\hat{T}_t))] \\
&+ \frac{1-e^{-a(T-t)}}{a}e^{at}(r_0 - m + \hat{W}_2(\hat{T}_t^2)) \\
&+ \exp\{\int_0^t (e^{as}(r_0 - m + \hat{W}_2(\hat{T}_s^2)) \\
&- e^{ks}(\delta_0 - a + \hat{L}(\hat{T}_s)))ds\}[\ln(S_0) + \hat{W}_1(\hat{T}_t^1)] \\
&+ C(T-t),
\end{aligned}$$

where $C(T-t)$ is a deterministic explicit function. If $\alpha > 1$, then we can calculate the value of futures contract.

Conclusion

- α-Stable Lévy Processes

- Change of Time Method for Lévy-Based Models

- Pricing of Financial Derivatives: Variance Swaps

- Pricing of Energy Contracts (Derivatives): Futures and Forwards

The End

Thank You for Your Time and Attention!