Variance Swap for Local Lévy based Stochastic Volatility with Delay *

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Outline of Presentation

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2. Variance Swap for Lévy based Stochastic Volatility with Delay
3. Examples: VG, Tempered Stable & Jump-Diffusion
4. Parameter Estimation
5. Numerical Example: $S\&P500(01/01/2000 - 12/31/2009)$

Abstract

The valuation of the variance swaps for local Lévy based stochastic volatility with delay (LLBSVD) is discussed in this paper. We provide some analytical closed forms for the expectation of the realized variance for the LLBSVD.

As applications of our analytical solutions, we fit our model to 10 years of $S\&P500$ data (2000-01-01–2009-12-31) with variance gamma model and apply the obtained analytical solutions to price the variance swap.

Introduction

The key risk factors considered in option pricing models, besides the diffusive price risk of the underlying asset, are stochastic volatility and jumps, both in the asset price and its volatility. Models that include some or all of these factors were developed, in particular, by Merton (1973), Heston (1993), Duffie et al (2000), Bakshi et al (1997), Bates (1996).

Introduction: Why Jumps?

The importance of jumps in volatility has become apparent in recent studies, which try to explain the time series properties of both stock and option prices, like Eraker et al (2003) or Broadie et al (2008). The jumps in stock market volatility are found to be so active that this discredits many recently proposed stochastic volatility models without jumps (see Bollerslev et al (2008)) .

Introduction: Why Jumps?

There is currently fairly compelling evidence for jumps in the level of financial prices. The most convincing evidence comes from recent nonparametric work using high-frequency data as in Barndorff-Nielsen et al (2005) and Ait-Sahalia et al (2008) among others. Also, paper Todorov et al (2008) conducts a non-parametric analysis of the market volatility dynamics using high-frequency data on the VIX index compiled by the CBOE and the $S\&P500$ index.

Introduction: Why Jumps?

The results in Eraker et al (2003) show that the jump-in-volatility models provide a significant better fit to the returns data. They use returns data to investigate the performance of models with jumps in volatility using the class of jump-in-volatility models proposed by Duffie et al (2000). Technical issues aside, jumps are important because they represent a significant source of non-diversifiable risk as discussed at length in Bollerslev et al (2008).

Introduction: Why Delay?

From the other side, some statistical studies of stock prices (see Sheinkman et al (1989) and Akrigay (2003)) indicate the dependence on past returns. A diffusion approximation result for processes satisfying some equations with past-dependent coefficients obtained in Kind et al (1997), and this result they applied to a model of option pricing, in which the underlying asset price volatility depends on the past evolution to obtain a generalized (asymptotic) Black-Scholes formula.

Introduction: Why Delay?

Hobson et al (1998) suggested a new class of nonconstant volatility models, which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model.

Introduction: Why Delay?

In this talk, we incorporate a jump part into the stochastic volatility model with delay (and without jumps) proposed in Swishchuk (2005). The stock price $S(t)$ satisfies the following equation

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t), \quad t > 0,$$

where $\mu \in R$ is the mean rate of return, the volatility term $\sigma > 0$ is a bounded function and $W(t)$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$.

Introduction: Why Delay?

We also let $r > 0$ be the risk-free rate of return of the market. We denote $S_t = S(t-\tau), \quad t > 0$ and the initial data of $S(t)$ is defined by $S(t) = \varphi(t)$, where $\varphi(t)$ is a deterministic function with $t \in [-\tau, 0], \quad \tau > 0$.

Introduction: Why Delay?

The volatility $\sigma(t, S_t)$ satisfies the following equation:

$$\begin{aligned} \frac{d\sigma^2(t,S_t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(u, S_u)dL(u)\right]^2 \\ &- (\alpha + \gamma)\sigma^2(t, S_t) \end{aligned}$$

where $L(t)$ is a Lévy process independent of $W(t)$ with Lévy triplet (a, γ, ν). Here, $V > 0$ is a mean-reverting level (or long-term equilibrium of $\sigma^2(t, S_t)$), $\alpha, \gamma > 0$, and $\alpha + \gamma < 1$.

Introduction: Why Delay?

Our model of stochastic volatility exhibits jumps and also past-dependence: the behavior of a stock price right after a given time t not only depends on the situation at t, but also on the whole past (history) of the process $S(t)$ up to time t. This draws some similarities with fractional Brownian motion models (see Mandelbrot (1997)) due to a long-range dependence property. Another advantage of this model is mean-reversion.

Introduction: Why Delay?

This model is also a continuous-time version of GARCH(1,1) model (see Bollerslev (1986)) with jumps:

$$\sigma_n^2 = \gamma V + \alpha \ln^2(S_{n-1}/S_{n-2}) + (1-\alpha-\gamma)\sigma_{n-1}^2$$

or, more general,

$$\sigma_n^2 = \gamma V + \frac{\alpha}{l} \ln^2(S_{n-1}/S_{n-1-l}) + (1-\alpha-\gamma)\sigma_{n-1}^2.$$

Introduction: Why Delay?

If we write down the last equation in differential form we can get the continuous-time GARCH with expectation of log-returns of zero:

$$\frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau}\ln^2\left(\frac{S(t)}{S(t-\tau)}\right) - (\alpha+\gamma)\sigma^2(t)$$

If we incorporate non-zero expectation of log-return (using Itô Lemma for $\ln\frac{S(t)}{S(t-\tau)}$), then we arrive to our continuous-time GARCH model for stochastic volatility with delay (see Swishchuk (2005)):

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sigma(s,S_s)dW(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t).$$

Introduction: Some Papers' Review on Delay and Swaps

We note, that paper Swishchuk (2005) studied the case when $\lambda = 0$ (model without jumps). Paper Swishchuk et al (2007) investigated the case with pure Poisson jumps in the form of $\int_{t-\tau}^{t} dN(s)$, compound Poisson jumps in the form of $\int_{t-\tau}^{t} y_s dN(s)$, and more general case with jump sizes y_t that have finite mean and variance.

Introduction: Some Papers' Review on Delay and Swaps

The paper Swishchuk (2009) incorporates the case of jumps into the model Swishchuk (2005) in the form of the following integral $\int_{t-\tau}^{t} \sigma(s, S_s) d\tilde{N}(s)$. As long as the stochastic volatility $\sigma(t, S_t)$ depends on t and S_t, has Lévy process as a random factor and delay we call it *local Lévy based stochastic volatility with delay* (LLBSVD).

The Model: Stock Price with Lévy-based Delayed Volatility

In this talk, we incorporate a jump part into the stochastic volatility model with delay (and without jumps) proposed in Swishchuk (2005). The stock price $S(t)$ satisfies the following equation

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The Model: Stock Price with Lévy-based Delayed Volatility

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The Model: Stock Price with Lévy-based Delayed Volatility

The volatility $\sigma(t, S_t)$ satisfies the following equation:

$$\begin{aligned} \frac{d\sigma^2(t,S_t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(u, S_u) dL(u)\right]^2 \\ &- (\alpha + \gamma)\sigma^2(t, S_t) \end{aligned}$$

where $L(t)$ is a Lévy process independent of $W(t)$ with Lévy triplet (a, γ, ν). Here, $V > 0$ is a mean-reverting level (or long-term equilibrium of $\sigma^2(t, S_t)$), $\alpha, \gamma > 0$, and $\alpha + \gamma < 1$.

Variance Swap for Lévy based Stochastic Volatility with Delay

Suppose the asset volatility is defined as the solution of the following delay equation,

$$\frac{\mathrm{d}\sigma^2(t,S_t)}{\mathrm{d}t} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(s,S_s)\mathrm{d}L(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t) \quad (1)$$

where L is a Lévy process (see Schoutens (2003)). From Protter (2003) we have

$$[\sigma\cdot L, \sigma\cdot L]_t = \int_0^t \sigma_s^2 \mathrm{d}[L,L]_s$$

hence

$$\mathbb{E}\left(\left[\int_{t-\tau}^{t} \sigma(s,S_s)\mathrm{d}L(s)\right]^2\right) = \mathbb{E}\left(\int_{t-\tau}^{t} \sigma_s^2 \mathrm{d}[L,L]_s\right). \quad (2)$$

Variance Swap for Lévy based Stochastic Volatility with Delay

From Cont et al (2003) if L is a Lévy process with characteristic triplet (a, γ, ν) we have

$$[L, L]_t = a^2 t + \int_{[0,t]} \int_{\mathbb{R}} y^2 J_L(\mathrm{d}s\,\mathrm{d}y) \tag{3}$$

where J_L is the jump measure of L. So we have

$$\mathbb{E}\left(\left[\int_{t-\tau}^{t} \sigma(s, S_s)\mathrm{d}L(s)\right]^2\right) = \mathbb{E}\left(\int_{t-\tau}^{t} a^2\sigma_s^2 \mathrm{d}s + \int_{[0,t]} \int_{\mathbb{R}} \sigma_s^2 y^2 J_L(\mathrm{d}s\,\mathrm{d}y)\right)$$

with $\sigma_s^2 \geq 0$ applying Fubini's Theorem we get

$$\begin{aligned} \mathbb{E}\left(\left[\int_{t-\tau}^{t} \sigma(s, S_s)\mathrm{d}L(s)\right]^2\right) &= \mathbb{E}\left(\int_{t-\tau}^{t} a^2\sigma_s^2 \mathrm{d}s + \int_{t-\tau}^{t} \sigma_s^2 \mathrm{d}s \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y)\right) \\ &= \mathbb{E}\left(\int_{t-\tau}^{t} \sigma_s^2 \mathrm{d}s\right)\left[a^2 + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y)\right] \end{aligned}$$

Variance Swap for Lévy based Stochastic Volatility with Delay

Taking the expectation of (1) and denoting $v(t) = \mathbb{E}[\sigma^2(t, S_t)]$ we get

$$\frac{\mathrm{d}v(t))}{\mathrm{d}t} = \gamma V + \alpha\tau(\mu - r)^2 + \frac{\alpha}{\tau}\left(a + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y)\right)\int_{t-\tau}^{t} v(s)\mathrm{d}s - (\alpha + \gamma)v(t). \tag{4}$$

This has a stable solution of

$$v(t) \equiv X = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\alpha + \gamma - \alpha\left(a + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y)\right)}.$$

Variance Swap for Lévy based Stochastic Volatility with Delay

As an approximate solution we assume $v(t) = X + C\mathrm{e}^{\rho t}$, substituting into (4), the characteristic equation for ρ is

$$\rho = \frac{\alpha\left(a + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y)\right)}{\rho\tau}\left(1 - \mathrm{e}^{-\rho\tau}\right) - \gamma - \alpha.$$

Approximating $\mathrm{e}^{-\rho\tau} \approx 1 - \rho\tau$ we get

$$\rho = \alpha\left(a + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) - 1\right) - \gamma.$$

Now with $v(0) = \sigma_0^2$ we have

$$C = \sigma_0^2 - \frac{\gamma V + \alpha\tau(\mu - r)^2}{\alpha + \gamma - \alpha\left(a + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y)\right)}.$$

Variance Swap for Lévy based Stochastic Volatility with Delay

In this way, we obtained the followinf result.

Theorem (Variance Swap for LLBSVD). The general approximated solution for (4) has the following form:

$$\begin{aligned} v(t) &\approx X + Ce^{\rho t} \\ &= \frac{\gamma V + \alpha\tau(\mu - r)^2}{\alpha + \gamma - \alpha\left(a + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y)\right)} + \left[\sigma_0^2 - \frac{\gamma V + \alpha\tau(\mu - r)^2}{\alpha + \gamma - \alpha\left(a + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y)\right)}\right] \\ &\times \exp\left[\alpha\left(a + \int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) - 1\right) - \gamma\right] t. \end{aligned}$$

Examples

Example 1 (Variance Gamma)

Consider a Variance Gamma process with the CGM parameterization (see Schoutens (2003)), that is, with characteristic function of the form

$$\phi_{\mathsf{VG}}(u; C, G, M) = \left(\frac{GM}{GM + (M - G)\mathrm{i}u + u^2}\right)^C$$

where $C > 0$, $G > 0$, and $M > 0$.

Example 1 (Variance Gamma)

With Lévy measure

$$\nu_{VG}(\mathrm{d}x) = C|x|^{-1}(\exp(Gx)1_{(X<0)} + \exp(-Mx)1_{(X>0)})\mathrm{d}x$$

we have

$$\int_{\mathbb{R}} x^2 \nu_{VG}(\mathrm{d}x) = \frac{C}{M^2} - \frac{C}{G^2}$$

so

$$X = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\alpha + \gamma - \alpha\left(\frac{C}{M^2} - \frac{C}{G^2}\right)}$$

and

$$\rho = \alpha\left(\frac{C}{M^2} - \frac{C}{G^2} - 1\right) - \gamma.$$

Example 2 (Tempered Stable)

The Tempered Stable distribution (see Schoutens (2003)) has the characteristic function

$$\phi_{\mathsf{TS}}(u;\kappa,a,b) = \exp(ab - a(b^{1/\kappa} - 2\mathrm{i}u)^{\kappa})$$

where $a > 0$, $b \geq 0$, and $0 < \kappa < 1$. Here

$$\nu_{TS}(\mathrm{d}x) = a2^{\kappa}\frac{\kappa}{\Gamma(1-\kappa)}x^{-\kappa-1}\exp\left(-\frac{1}{2}b^{1/\kappa}x\right)1_{(x>0)}\mathrm{d}x$$

and hence if $b > 0$ then

$$\int_{\mathbb{R}} x^2\nu_{TS}(\mathrm{d}x) = \frac{2^{\kappa+4}\,a\,\Gamma\,(\kappa+1)\,\sin(\pi\,\kappa)}{\pi\,b^{\frac{3}{\kappa}}}$$

Example 2 (Tempered Stable)

so

$$X = \frac{\pi\, b^{\frac{3}{\kappa}}(\gamma\, V + \alpha\, \tau\, (\mu - r)^2)}{\pi\, b^{\frac{3}{\kappa}}(\alpha + \gamma) - 2^{\kappa+4}\, \alpha\, a\, \sin(\pi\, \kappa)\ \Gamma\,(\kappa + 4)}$$

with

$$\rho = \alpha \left(\frac{2^{\kappa+4}\, a\, \Gamma\,(\kappa + 1)\ \sin(\pi\, \kappa)}{\pi\, b^{\frac{3}{\kappa}}} - 1 \right) - \gamma.$$

Example 3 (Jump-diffusion)

Consider a process with characteristic triplet $(1, 0, \lambda\delta(1))$ we then have

$$\int_{\mathbb{R}} x^2 \nu(\mathrm{d}x) = \lambda$$

so

$$X = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\alpha + \gamma - \alpha(1 + \lambda)} = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}$$

with

$$\rho = \alpha\lambda - \gamma.$$

We got the result obtained in Swishchuk (2005).

Parameter Estimation

As in Kazmerchuk et al (2005) we consider a Maximum Likelihood for the estimation of the parameters in (1). The discrete time analogue of (1) is given by

$$\sigma_n^2 = \omega + \frac{\alpha}{l}\left(\sum_{i=1}^{l} \epsilon_{n-i}\right)^2 + \beta\sigma_{n-1}^2$$

where

$$\epsilon_n = y_n - \mu$$

and $\epsilon = \ln(S_n/S_{n-1})$ are the log-returns.

Parameter Estimation

Furthermore as in the GARCH model we have $\alpha + \beta + \gamma < 1$ and $l \geq 1$ is our discrete delay parameter. Here we assume ϵ_n follows a distribution with a probability density function given by $f(x; \theta)$ where θ is a vector of parameters introduced by the distribution of ϵ_n.

Parameter Estimation

As in Konlach et al (2009) estimation of our model parameters then becomes an exercise in maximizing the likelihood function

$$L(\mu, \alpha, \beta, \omega, \theta) = \sum_{t=1}^{T} \left[\ln f(\epsilon_t \sigma_t^{-1}; \theta) - \ln \sigma_t \right]$$

for a given lag l.

Parameter Estimation

Following Kazmerchuk et al (2005) we use the *Akaike's information criterion* to select an l. With L_{max} being the maximum likelihood we have

$$AICc = 2k - 2\ln(L_{max}) + \frac{2k(k+1)}{n-k-1}$$

where k is the number of parameters $k = 3 + (l-1) + v)$ with $\theta \in \mathbb{R}^v$.

Numerical Example: $S\&P500(2000-01-01--2009-12-31)$

Here we fit our model to 10 years of $S\&P500$ data (2000-01-01–2009-12-31) and apply the obtained analytical solutions to price the variance swap.

Statistics for $S\&P500(2000-01-01--2009-12-31)$

Observations	2514
Mean	-0.0001058922
Maximum	0.1095720
Minimum	-0.09469514
Std. Dev.	0.01400746
Skewness	-0.1036433
Kurtosis	7.635567

Numerical Example: Variance Gamma

As an example we'll assume a variance gamma distribution with probability density function

$$f_{VG}(x;\theta) = \frac{\gamma^{2\lambda}|x-\mu|^{\lambda-1/2}K_{\lambda-1/2}\left(\alpha|x-\mu|\right)}{\sqrt{\pi}\Gamma(\lambda)(2\alpha)^{\lambda-1/2}}\, e^{\beta(x-\mu)}$$

where $\theta = [\mu, \alpha, \gamma, \lambda]$, $\lambda > 0$, and $\gamma = \sqrt{\alpha^2 - \beta^2} > 0$.

Numerical Example: Variance Gamma

We can see from these data that for this set of data the gaussian case is unstable for most l aside from $l \in \{1, 2, 4, 15\}$ and selecting a model using the minimum AICc we select the variance gamma case with a discrete lag of 6.

Numerical Example: Variance Gamma

A note should be made on the estimation of the added Variance Gamma parameters. We can see that by initializing the model parameters α, β and ω to the gaussian case we can fit the Variance Gamma model under these parameters such that they provide no additional increase in the likelihood.

Numerical Example: Variance Gamma

This is an indication of over-fitting of the data, we could choose any α, β and ω and simply fit the Variance Gamma parameters to achieve a high likelihood. However by selecting Variance Gamma parameters to introduce certain features not present in the normal distribution we could attain stability in the optimization and reduce the problem of over-fitting by reducing the degrees of freedom.

Numerical Example: Variance Gamma

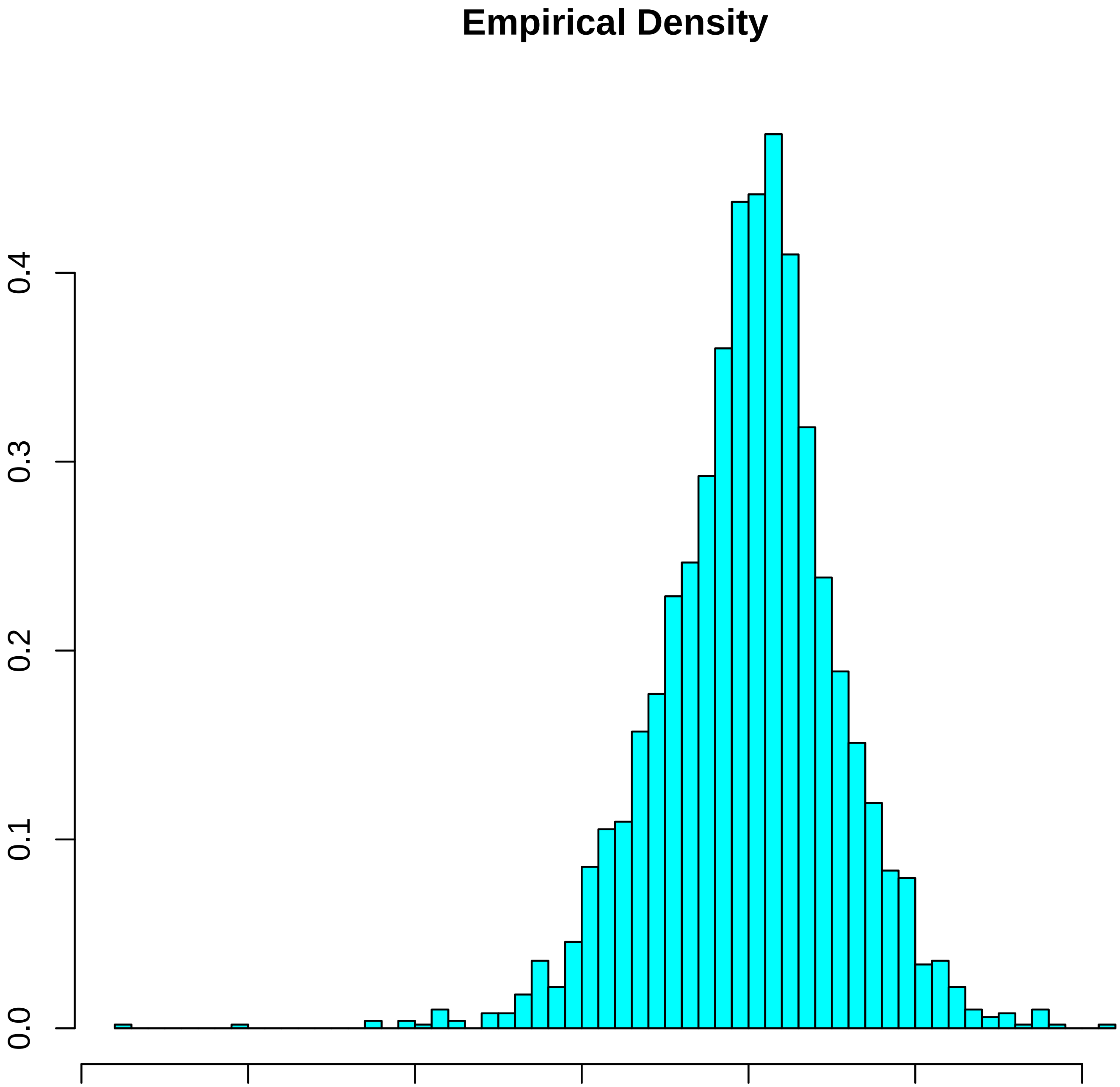
To price the variance swap we can change to the CGM parameterization by

$$
\begin{aligned}
C &= 1/v, \\
G &= \left(\sqrt{\frac{1}{4}\theta^2 v^2 + \frac{1}{2}\sigma^2 \nu} - \frac{1}{2}\theta v\right)^{-1}, \\
M &= \left(\sqrt{\frac{1}{4}\theta^2 v^2 + \frac{1}{2}\sigma^2 \nu} + \frac{1}{2}\theta v\right)^{-1}.
\end{aligned}
$$

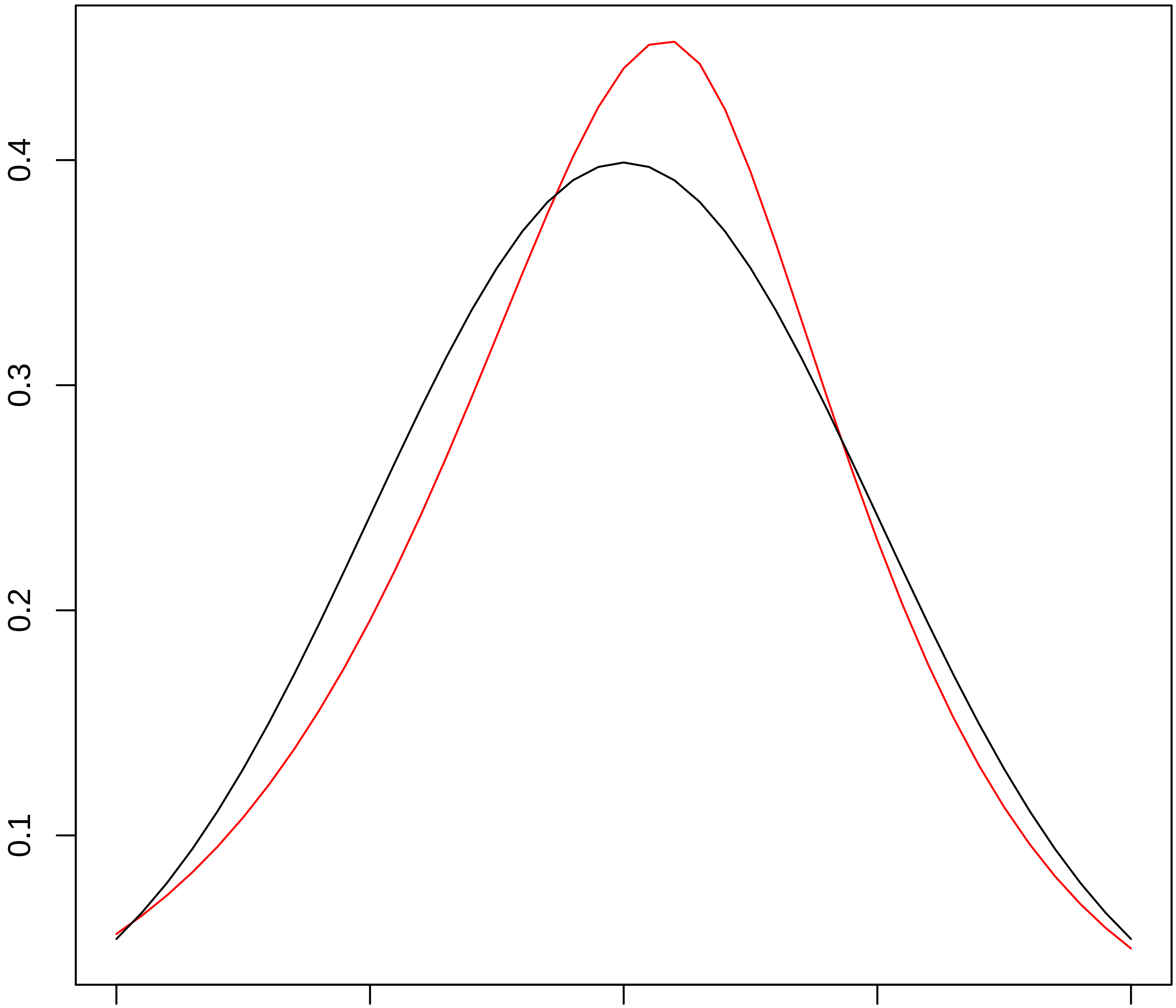
where a change of parameters gives us $c = 0.2508737$, $\sigma = 1.058981$, $\theta = -0.2490418$, and $\nu = 0.4459436$ we then get $C = 2.242436$, $G = 1.790021$, $M = 2.234167$.

Numerical Example: Variance Gamma

Now assuming $r = 0.02$ and a maturity of $T = 1$ we get a price of 0.0002104639 under the gaussian model with a lag of 1 and a price of 0.0002048042 under the variance gamma model. Using the AICc selected model, variance gamma with lag of 6, we get a price of 0.0002879282.


Empirical Density
0.0
0.1
0.2
0.3
0.4

Probability Density Functions




Delivery Price vs. Maturity
Delivery Price
0.0004
0.0006
0.0008
0.0010
0.0012
0.0014

Conclusion

1. Introduction

2. Variance Swap for Lévy based Stochastic Volatility with Delay

3. Examples: VG, Tempered Stable & Jump-Diffusion

4. Parameter Estimation

5. Numerical Example: $S\&P500(01/01/2000-12/31/2009)$

The End of Chapter 1

Thank You for Your Time and Attention!

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