Pricing Variance Swaps for Stochastic Volatilities with Delay and Jumps

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Outline

• Stochastic Volatilities (SV) with Delay
• Multi-Factor SV with Delay
• SV with Delay and Jumps (joint work with Xu, Li (Dept of Math & Stat, U of C))
• Swaps
• Numerical Examples
Volatility

- **Volatility** is the standard deviation of the change in value of a financial instrument with specific time horizon
- It is often used to quantify the **risk** of the instrument over that time period
- **The higher volatility, the riskier the security**
Types of Volatilities

- **Historical V**: standard deviation (uses historical (daily, weekly, monthly, quarterly, yearly)) price data to empirically measure the volatility of a market or instrument in the past

- **Implied V**: volatility implied by the market price of the option based on an option pricing model (smile and skew-varying volatility by strike)
Volatility Smile

• The models by Black & Scholes (continuous-time (B,S)-security market, 1973) and Cox & Rubinstein (discrete-time (B,S)-security market (binomial tree), 1979) are unable to explain the negative skewness and leptokurticity (fat tail) commonly observed in the stock markets.

• The famous implied-volatility smile would not exist under their assumptions.
Coffee Options

Coffee Call Option

- CSCE May 2001 coffee call option implied volatilities as of March 12, 2001
Implied Volatility: Volatility Smile

• Graph indicates implied volatilities at various strikes for the May 2001 calls based upon their March 12, 2001 settlement prices. The pattern of implied volatilities form a "smile" shape, which is called a volatility smile.
Most derivatives markets exhibit persistent patterns of volatilities varying by strike. In some markets, those patterns form a **smile**. In others, such as **equity index options** markets, it is more of a skewed curve. This has motivated the name **volatility skew**. In practice, either the term "volatility smile" or "volatility skew" (or simply **skew**) may be used to refer to the general phenomena of volatilities varying by strike.
Implied Volatility: Volatility Surface

- Another dimension to the problem of volatility skew is that of volatilities varying by expiration. This is illustrated for CSCE coffee options. It indicates what is known as a volatility surface.
Types of Volatilities II

- **Level-Dependent Volatility** (CEV or Firm Model)- function of the spot price alone
- **Local Volatility**- function of the spot price and time (Dupire formulae, 1994)
- **Stochastic Volatility**: volatility is not constant, but a stochastic process (explains smile and skew)
Two Approaches to Introduce SV

• One approach-to change the clock time \( t \) to a random time \( T(t) \) (change of time)

\[ \sigma W(t) \Rightarrow W(T(t)) \]

• Another approach-
  change constant volatility into a positive stochastic process

\[ \sigma = \sigma(t), \quad \int_0^t \sigma^2(s)ds < +\infty \]
Stochastic Volatility: Some Models

- **ARCH model** (Engle (1982))
  \[ \ln(S_t/S_{t-1}) = \sigma_t \xi_t, \quad \xi_t \sim i.i.d. N(0,1) \]

- **Discrete SV: GARCH model** (Bollerslev (1986))
  \[ \sigma^2_t = \gamma \ln^2(S_{t-1}/S_{t-1-1}) + (1-\alpha-\gamma)\sigma^2_{t-1} \]

- **Heston SV model** (1993)

- **Mean-Reverting SV model** (Wilmott, Haug, Javaheri (2000))

- **Elliott and Chan** 'Option Pricing with Stochastic Volatility Driven by a Fractional Ornstein-Ohlenbeck Process'.

  \[ d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t^2dw_t \]
SV with Delay

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]

Kazmerchuk, Swishchuk & Wu (2002)
One-Factor SV with Delay

The underlying asset $S(t)$ follows the process

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t)$$

$$S_t := S(t - \tau) \quad S(t) = \phi(t), \quad t \in [-\tau, 0], \quad \tau > 0.$$ 

The asset volatility is defined as the solution of the following equation

$$\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t).$$

Why this equation?
From GARCH to SV with Delay

\[ \sigma_n^2 = \gamma V + \alpha \ln^2 \left( \frac{S_{n-1}}{S_{n-2}} \right) + (1 - \alpha - \gamma)\sigma_{n-1}^2 \]

\[ \sigma_n^2 = \gamma V + \frac{\alpha}{l} \ln^2 \left( \frac{S_{n-1}}{S_{n-1-l}} \right) + (1 - \alpha - \gamma)\sigma_{n-1}^2 \]

\[ \frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau} \ln^2 \left( \frac{S(t)}{S(t - \tau)} \right) - (\alpha + \gamma)\sigma^2(t) \]

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t) \]

-discrete-time GARCH(1,1)

-discrete-time GARCH(1,1) (l=1)

-continuous-time GARCH (expectation of log-returns is zero)

-continuous-time GARCH (non-zero expectation of log-return)
Comparison with GARCH (1,1)

GARCH (1,1)

\[
\ln(S_n/S_{n-1}) = m + \sigma_n \xi_n, \quad \{\xi_n\} \sim \text{i.i.d. } N(0,1),
\]

\[
\sigma_n^2 = \gamma V + \alpha (\sigma_{n-1} \xi_{n-1})^2 + (1 - \alpha - \gamma) \sigma_{n-1}^2
\]

\[
= \gamma V + \alpha (\ln(S_{n-1}/S_{n-2}) - m)^2 + (1 - \alpha - \gamma) \sigma_{n-1}^2
\]

Log-returns for \(S(t)\) (Ito formula)

\[
\ln \frac{S(t)}{S(t-\tau)} = \int_{t-\tau}^{t} \left( r - \frac{1}{2} \sigma^2(u, S(u)) du + \int_{t-\tau}^{u} \sigma(u, S(u)) dW(u) \right)
\]

Continuous-Time GARCH for SV with Delay

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma) \sigma^2(t, S_t)
\]
Main Features of 1-Factor SV with Delay

1) continuous-time analogue of discrete-time GARCH model
   \[
   \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t).
   \]

2) Mean-reversion
   \[
   \frac{d\sigma^2}{dt} = (\alpha + \gamma) \left( \frac{\gamma}{\alpha + \gamma} V - \sigma^2 \right) dt + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma_s dw_s \right]^2
   \]

3) Market is complete (W is the same as for the stock price)
   \[
   dS(t) = \mu S(t) dt + \sigma(t, S_t) S_t dW(t)
   \]

4) Incorporate the expectation of log-returns
   \[
   \ln \frac{S(t)}{S(t-\tau)} = \int_{t-\tau}^{t} (r - \frac{1}{2} \sigma^2(u, S(u))) du + \int_{t-\tau}^{t} \sigma(u, S(u)) dW(u)
   \]
Equation for the Expectation of Variance

Equation for the Variance

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]

Equation for the Expectation

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t)
\]

\[v(t) := E_{P^*}[\sigma^2(t, S_t)]\]
Solution of the Equation for the Expectation of Variance (1FSV)

Equation to be solved

\[ \frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t) \]

Stationary solution

\[ v(t) \equiv X = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} \]

General solution

\[ v(t) \approx X + Ce^{-\gamma t} \]

\[ C = v(0) - X = \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \]
General Solution

Integro-differential equation with delay

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t)
\]

General solution

\[
v(t) = E_p^* [\sigma^2(t, S_t)] \\ \approx V + \frac{\alpha \tau (\mu - r)^2}{\gamma} + \left(\sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma}\right) e^{-\gamma t}
\]
Multi-Factor SV Models

- **One-Factor SV Models** *(all above-mentioned)*:  
  1) incorporate the leverage between returns and volatility and  
  2) reproduce the ‘skew’ of implied volatility

- However, it *fails to match either the high conditional kurtosis of returns* *(Chernov et. al. (2003)) or the full term structure of implied volatility surface* *(Cont & Tankov (2004))*

- **Adding jump components** in returns and/or volatility process, or considering **multi-factor SV models** are two primary generalizations of one-factor SV models
Multi-Factor SV Models

- **Chernov et al. (2003):** used efficient method of moments to obtain comparable empirical-of-fit from affine jump-diffusion models & two-factor SV family models
- **Molina et al. (2003):** used a Markov Chain Monte Carlo method to find strong evidence of two-factor SV models with well-separated time scales in foreign exchange data
- **Cont & Tankov (2004):** found that jump-diffusion models have a fairly good fit to the implied volatility surface
- **Fouque et al. (2000):** found that two-factor SV models provide a better fit to the term structure of implied volatility than one-factor SV models by capturing the behaviour at short and long maturities
- **Swishchuk (2006):** introduced two-factor and three-factor SV models with delay (incorporating mean-reverting level as a random process (GBM, OU, Pilipovich or continuous-time GARCH(1,1) model))
Multi-Factor SV with Delay

One-Factor SV with Delay

\[
\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s,S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t,S_t)
\]

- Multi-Factor-Mean SV with Delay-mean-reversion level V is a stochastic process

**V->V (t)-stochastic process**

- V (t) - geometric Brownian motion (GBM) (two-factor)
- V (t) - Ornstein-Uhlenbeck (UE) process (two-factor)
- V (t) - Pilipovich one-factor (two-factor)
- V (t) – Pilipovich two-factor process (three-factor)
2-Factor SV with Delay: GBM Mean-Reversion (GBM-MR)

\[
\begin{align*}
\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 \\
&\quad - (\alpha + \gamma)\sigma^2(t, S_t), \\
\frac{dV_t}{V_t} &= \xi dt + \beta dW_1(t).
\end{align*}
\]
2-Factor SV with Delay (GBMMR): Equation for the E and Solution

Equation for the E

\[
\frac{dv(t)}{dt} = \gamma V_0 e^{(\xi - \lambda \beta)t} + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t).
\]

where \( v(t) := E_p \sigma^2(t, S_t) \).

Solution

\[
v(t) \approx X + C e^{-\gamma t} + (\xi - \lambda \beta) \gamma V_0 \times \left[ \frac{X}{\xi - \lambda \beta} (e^{(\xi - \lambda \beta)t} - 1) + \frac{C}{\xi - \lambda \beta - \gamma} (e^{(\xi - \lambda \beta)t} - e^{\gamma t}) \right]
\]
2-Factor SV with Delay: OU Mean-Reversion (OUMR)

\[
\begin{aligned}
\left\{ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) \, dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t). \right.

\left. dV_t = \xi (L - V_t) \, dt + \beta \, dW_1(t). \right.
\end{aligned}
\]
2-Factor SV with Delay (OUMR):
Equation for the E and Solution

Equation for E

\[
\frac{dv(t)}{dt} = \gamma \left( e^{-\xi t} \left( V_0 - \left( L - \frac{\lambda \beta}{\xi} \right) \right) + \left( L - \frac{\lambda \beta}{\xi} \right) \right) \\
+ \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) \, ds - (\alpha + \gamma) v(t)
\]

Solution

\[
v(t) \approx X + Ce^{-\gamma t} + \xi \gamma \left( V_0 - \left( L - \frac{\lambda \beta}{\xi} \right) \right) \\
\times \left[ \frac{X}{\xi} (e^{-\xi t} - 1) + \frac{C}{\xi + \gamma} (e^{-\xi t} - e^{\gamma t}) \right]
\]
2-Factor SV with Delay: Pilipovich 1-Factor Mean-Reversion (OFMR)

\[
\left\{ \begin{array}{l}
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V_t + \frac{\alpha}{T} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) \, dW(s) \right]^2 \\
-(\alpha + \gamma)\sigma^2(t, S_t), \\
dV_t = \xi(L - V_t) \, dt + \beta V_t \, dW_1(t).
\end{array} \right.
\]
2-Factor SV with Delay (Pilipovich 1FMR): Equation for the E and Solution

Equation for E

\[
\frac{dv(t)}{dt} = \gamma \left( e^{-(\xi+\lambda\beta)t} \left( V_0 - L \frac{\xi}{\xi + \lambda\beta} \right) + L \frac{\xi}{\xi + \lambda\beta} \right) + \alpha\tau(\mu - r)^2 \\
+ \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) \, ds - (\alpha + \gamma)v(t)
\]

Solution

\[
v(t) \approx X + Ce^{-\gamma t} + \frac{\gamma\xi}{\xi + \lambda\beta} \left[ \left( X \left( \frac{V_0(\xi + \lambda\beta)}{\xi} - L \right) \\
\times (1 - e^{-(\xi+\lambda\beta)t}) + XLt \right)
+ \frac{C(V_0(\xi+\lambda\beta)}{\xi} - L \right)
+ \frac{CL}{\gamma}(e^{\gamma t} - 1) \right].
\]
3-Factor SV with Delay: Pilipovich 2-Factor Mean-Reversion (2FMR)

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 \\
- (\alpha + \gamma)\sigma^2(t, S_t),
\]

\[
dV_t = \xi(L_t - V_t) dt + \beta V_t dW_1(t),
\]

\[
dL_t = \beta_1 L_t dt + \eta L_t dW_2(t).
\]
3-Factor SV with Delay (Pilipovich 2FMR): Equation for the E and Solution

**Equation for E**

\[
\frac{dv(t)}{dt} = \gamma(e^{-(\xi+\lambda\beta)t}V_0 + \frac{\xi + \lambda\beta}{\xi + \lambda\beta + \beta_1}L_0(e^{(\beta_1-\lambda_1\eta)t} - e^{-(\xi+\lambda\beta)t})) + \alpha\tau(\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) \, ds - (\alpha + \gamma)v(t)
\]

**Solution**

\[
v(t) \approx X + Ce^{-\gamma t} - (\xi + \lambda\beta)\gamma V_0 \left[ \frac{X}{\xi + \lambda\beta} (1 - e^{-(\xi+\lambda\beta)t}) + \frac{C}{\xi + \lambda\beta + \gamma} (e^{\gamma t} - e^{-(\xi+\lambda\beta)t}) \right] + L_0 \frac{\xi + \lambda\beta}{\xi + \lambda\beta + \beta_1} \\
\times \left[ X(e^{(\beta_1-\lambda_1\eta)t} - e^{-(\xi+\lambda\beta)t}) + \frac{C(\beta_1 - \lambda_1\eta)}{(\beta_1 - \lambda_1\eta - \gamma)} \right] \\
\times (e^{(\beta_1-\lambda_1\eta)t} - e^{\gamma t}) + \frac{C(\xi + \lambda\beta)}{(\xi + \lambda\beta + \gamma)} (e^{\gamma t} - e^{-(\xi+\lambda\beta)t})
\]
Main Features of All the Solutions for MFSVD

• 1) Contains solution of one-factor SV with Delay

• 2) Contains additional terms due to the new parameters (more factors-more parameters)

• Solution (MFSVD)=Solution (1FSVD) + Additional Terms (Due to the stochastic mean-reversion)
Variance Swaps

Forward contract—*an agreement to buy or sell something at a future date for a set price* (forward price)

**Variance swaps** are forward contract on future realized stock variance
Realized Continuous Variance

Realized (or Observed) Continuous Variance:

\[ \sigma^2_R(S) := \frac{1}{T} \int_0^T \sigma^2(s) \, ds, \]

where \( \sigma(t) \) is a stock volatility,

\( T \) is expiration date or maturity.
Why Trade Volatility (Variance)?

- Volatility Swaps allow investors to **profit** from the risks of an increase or decrease in future volatility of an index of securities or to **hedge** against these risks.

- If you think current volatility is low, for the right price you might want to take a position that **profits if volatility increase.**
How does the Volatility Swap Work?

Fixed leg = strike price
Floating leg = realized volatility

SCENARIOS

A – The volatility increases:

B – The volatility decreases:
Payoff of Variance Swaps

A Variance Swap is a forward contract on realized variance.

Its payoff at expiration is equal to

\[ N \left( \sigma^2_R(S) - K_{var} \right) \]

\( N \) is a notional amount ($/variance);
\( K_{var} \) is a strike price
Valuing of Variance Swap for Stochastic Volatility with Delay

Value of Variance Swap (present value):

\[ P = e^{-rT} E_{P^*} \left[ \sigma_R^2(S) - K_{var} \right] \]

where \( E_{P^*} \) is an expectation (or mean value), \( r \) is interest rate.

To calculate variance swap we need only \( E_{P^*}[\sigma^2(t, S_t)] \),

\[ \sigma_R^2(S) := \frac{1}{T} \int_0^T \sigma^2(u, S(u - \tau))du. \]
Valuing of Variance Swap for One-Factor SV with Delay in Stationary Regime

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s)ds - (\alpha + \gamma)v(t)
\]

\[v(t) = E_{P^*}[\sigma^2(t)] = V + \frac{\alpha \tau (\mu - r)^2}{\gamma}.
\]

\[E_{P^*}[Var(S)] = \frac{1}{T} \int_{0}^{T} E_{P^*}[\sigma^2(t)]dt = V + \frac{\alpha \tau (\mu - r)^2}{\gamma}.
\]

\[P^* = e^{-rT}[V - K + \frac{\alpha \tau (\mu - r)^2}{\gamma}].
\]
Valuing of Variance Swap for One-Factor SV with Delay in General Case

\[ \frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t) \]

\[ v(t) \approx X + Ce^{-\gamma t} = V + \alpha \tau (\mu - r)^2 / \gamma + Ce^{-\gamma t} \]

\[ C = v(0) - X = \sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma. \]

In this way

\[ v(t) = E_P [\sigma^2(t)] \approx V + \alpha \tau (\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma)e^{-\gamma t}. \]
Valuing of Variance Swap for One-Factor SV with Delay in General Case

We need to find $E_{P^*}[\text{Var}(S)]$:

$$E_{P^*}[\text{Var}(S)] = \frac{1}{T} \int_0^T E_{P^*}[\sigma^2(t)]dt$$

$$\approx \frac{1}{T} \int_0^T [V + \alpha \tau (\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma)e^{-\gamma t}]dt$$

$$= V + \alpha \tau (\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma) \frac{1 - e^{-\gamma T}}{T \gamma}.$$
Comparison of SV in Heston Model with SV with Delay

Heston Model (1993)

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sigma_t dw_t^1 \\
\sigma_t^2 &= k(\theta^2 - \sigma_t^2) dt + \gamma \sigma_t dw_t^2,
\end{align*}
\]
Comparison of SV in Heston Model with SV with Delay II

Swap for SV in Heston Model

\[ E\{V\} = \frac{1 - e^{-kT}}{kT} (\sigma_0^2 - \theta^2) + \theta^2, \]

Swap for SV with Delay

\[ E\{V\} \approx \frac{1 - e^{-\gamma T}}{\gamma T} (\sigma^2(0, \phi(-\tau)) - V - \alpha \tau (\mu - r)^2 / \gamma) + [V + \alpha \tau (\mu - r)^2 / \gamma] \]

When \( \tau = 0 \) (the same expression as above):

\[ E\{V\} = \frac{1 - e^{-\gamma T}}{\gamma T} (\sigma^2(0, \phi(-\tau)) - V) + V. \]
### Numerical Example 1: S&P60 Canada Index (1997-2002)

#### Table 1

<table>
<thead>
<tr>
<th>Statistics on Log Returns S&amp;P60 Canada Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Series: LOG RETURNS S&amp;P60 CANADA INDEX</td>
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<tr>
<td>Sample: 1  1300</td>
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<tr>
<td>Observations: 1300</td>
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<td>Mean: 0.000235</td>
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<td>Median: 0.000593</td>
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<td>Maximum: 0.051983</td>
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<td>Minimum: -0.101108</td>
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<tr>
<td>Std. Dev.: 0.013567</td>
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<td>Skewness: -0.665741</td>
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<tr>
<td>Kurtosis: 7.787327</td>
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</table>

\[
E\{Var(S)\} = V + \frac{1-\gamma}{\gamma} \alpha \tau (\mu - r)^2 + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma) \frac{1-e^{-\gamma T}}{T \gamma}
\]

\[
= 0.0002 + ((1 - 0.0124)/0.0124) \times 0.0604 \times (0.0002 - 0.02)^2
+ (0.0001 - 0.0002 - 0.0604 \times (0.0002 - 0.02)^2/0.0124) \times \frac{1-e^{-0.0124}}{0.0124}
\]

\[
= 0.000102.
\]
Dependence of Variance (Realized) Swap for One-Factor SV with Delay on Maturity (S&P60 Canada Index)
Variance (Realized) Swap for One-Factor SV with Delay (S&P60 Canada Index)

Table 3

<table>
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<th>Statistics on Log Returns S&amp;P500 Index</th>
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</table>

\[
E\{Var(S)\} = V + \frac{1-\gamma}{\gamma} \alpha \tau (\mu - r)^2 + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2/\gamma) \frac{1-e^{-\gamma T}}{T\gamma} \\
= 0.004038144 + ((1 - 0.511)/0.511) \times 0.3828 \times 14 \times (0.000263 - 0.02)^2 \\
+ (0.000063 - 0.04038144 - 0.3828 \times 14 \times (0.000263 - 0.02)^2/0.511) \\
\times \frac{1-e^{-0.611}}{0.511} \\
= 0.00774376584.
\]
Dependence of Variance (Realized) Swap for One-Factor SV with Delay on Maturity (S&P500)
Variance (Realized) Swap for One-Factor SV with Delay (S&P500 Index)
Numerical Example 1: S&P60 Canada Index

\[ X = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} = 0.0002 \]
\[ C = \sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma = 0.007. \]
E of Variance and the Price of Variance Swap for 1-Factor SV with Delay

\[ v(t) \approx X + Ce^{-\gamma t} = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} + Ce^{-\gamma t} \]

\[ P^* \approx e^{-rT} \left[ V - K + \frac{\alpha \tau (\mu - r)^2}{\gamma} \right. \]

\[ + \left( \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \right) \frac{1 - e^{-\gamma T}}{T\gamma} \]

FIGURE 1: Variance of one-factor SV with delay (formula (10)).

FIGURE 2: The price of variance swap for one-factor SV with delay (formula (13)).
Variance and The Price of Variance Swap for SV with Delay (GBMMR)

\[ v(t) \approx X + C e^{-\gamma t} + (\xi - \lambda \beta)\gamma V_0 \times \left[ \frac{X}{\xi - \lambda \beta} (e^{(\xi - \lambda \beta)t} - 1) + \frac{C}{\xi - \lambda \beta - \gamma} (e^{(\xi - \lambda \beta)t} - e^{\gamma t}) \right] \]

\[ \mathcal{P}^* \approx e^{-rT} \left\{ X - K + C \frac{1 - e^{-\gamma T}}{T \gamma} \left[ \frac{X}{\xi - \lambda \beta} \left( \frac{e^{(\xi - \lambda \beta)T} - 1}{(\xi - \lambda \beta)} - T \right) \right. \right. \]
\[ + \frac{(\xi - \lambda \beta)\gamma V_0}{T} \left[ \frac{X}{(\xi - \lambda \beta)} \left( \frac{e^{(\xi - \lambda \beta)T} - 1}{(\xi - \lambda \beta)} - T \right) \right. \]
\[ + \frac{C (e^{(\xi - \lambda \beta)T} - 1)}{(\xi - \lambda \beta)(\xi - \lambda \beta - \gamma)} - \frac{C (e^{\gamma T} - 1)}{\gamma (\xi - \lambda \beta - \gamma)} \} \right\} \]

FIGURE 3: Variance of two-factor SV with delay and with GBM mean-reversion (formula (17)).

FIGURE 4: Price of variance swap for two-factor SV with delay and with GBM mean-reversion (formula (19)).
Variance and The Price of Variance Swap SV with Delay (OUMR)

\[ v(t) \approx X + Ce^{-\gamma t} + \xi \gamma \left( V_0 - \left( L - \frac{\lambda \beta}{\xi} \right) \right) \times \left[ \frac{X}{\xi} (e^{-\xi t} - 1) + \frac{C}{\xi + \gamma} (e^{-\xi t} - e^{-\gamma t}) \right] \]

\[ \mathcal{P}^* \approx e^{-rT} \left\{ \left[ X - K + C \frac{1 - e^{-\gamma T}}{T \gamma} \right] + \frac{\xi \gamma (V_0 - (L - \frac{\lambda \beta}{\xi}))}{T} \right\} \times \left[ \frac{X}{\xi} \left( \frac{e^{-\xi T} - 1}{\xi} + T \right) + \frac{C(e^{-\xi T} - 1)}{\xi (\xi + \gamma)} + \frac{C(e^{-\gamma T} - 1)}{\gamma (\gamma + \xi)} \right] \]

FIGURE 5: Variance of two-factor SV with delay and with OU mean-reversion (formula (22)).

FIGURE 6: Price of variance swap for two-factor SV with delay and with OU mean-reversion (formula (24)).
Variance and The Price of Variance Swap for 
SV with Delay (Pilipovich 1FMR)

\[ v(t) \approx X + C e^{-\gamma t} + \frac{\gamma \xi}{\xi + \lambda \beta} \left[ X \left( \frac{V_0(\xi + \lambda \beta)}{\xi} - L \right) \times (1 - e^{-(\xi + \lambda \beta)t}) + XLT \right] + \frac{C(V_0(\xi + \lambda \beta)}{\xi + \lambda \beta + \gamma} + \frac{CL}{\gamma} \left( e^{\gamma t} - 1 \right) \]

**FIGURE 7:** Variance of two-factor SV with delay and with Pilipovich one-factor mean-reversion (formula (27)).

\[ \mathcal{P}^* \approx e^{-rT} \left\{ \left[ X - K + \frac{C}{T \gamma} \left[ \frac{1 - e^{-\gamma T}}{T \gamma} \right] \right] + \frac{\gamma \xi}{(\xi + \lambda \beta)T} \times \left[ X \left( \frac{V_0(\xi + \lambda \beta)}{\xi} - L \right) \left( \frac{e^{-(\xi + \lambda \beta)T} - 1}{\xi + \lambda \beta} - T \right) \right] \right. 
\times \frac{XLT^2}{2} + \frac{C(V_0(\xi + \lambda \beta)}{\xi + \lambda \beta + \gamma} \left. \left( \frac{e^{\gamma T} - 1}{\gamma + \lambda \beta} + \frac{e^{-(\xi + \lambda \beta)T - 1}}{\xi + \lambda \beta} \right) + CL \left( \frac{e^{\gamma T} - 1}{\gamma} - T \right) \right] \}

**FIGURE 8:** Price of variance swap for two-factor SV with delay and with Pilipovich one-factor mean-reversion (formula (28)).
Variance and The Price of Variance Swap for SV with Delay (Pilipovich 2FMR)

\[ v(t) \approx X + Ce^{-\gamma t} - (\xi + \lambda \beta)\gamma V_0 \left[ \frac{X}{\xi + \lambda \beta} (1 - e^{-*(\xi + \lambda \beta)t}) + \frac{C}{\xi + \lambda \beta + \gamma} (e^{\gamma t} - e^{-*(\xi + \lambda \beta)t}) \right] + L_0 \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \beta_1} \times \left[ X(e^{(\beta_1 - \lambda_1 \eta)t} - e^{-*(\xi + \lambda \beta)t}) + \frac{C(\beta_1 - \lambda_1 \eta)}{(\beta_1 - \lambda_1 \eta - \gamma)} \times (e^{(\beta_1 - \lambda_1 \eta)t} - e^{\gamma t}) + \frac{C(\xi + \lambda \beta)}{(\xi + \lambda \beta + \gamma)} (e^{\gamma t} - e^{-*(\xi + \lambda \beta)t}) \right] \]

\[ P^* \approx e^{-rT} \left\{ \left[ X - K + C \frac{1 - e^{-\gamma T}}{T} \right] - (\xi + \lambda \beta)\gamma V_0 \right. \\
\left. \times \left[ \frac{X}{(\xi + \lambda \beta)} \left( \frac{e^{-(\xi + \lambda \beta)T} - 1}{(\xi + \lambda \beta)} + T \right) \right. \\
\left. + \frac{C(e^{-(\xi + \lambda \beta)T} - 1)}{(\xi + \lambda \beta)(\xi + \lambda \beta + \gamma)} + \frac{C(e^{\gamma T} - 1)}{\gamma(\gamma + \xi + \lambda \beta)} \right] \\
\left. + \frac{(\xi + \lambda \beta)L_0}{(\xi + \lambda \beta + \beta_1)T} \left[ X(e^{(\beta_1 - \lambda_1 \eta)T} - 1 - (\beta_1 - \lambda_1 \eta)T) \right. \\
\left. + \frac{X(e^{-(\xi + \lambda \beta)T} - 1 + (\xi + \lambda \beta)T)}{(\xi + \lambda \beta)} \right. \\
\left. + \frac{C(\beta_1 - \lambda_1 \eta)}{(\beta_1 - \lambda_1 \eta - \gamma)} \left( \frac{e^{(\beta_1 - \lambda_1 \eta)T} - 1}{\beta_1 - \lambda_1 \eta} - \frac{e^{\gamma T} - 1}{\gamma} \right) \right. \\
\left. + \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \gamma} \left( \frac{e^{-(\xi + \lambda \beta)T} - 1}{(\xi + \lambda \beta)} + \frac{e^{\gamma T} - 1}{\gamma} \right) \right\}. \]

FIGURE 9: Variance of three-factor SV with delay and with Pilipovich two-factor mean-reversion (formula (31)).

FIGURE 10: Price of variance swap for three-factor SV with delay and with Pilipovich two-factor mean-reversion (formula (33)).
Comparison

One-Factor

2-F(GBMMR)

2-F(OUMR)

2-F(Pilipovich 1FMR)

3-F(Pilipovich 2FMR)

FIGURE 1: Variance of one-factor SV with delay (formula (10)).

FIGURE 2: Variance of two-factor SV with delay and with GBM mean-reversion (formula (17)).

FIGURE 3: Price of variance swap for two-factor SV with delay and with GBM mean-reversion (formula (10)).

FIGURE 4: Variance of two-factor SV with delay and with OUM mean-reversion (formula (29)).

FIGURE 5: Price of variance swap for two-factor SV with delay and with OUM mean-reversion (formula (24)).

FIGURE 6: Variance of two-factor SV with delay and with Pilipovich one-factor mean-reversion (formula (27)).

FIGURE 7: Variance of two-factor SV with delay and with Pilipovich one-factor mean-reversion (formula (19)).

FIGURE 8: Price of variance swap for two-factor SV with delay and with Pilipovich one-factor mean-reversion (formula (28)).

FIGURE 9: Variance of three-factor SV with delay and with Pilipovich two-factor mean-reversion (formula (51)).

FIGURE 10: Price of variance swap for three-factor SV with delay and with Pilipovich two-factor mean-reversion (formula (59)).
Conclusion

• There is no big difference between One-Factor SV with Delay and Multi-Factor SV with Delay

• One-Factor SV with Delay catches almost all the features of Multi-Factor SV with Delay

• One-Factor SV with Delay is Similar to the SV in Heston Model (at least for variance swaps)
SV with Delay and Jumps

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) + \int_{t-\tau}^{t} y_s dN(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t) \]

SV with Delay without Jumps

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t). \]
SV with Delay and Jumps
(Simple Poisson Process Case)

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) + \int_{t-\tau}^{t} dN(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]
Equation for The Mean of Variance

\[ v(t) = E^*[\sigma^2(t, S_t)] \]

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \lambda + \alpha \lambda^2 \tau - 2\alpha \lambda \tau (\mu - r) + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s)ds - (\alpha + \gamma)v(t)
\]
Stationary Solution

\[ v(t) \equiv X = V + \left[ \alpha \lambda + \alpha \lambda^2 \tau - 2\alpha \lambda \tau (\mu - r) + \alpha \tau (\mu - r)^2 \right] / \gamma \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda + \tau (\lambda - \mu + r)^2 \right] \]

Stationary Solution without Jumps

\[ v(t) \equiv X = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} \]
Price of Var Swap in Stationary Case

\[ P = e^{-r(T-t)} \left\{ V - K + \frac{\alpha}{\gamma} \left[ \lambda + \tau (\lambda - \mu + r)^2 \right] \right\} \]
General Solution

\[ v(t) \approx X + Ce^{-\gamma t} \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda + \tau (\lambda - \mu + r)^2 \right] + Ce^{-\gamma t} \]

\[ C = \sigma_0^2 - V - \frac{\alpha}{\gamma} \left[ \lambda + \tau (\lambda - \mu + r)^2 \right] \]
Price of Var Swap in General case

\[ P \approx e^{-r(T-t)} \left[ X - K + \left( \sigma_0^2 - X \right) \frac{1 - e^{-\gamma T}}{\gamma T} \right] \]

\[ X = V + \left[ a\lambda + a\lambda^2 \tau - 2a\lambda \tau (\mu - r) + a\tau (\mu - r)^2 \right] / \gamma \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda + \tau (\lambda - \mu + r)^2 \right] \]
Compound Poisson Process Case

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW^*(s) + \int_{t-\tau}^{t} y_s dN(s) - (\mu - r)\tau \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]
Equation for the Mean of Variance

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \lambda + \alpha \lambda^2 \tau - 2\alpha \lambda \tau (\mu - r) + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds
\]

\[-(\alpha + \gamma)v(t)\]
Stationary Solution

\[ v(t) \equiv X = V + \left[ \alpha \lambda (\xi^2 + \eta) + \alpha \lambda^2 \tau \xi^2 - 2\alpha \lambda \tau \xi (\mu - r) + \alpha \tau (\mu - r)^2 \right] / \gamma \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \]
Price of Swap in Stationary Case

\[ P = e^{-r(T-t)} \{ V - K + \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \} \]
\[ v(t) \approx X + Ce^{-\gamma t} \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda(\xi^2 + \eta) + \tau(\lambda \xi - \mu + r)^2 \right] + Ce^{-\gamma t} \]

\[ C = \sigma_0^2 - V - \frac{\alpha}{\gamma} \left[ \lambda(\xi^2 + \eta) + \tau(\lambda \xi - \mu + r)^2 \right] \]
Price of Swap in General case

\[ P \approx e^{-r(T-t)} [X - K + (\sigma_0^2 - X) \frac{1-e^{-\gamma T}}{\gamma T}] \]

\[ X = V + \left[ \alpha \lambda (\xi^2 + \eta) + \alpha \lambda \tau \xi^2 - 2 \alpha \lambda \tau \xi (\mu - r) + \alpha \tau (\mu - r)^2 \right] / \gamma \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \]
More General case

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW^*(s) + \int_{t-\tau}^{t} y_s dN(s) - (\mu - \lambda) \tau \right]^2
\]

\[-(\alpha + \gamma)\sigma^2(t, S_t)\]  

where \(W^*(t)\) is a Brownian motion, \(N(t)\) is a Poisson process with intensity \(\lambda\) and \(y_t\) is the jump size at time \(t\). We assume that \(E[y_t] = A(t)\), \(E[y_s y_t] = C(s, t), s < t\) and \(E[y_t^2] = B(t) = C(t, t)\), where \(A(t), B(t), C(s, t)\) are all deterministic functions. Note that the change of measure do not change the Poisson intensity \(\lambda\) and the distribution of jump size \(y_t\), since they are independent to the Brownian motion.
Equation for the Mean of Variance

\[
\frac{dv(t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} v(s)ds + \lambda \int_{t-\tau}^{t} B(s)ds + \lambda^2 (K(t, \tau) + G) \right] \\
+ (\mu - r)^2 \tau^2 - 2\lambda \tau (\mu - r) \int_{t-\tau}^{t} A(s)ds \right] - (\alpha + \gamma)v(t)
\]
General Solution

\[
v(t) \approx \frac{1-e^{-\gamma t}}{\gamma} v'(0) + \left[ \frac{\alpha}{\gamma} (1 - e^{-\gamma t}) + 1 \right] v(0) - \frac{\alpha}{\gamma \tau} \int_{-\tau}^{t} v(s) [1 - e^{-\gamma(t-s-\tau)}] ds
\]

\[
+ \frac{1}{\gamma} \int_{0}^{t} h(s, \tau) [1 - e^{-\gamma(t-s)}] ds + C. \tag{49}
\]

\[
C = \frac{\alpha}{\gamma \tau} \int_{-\tau}^{0} v(s) [1 - e^{\gamma(s+\tau)}] ds.
\]
Numerical Example

Table 1

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</table>
Constant $V$ and $E^*[v]$

\[
V + \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \\
= 0.0002 + 0.0604/0.0124 \times \left[ 0.0115 \times \left[ (-0.003)^2 + 0.0035 \right] \\
+ (0.0115 \times (-0.003) - 0.0002 + 0.0124)^2 \right] \\
= 0.0023.
\]

\[
E^*[v] \approx V + \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \\
+ \left\{ \sigma_0^2 - V - \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \right\} \frac{1-e^{-\gamma T}}{\gamma T} \\
= 0.0023 + (0.0001 - 0.0023) \times \frac{1-e^{-0.0124}}{0.0124} \\
= 0.0001136.
\]
Delivery Price and Maturity

Figure 1: Dependence of Delivery Price on Maturity (S&P60 Canada Index).
Figure 2: Dependence of Delivery Price on Delay (S&P60 Canada Index).
Delivery Price and Jump Intensity

Figure 3: Dependence of Delivery Price on Jump Intensity (S&P60 Canada Index).
Delivery Price, Delay and Jump Intensity

Figure 4: Dependence of Delivery Price on Delay and Jump Intensity (S&P60 Canada Index).
Delivery Price, Delay and Maturity

Figure 5: Dependence of Delivery Price on Delay and Maturity (S&P60 Canada Index).
Figure 6: Dependence of Delivery Price on Jump Intensity and Maturity ($S&P60$ Canada Index).
Publications

• ‘Continuous-time GARCH model for Stochastic Volatility with Delay’ (Kazmerchuk, Sw, Wu), CAMQ, 2005, v. 3, No. 2
• ‘Modeling and Pricing of Variance Swap for Stochastic Volatility with Delay’ (Sw), Wilmott Magazine, Issue 19, September 2005
• ‘Modeling and Pricing of Variance Swaps for Multi-Factor Stochastic Volatilities with Delay’ (Sw), CAMQ, 2006, v. 14, No. 4
• ‘Pricing Variance Swaps for Stochastic Volatilities with Delay and Jumps’ (Sw, Xu, L.), Quantitative Finance (submitted), 2007
The End

• Thank You for Your Attention!
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