Chapter 10.3-10.4: Market Making with Utility and Adverse Selection

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Consider an exponential utility function and corresponding performance criteria,

\[ u(x) = e^{-\gamma x} \]

\[ G^\delta(t, x, S, q) = \mathbb{E}_{t, x, s, q}[ - \exp \{-\gamma(X^\delta_T + Q^\delta_T(S_T - \alpha Q^\alpha_T))\} ] \]

With the ansatz \( G(t, x, S, q) = -e^{-\gamma(x+qS+g(t,q))} \), we get the HJB equation:

\[
\partial_t g - \frac{1}{2} \sigma^2 \gamma q^2 + \sup_{\delta_+} \lambda^+ e^{-\kappa_+\delta_+} \frac{1 - e^{-\gamma(\delta_+ + g(t,q-1) - g(t,q))}}{\gamma} \\
+ \lambda^- e^{-\kappa_-\delta_-} \frac{1 - e^{-\gamma(\delta_- + g(t,q+1) - g(t,q))}}{\gamma} = 0
\]

\[ g(t, 0) = 0 \quad g(T, q) = -\alpha q^2 \]
Solving the HJB with calculus leads to the optimal posting depth that maximizes utility,

$$\delta^{\pm,*} = \begin{cases} 
\delta^{\pm,*}_0 + \left( \gamma \left[ \log \left( 1 + \frac{\gamma}{\kappa^{\pm}} \right) \right]^{-1} - \left[ \kappa^{\pm} \right]^{-1} \right) & \gamma > 0 \\
\delta^{\pm,*}_0 & \gamma = 0
\end{cases}$$

Where the 0 subscript indicates the parameter of the base model in 10.2. This shows us that a model using an exponential utility function is just a constant shift of the model using an inventory penalty.
We suppose that the midprice follows the dynamics,

\[ dS_t = \sigma dW_t + \epsilon^+ dM^+_t - \epsilon^- dM^-_t \]

Where \( M^\pm_t \) are Poisson processes with intensities \( \lambda^+ \) and \( \lambda^- \) respectively. Let \( \epsilon^\pm = \mathbb{E}[\epsilon^\pm] < \infty \).

We consider the running inventory penalty objective function,

\[ H^\delta(t, x, S, q) = \mathbb{E}_{t, x, S, q} \left[ X^\delta_T + Q^\delta_T (S_t - \alpha Q^\delta_T) - \phi \int_t^T (Q^\delta_u)^2 du \right] \]

We use the ansatz,

\[ H(t, x, S, q) = x + qS + h(t, q) \]
We get the HJB, where \(\underline{q}\) and \(\overline{q}\) are the minimum and maximum inventory levels respectively,

\[
\phi q^2 = \partial_t h + \lambda^+ \sup_{\delta^+} \left\{ e^{-\kappa^+ \delta^+} (\delta^+ - \varepsilon^+ + h_{q-1} - h_q) \right\} 1_{q > \underline{q}}
\]

\[
+ \lambda^- \sup_{\delta^-} \left\{ e^{-\kappa^- \delta^-} (\delta^- - \varepsilon^- + h_{q+1} - h_q) \right\} 1_{q < \overline{q}}
\]

\[
+ (\varepsilon^+ \lambda^+ - \varepsilon^- \lambda^-) q
\]

\[h(T, q) = -\alpha q^2\]

Define the vector \(z \in \mathbb{R}^{\overline{q} - q + 1}\) such that \(z_j = e^{-\alpha \kappa j^2}\) \(\quad j = \overline{q} \ldots, q\). Define the matrix \(A \in \mathbb{M}^{(\overline{q} - q + 1) \times (\overline{q} - q + 1)}\) where,

\[
A_{i,q} = \begin{cases} 
q \kappa (\varepsilon^+ \lambda^+ - \varepsilon^- \lambda^-) - \phi q^2, & i = q \\
\lambda^+ e^{-1 - \kappa \varepsilon^+}, & i = q - 1 \\
\lambda^- e^{-1 - \kappa \varepsilon^-}, & i = q + 1 \\
0, & \text{otherwise}
\end{cases}
\]
10.4.1- Impact of Market Orders on Midprice

Define $w(t) = e^{A(T-t)}z \in \mathbb{R}^{q-q+1}$. We find the optimal control to be, assuming $\kappa^+ = \kappa^- = \kappa$,

$$
\delta^{+,*}(t, q) = \varepsilon^+ + \frac{1}{\kappa} \left( 1 + \log \left( \frac{w_q(t)}{w_{q-1}(t)} \right) \right) \quad q \neq q
$$

$$
\delta^{-,*}(t, q) = \varepsilon^- + \frac{1}{\kappa} \left( 1 + \log \left( \frac{w_q(t)}{w_{q+1}(t)} \right) \right) \quad q \neq \bar{q}
$$
Behaviour of the Strategy

We see that more risk aversion results in a larger fill rate and a higher magnitude of inventory drift.

Figure 10.9 Long-term inventory level. Model parameters are: $\lambda^+ = 2$, $\lambda^- = 1$, $\varepsilon = 0.005$, $\kappa^\pm = 100$, $\bar{q} = -q = 10$, $\phi = \{0.2, 0.1, 0.05\}$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$. 

We see that more risk aversion results in a larger fill rate and a higher magnitude of inventory drift.
Suppose our midprice includes the alpha of the stock,

\[ dS_t = (\nu + \alpha_t) dt + \sigma dW_t \]

\[ d\alpha_t = -\zeta \alpha_t dt + \eta dW_t + \epsilon^+_{1+M_t^+} dM_t^+ - \epsilon^-_{1+M_t^-} dM_t^- \]

Where \( \epsilon^\pm_i, i \in \mathbb{N} \) are i.i.d. random variables representing the size of market impact. Define \( \ell_t^\pm \in \{0, 1\} \) denote whether or not a limit order is posted on the respective side of the book at time \( t \). The agent has the cash process,

\[ dX^\ell_t = \left( S_t + \frac{\Delta}{2} \right) dN^+_{t,\ell} - \left( S_t - \frac{\Delta}{2} \right) dN^-_{t,\ell} \]

Where \( \Delta \) is the spread and \( N_t^\pm \) is the counting process for filled limit orders.
Short-term alpha and Adverse Selection

We use a running penalty performance criteria,

\[
H^\ell(t, x, S, \alpha, q) = \mathbb{E} \left[ X_T^\ell + Q_T^\ell (S_T - \left( S_T - \left( \frac{\Delta}{2} + \varphi Q_T^\ell \right) \right) - \phi \int_t^T (Q_u^\ell)^2 du \right]
\]

The ansatz of choice is,

\[
H(t, x, S, \alpha, q) = x + qS + h(t, \alpha, q)
\]

with the terminal condition \( h(T, \alpha, q) = -q \left( \frac{\Delta}{2} + \varphi q \right) \). An implicit optimal control is found to be,

\[
l^{+,*}(t, q) = 1 \left\{ \frac{\Delta}{2} + \mathbb{E}[h(t, \alpha+\epsilon^+, q-1) - h(t, \alpha+\epsilon^+, q)] > 0 \right\} \cap \{ q > q \}
\]

\[
l^{-*,}(t, q) = 1 \left\{ \frac{\Delta}{2} + \mathbb{E}[h(t, \alpha-\epsilon^-, q+1) - h(t, \alpha-\epsilon^-, q)] > 0 \right\} \cap \{ q < q \}
\]
We see that $\alpha < 0$ encourages sell orders, and $\alpha > 0$ encourages buy orders.
Figure 10.11 Sample path of the optimal strategy. Green lines show when and at what price the agent is posted. Solid red circles indicate MOs that arrive and hit/lift the posted bid/offer. Open red circles indicate MOs that arrive but do not hit/lift the agent's posts. Shaded region is the bid-ask spread.