Market Making
Activity and Market Quality, part I.

Jonathan A. Chávez Casillas

1University of Calgary
Department of Mathematics and Statistics

LOBster Seminar
A Market Maker (MM) maximizes terminal wealth by trading in and out positions using LOs.

1) \( S = (S_t := S_0 + \sigma W_t)_{0 \leq t \leq T} \) denotes the midprice.

2) \( \delta^\pm = (\delta^\pm)_{0 \leq t \leq T} \) denotes the depth for posting LO. Sell LO are posted at \( S_t + \delta^+_t \) and buy at \( S_t - \delta^-_t \).

3) \( M^\pm = (M^\pm_t)_{0 \leq t \leq T} \) denotes counting processes corresponding to the arrival of other participants buy (+) and sell (-) MO.

4) \( N^{\delta, \pm} = (N^{\delta, \pm}_t)_{0 \leq t \leq T} \) denote the controlled counting process for the agent’s filled sell (+) and buy (-) LOs.

5) Conditional on a MO arrival, the posed LO is filled with probability \( e^{-\kappa \delta^\pm_t} \).

6) \( X^\delta = (X^\delta_t)_0 \leq t \leq T \) denotes the MM’s cash process and satisfies the SDE

\[
dX^\delta_t = (S^-_t + \delta^+_t) dN^{\delta, +}_t - (S^-_t - \delta^-_t) dN^{\delta, -}_t
\]

7) \( Q^\delta = (Q^\delta_t)_{0 \leq t \leq T} \) denotes the inventory process and

\[
Q^\delta_t = N^{\delta, -}_t - N^{\delta, +}_t
\]
Market Making

The MM caps his inventory. That is, $q$ is between $\underline{q} < 0$ and $\overline{q} > 0$. Moreover, the performance criterion of the MM is

$$H^\delta(t, x, S, q) = \mathbb{E}_{t, x, q, S} \left[ X_T + Q_T^\delta (S_T - \alpha Q_T^\delta) - \phi \int_t^T (Q_u)^2 du \right]$$

Then, the MM value function is

$$H(t, x, S, q) = \sup_{\delta \pm \varepsilon \in A} H^\delta(t, x, S, q)$$

After formulating the DPP (see equation 10.4 and 10.5) we assume that $H$ has the ansatz,

$$H(t, x, q, S) = x + qS + h(t, q)$$

As before, we obtain the optimal control in feedback form:

$$\delta^{+, *}(t, q) = \frac{1}{\kappa^+} - h(t, q - 1) + h(t, q), \quad q \neq \underline{q}$$

$$\delta^{-, *}(t, q) = \frac{1}{\kappa^-} - h(t, q + 1) + h(t, q), \quad q \neq \overline{q}$$
Substituting the optimal controls into de DPE:

\[ \phi q^2 = \partial_t h(t, q) + \frac{e^{-\lambda^+}}{\kappa^+} e^{-\kappa^+(-h(t,q-1)+h(t,q))} \mathbb{1}_{\{q>\bar{q}\}} + \frac{e^{-\lambda^-}}{\kappa^-} e^{-\kappa^-(-h(t,q+1)+h(t,q))} \mathbb{1}_{\{q<\bar{q}\}} \]

Assuming that \( \kappa = \kappa^+ = \kappa^- \) then the solution of the DPE is given by

\[ h(t, q) = \frac{1}{\kappa} \log(\omega(t, q)) , \]

where \( \omega(t, q) \) is an entry on the vector \( \omega(t) = [\omega(t, \bar{q}), \omega(t, \bar{q} - 1, \ldots, \omega(t, q)]^T \). Moreover, the vector \( \omega(t) \) satisfies the equation

\[ \omega(t) = e^{A(T-t)} z, \]

where \( A \) is the tridiagonal matrix whose entries range from \( q \) to \( \bar{q} \), its diagonal entries are \( -\phi \kappa q^2 \), the entries below the diagonal are \( \lambda^+ e^{-1} \) and the entries above the diagonal are \( \lambda^- e^{-1} \) and \( z \) is the vector whose components are \( z_j = e^{-\alpha \kappa j^2} \), where \( j \) ranges from \( q \) to \( \bar{q} \).
When the strategy is far away from expiry and inventories are close to the allowed minimum, the optimal sell posting is furthest away from the midprice at a very high price (need high incentive to even go further down) but the optimal buy posting is close to the midprice to complete round-trips. Moreover, as the strategy approaches $T$ and $q_t < 0$, the optimal sell depth $\delta^+$ decreases.
The figure shows the inventory and price path for one simulation of the strategy. We can see in the left that the inventory is mean reverting and that in this particular path, although the cap is set to be 10, the inventory never surpasses 5. In the right, it is displayed a window of the midprice along with MM’s buy and sell LOs. Solid circles show incoming MO that are filled by the resting LO of the MM.

**Figure 10.3** Inventory and midprice path. Model parameters are: $\lambda^\pm = 1$, $\kappa^\pm = 100$, $\bar{q} = -q = 10$, $\phi = 2 \times 10^{-4}$, $\alpha = 0.0001$, $\sigma = 0.01$, $S_0 = 100$. 
The figure shows the Profit and Loss (P& L) of the optimal strategy on the left and on the right the lifetime of the inventory for different values of the running penalty $\phi$. When $\phi$ increases, the P& L histogram shifts to the left because the strategy does not allow inventory positions to stray away from 0, and hence profits decrease.
Market Making with no Inventory Restrictions

Assume that $\phi = \alpha = 0$, so that there no terminal inventory penalty and no penalzation on running inventories. Then the optimal strategy is

$$\delta^{+,\ast} = \frac{1}{\kappa^+}, \quad \text{and} \quad \delta^{-,\ast} = \frac{1}{\kappa^-}$$

Here, it is just maximized the probability of being filled.
Market Making at the touch

In very liquid markets, most of the MO do not walk the book. Thus, the MM just posts LO at the touch. For this model, assume that the spread is constant to $\Delta$. Let $l^\pm_t \in \{0,1\}$ denote whether the agent is posted on the sell side (+) or the buy side (-) of the LOB. In this way the agent may post at both, any or none of the sides of the book. The performance criteria of the agent is given by:

$$H^l(t,x,S,q) = E_{t,x,q,S}\left[ X_T^l + Q_T^l(S_T(\frac{\Delta}{2} + \phi Q_T^l)) - \phi \int_t^T (Q_u^l)^2 du \right]$$

and the safer solving the DPE, the optimal control has the form:

$$l^+,*(t,q) = 1_{\left\{ \{ \frac{\Delta}{2} + h(t,q-1) - h(t,q) > 0 \} \cap \{ q > q \} \right\}}$$

$$l^-,*(t,q) = 1_{\left\{ \{ \frac{\Delta}{2} + h(t,q+1) - h(t,q) > 0 \} \cap \{ q > \bar{q} \} \right\}}$$

Thus, the agent posts a LO on the appropriate side of the LOB by ensuring that she only posts if the arrival of a MO, which hits his LO, produces a change in her value by more than $-\Delta/2$. 
In the figure, it is shown that the agent’s optimal posting varies with time and running penalty. In the left, we see that the agent only posts sell LO when her inventory is very high and buy LO when is very low. In the central region she posts both. On the right we see that when the running penalty increases, the region over which the agent constraints her inventory shrink and eventually arrives to a region where she only posts one unit of the asset at a time and then immediately liquidates it.

Figure 10.5 The optimal strategy for the agent who posts only at-the-touch.