Algorithmic and High-Frequency Trading
A Primer on the Microstructure of Financial Markets

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Overview

- Introduction
- Market Making
  - Grossman-Miller Market Making Model
  - Trading Costs
  - Measuring Liquidity
  - Market Making using Limit Orders
- Trading on an Informational Advantage
- MM with an Informational Disadvantage
  - Price Dynamics
  - Price Sensitive Liquidity Traders
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Market Microstructure
   — subfield of finance
   — “study of the process and outcome of exchanging assets under explicit trading rules” (O’Hara(1995))

Key Dimension of trading and pricing: Information

Price Efficiency
   — “market prices are an efficient way of transmitting the information required to arrive at a Pareto optimal allocation of resources” (Grossman&Stiglitz (1976))

Trading: many possible ways
   — focus on trading in large electronic markets
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Market Making

- Market participants
  - Market Maker (MM)
  - Liquidity Trader (LT)
- Market Maker
  - Competition
  - Provide liquidity → immediacy
  - Bid and ask prices → Limit Orders
- Liquidity Trader
  - Take liquidity
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 Providing liquidity $\rightarrow$ accept one side of trade $\rightarrow$ hold assets until person with matching demand enters market $\rightarrow$ risk of prices moving against MM $\rightarrow$ Risk Premium/Liquidity Premium

Model
- $n$ identical Market Makers
- three dates $t \in \{1, 2, 3\}$
- no uncertainty about arrival of matching orders

\[ LT_1 \text{ wants to sell } i \text{ units of the asset} \]
\[ LT_2 \text{ wants to buy } i \text{ units of the asset} \]
Beginning (t=0)

- $W_0$: initial cash amount
- MM: no assets
- $LT_1$: $i$ units
- $LT_2$: -$i$ units

- No trading or direct costs for holding inventory
Grossman-Miller Market Making Model

- $S_t$ : cash value of asset in $t$
  \[ S_3 = \mu + \epsilon_2 + \epsilon_3 \]
  - $\mu$ : constant
  - $\epsilon_2$ and $\epsilon_3$ : independent, $N(0, \sigma^2)$, random variables
  - $\epsilon_2$ : announced between $t=1$ and $t=2$
  - $\epsilon_3$ : announced between $t=2$ and $t=3$

$LT_1$ : sell $i$ units

$LT_2$ : buy $i$ units
Risk averse participants

- Expected utility: $E[U(X_3)]$ where $U(X) = -\exp(-\gamma X)$
- $\gamma$: utility penalty for taking risk $\rightarrow$ risk aversion parameter

$q_t^j$: asset holdings at the end of period $t$

$j \in \{MM, LT1, LT2\}$

$LT_1$: sell $i$ units

$LT_2$: buy $i$ units
Grossman-Miller Market Making Model

- \( t=3 \)
  \[ S_3 = \mu + \epsilon_2 + \epsilon_3 \]

- \( t=2 \)
  \[
  \max_{q_2^j} \mathbb{E}
  \left[
  U\left(X_3^j\right) \bigg| \epsilon_2 \right]
  \text{ s.t. } X_3^j = X_2^j + q_2^j S_3 \quad \text{and} \quad X_2^j + q_2^j S_2 = X_1^j + q_1^j S_2
  \]
  \[ q_2^{j,*} = \frac{\mathbb{E}\left[S_3 \big| \epsilon_2 \right] - S_2}{\gamma \sigma^2} \quad \forall j \]
  equal demand and supply in \( t=2 \), i assets are held by MM & LT1, \( q_1^{LT2} = -i \)
  \[ q_2^{j,*} = 0, \quad S_2 = \mu + \epsilon_2 \]

LT_1: sell \( i \) units
LT_2: buy \( i \) units
\( t=1 \) \((X_3 = X_2)\)

\[
\max_{q_1^j} \mathbb{E} \left[ U(X_2^j) \right] \text{ s.t. } X_2^j = X_1^j + q_1^j S_1 \text{ and } X_1^j + q_1^j S_1 = X_0^j + q_0^j S_0
\]

\( q_1^{j,*} = \frac{\mathbb{E}[S_2] - S_1}{\gamma \sigma^2} \) for MMs and LT1

equal demand and supply in \( t=1 \) and \( q_0^{MM} = 0 \)

\( q_1^{j,*} = \frac{i}{n+1} \) and \( S_1 = \mu - \gamma \sigma^2 \frac{i}{n+1} \)
Grossman-Miller Market Making Model

- Influence Factors on Risk Premium
  - size of the liquidity demand $|i|$
  - amount of competition between MMs $n$
  - market risk aversion $\gamma$
  - volatility of the underlying asset $\sigma^2$

- $n \to \infty \Rightarrow$ Liquidity Premium goes to zero
  $\Rightarrow$ Price converges to efficient level $S_1 = \mu$
  $\Rightarrow$ LT1 net trade converges to liquidity need
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Model with participation costs $c$

- $c$ : proxy for time and investments needed to keep a constant, active and competitive presence in the market + opportunity costs
- Result:
  - Level of competition decreases monotonically with supplier’s participation costs
  - Increase the liquidity premium
Model with trading cost $\eta$
- Depend on level of activity in the market (proportional to shares traded)
- Act like participation costs for liquidity traders (fees are known)
  → Relatively small trades are too expensive

$t=2$

$$q_{2}^{j,*} = \frac{\mathbb{E}[S_{3}-\eta | \epsilon_{2}]- (S_{2}-\eta)}{\gamma \sigma^{2}}$$ for LT1 & MM

$$q_{2}^{LT2,*} = \frac{\mathbb{E}[S_{3}+\eta | \epsilon_{2}]- (S_{2}+\eta)}{\gamma \sigma^{2}}$$

$$S_{2} = \mu + \epsilon_{2},$$
Trading Costs

\[ t=1 \]

- LT1: any quantities he doesn’t sell now he has to sell later
  \[ q_{1}^{LT1} = \frac{\mathbb{E}[S_{2} - \eta | \epsilon_{2}] - (S_{1} - \eta)}{\gamma \sigma^{2}} \]

- MM: whatever they buy they have to sell later
  \[ q_{1}^{MM} = \frac{\mathbb{E}[S_{2} - \eta | \epsilon_{2}] - (S_{1} + \eta)}{\gamma \sigma^{2}} \]

- Equal demand and supply in \( t=1 \)
  \[ S_{1} = \mu - \gamma \sigma^{2} \frac{i}{n+1} - 2 \frac{n}{n+1} \eta \]
  and
  \[ q_{1}^{LT1,*} = \frac{i}{n+1} + 2 \frac{n}{n+1} \frac{\eta}{\gamma \sigma^{2}} \]

- Extra liquidity discount
- Almost all trading fees are paid by the LT initiating the transaction
- Increase in trading fees has smaller effect via competition but greater effect on immediacy and the liquidity discount than participation costs
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Transformation to electronic asset markets

- Trading does not take place at once and not to a single price
- LT post MO into exchange → meet LO of MM, which are resting in LOB
- Here:
  - LT1’s MO enters market and is executed against LO in LOB
  - Possibilities: one large order or many small orders
  - As execution price changes, so does LT1’s strategy and eventually, after selling \( i \frac{n}{n+1} \) shares, the price has moved too far and LT1 stops trading
  - Overall: execution at average price \( S_1 \)
  - Risk premium: \( S_1 - \text{midprice of first MO} \)
- Effect of traded quantity $\rightarrow$ Rewrite $S_1$
  \[
  S_1 = \mu + \lambda q^{LT1}
  \]

- Grossman-Miller model:
  \[
  \lambda = -\frac{1}{n}\gamma \sigma^2 \text{ and } q^{LT1} = i \frac{n}{n+1}
  \]
  - $\lambda$ : market price reaction to LT1’s total order (price impact)
  - Describes liquidity of the market for this asset
  - The more liquid, the lower the absolute lambda
Other way to measure liquidity: autocovariance in asset price changes (or returns)

Introduce new date $t=0$ and a random public event $\epsilon_1$ announced between 0 and 1

- $S_3 = \mu + \epsilon_1 + \epsilon_2 + \epsilon_3$
- $\mu_0 = \mathbb{E}[S_3]$, $\mu_1 = \mathbb{E}[S_3|\epsilon_1]$, $\mu_2 = \mathbb{E}[S_3|\epsilon_1, \epsilon_2]$, $\mu_3 = S_3$
- $\epsilon_1, \epsilon_2, \epsilon_3$ i.i.d. $\sim N(0, \sigma^2)$
- Discrete process $\mu_t$ is a martingale, efficient price process
- No trades in $t=0 \Rightarrow S_0 = \mathbb{E}[S_3] = \mu_0$
- $S_1 = \mu + \lambda q^{LT_1}$, $S_2 = \mu_2$
- $\Delta_1 = S_1 - S_0$, $\Delta_2 = S_2 - S_1$
- $\text{Cov}[\Delta_1, \Delta_2] = -\lambda^2 \text{Var}[q^{LT_1}] < 0$

→ Autocovariance captures liquidity just like price impact does
→ $\lambda \to 0 \Rightarrow \text{Cov} \to 0 \Rightarrow$ price process converges to martingale process $\mu_t$
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Small risk-neutral trader with costless inventory management and infinite patience
- Other MMs do not react to our MMs decision
- Don’t know time and size of incoming MOs
- $S_t$: current value of asset, midprice
- Liquidates her inventory at midprice at no costs

- $\delta^\pm$: depth, distance from mid price
- $p^\pm$: probability that an MO arrives
- $P^\pm$: cdf, probability that price walks to MM’s LO after arrival of MO
  $\Rightarrow p_P^-(\delta^-)$: prob that buy LO is filled
- Distribution of other LO’s: exponential with parameter $\kappa^\pm 
  \Rightarrow p_P^-(\delta^-) = p^-e^{-\kappa^-\delta^-}$
\[ \Pi: \text{MM's profit per trade} \]

\[
\max_{\delta^+, \delta^-} \mathbb{E}[\Pi (\delta^+, \delta^-)] = \max_{\delta^+, \delta^-} \{ p^+ e^{-\kappa^+ \delta^+} + p^- e^{-\kappa^- \delta^-} \}
\]

\[ \rightarrow \delta^{\pm,*} = \frac{1}{\kappa^\pm} \]

- Given our parametric choice of \( P_{\pm} \), the optimal depth is equal to the mean depth in the LOB
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Trading on an Informational Advantage

- Market for an asset, opens only at one time point
- \( S \): trading price
- \( v \): future cash value after trading \( \sim N(\mu, \sigma^2) \)
- Traders:
  - Informed trader: knows exact value of \( v \)
  - Anonymous mass of price insensitive liquidity traders (LT)
  - Large number of MMs (observe and compete for order flow)
    - Risk neutral \( \rightarrow \) do not need a liquidity premium for taking risk
    - Liquidity premium from informational disadvantage (will be borne by LTs)
    - Know that there is one informed trader, don’t know who it is
- \( u \): net demand of LTs \( \sim N(0, \sigma_u^2) \) independent of \( v \)
- \( x(v) \): number of shares traded by the informed trader
- \( x(v) + u \): net order flow observed by MMs
Solution: Bayesian Nash equilibrium

- All agents optimize given the decisions of all other players according to their beliefs

\[ S = \mathbb{E}[v|\mathcal{F}] \text{ where } \mathcal{F}: \text{all information available to MMs (semi-strong efficiency)} \]

\[ S(x + u) = \mathbb{E}[v|x + u] \]

- Because of the normality of \( v \) and \( u \) insiders hypothesize

\[ S(x + u) = \mu + \lambda(x + u) \]

- \( \lambda \): linear sensitivity of the market price to order flow

\[ \max_x \mathbb{E}[x(v - S(x + u))] \rightarrow x^*(v) = \beta(v - \mu); \quad \beta = (2\lambda)^{-1} \]

\[ S = \mathbb{E}[v|x + u] = \mu + \frac{2(x+u)}{\sigma} \sigma_u \rightarrow \text{premium } (\lambda = \frac{2\sigma_u}{\sigma}) \]
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Many informed traders, can only trade 1 unit
\[ \Delta_a, \Delta_b : \text{ask- and bid-halfspreads} \rightarrow \text{spread} = \Delta_a + \Delta_b \]

MM chooses \( a = \mu + \Delta_a \) and \( b = \mu - \Delta_b \) \((\mu = \mathbb{E}[v|\mathcal{F}])\)

- Buy order comes in:
  - From uninformed LT \( \rightarrow \) expected profit \( a - \mu = \Delta_a \)
  - From informed Trader \( \rightarrow \) expected loss \( a - V_H = \Delta_a - (V_H - \mu) \)

- Expected profit of posting price \( a \):
  \[
  \frac{(1-\alpha)/2}{\alpha p + (1-\alpha)/2} \Delta_a + \frac{\alpha p}{\alpha p + \frac{1-\alpha}{2}} (\Delta_a - (V_H - \mu)) \overset{!}{=} 0
  \]

\[ \rightarrow \Delta_a = \frac{1}{1 + \frac{1-\alpha}{\alpha} \frac{1/2}{p}} (V_H - \mu) \quad \Delta_b = \frac{1}{1 + \frac{1-\alpha}{\alpha} \frac{1/2}{1-p}} (\mu - V_L) \]
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Incorporate time dimension, for simplicity: interest = zero

Determination of cash value at t=T

\( \mathcal{F}_t \): Public information in t

\( p_t = \mathbb{P}(v = V_H | \mathcal{F}_t), \quad \mu_t = \mathbb{E}[v | \mathcal{F}_t] \)

\[ a_t = \mu_t + \Delta_{a,t} = \mu_t + \frac{1}{1 + \frac{1-\alpha}{\alpha} \frac{1}{p_t}} (V_H - \mu_t) \]

\[ b_t = \mu_t + \Delta_{b,t} = \mu_t - \frac{1}{1 + \frac{1-\alpha}{\alpha} \frac{1}{1-p_t}} (\mu_t - V_L) \]

At every execution, the execution price is equal to the expectation of the underlying asset conditional on the history of order flow and also on the information in the execution (buy or sell) \( \Rightarrow \) realized price process is a martingale.
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- LTs avoid trading if the half-spread is too high
- $LT_i$ gets a cash equivalent utility gain of $c_i$ (urgency parameter) if he executes his desired trade $\Rightarrow$ if $c_i < \Delta$ the trade won’t be executed
- $F(c) = \mathbb{P}(c_i < c)$

\[ \Delta_a = \frac{1}{1 - F(\Delta_a)} \left( V_H - \mu \right) \]
\[ \Delta_b = \frac{1}{1 - F(\Delta_b)} \left( \mu - V_L \right) \]

- MM increases halfspread $\Rightarrow$ smaller population of LTs trades
- $c_i$ small $\Rightarrow \Delta_a = V_H - \mu$ and $\Delta_b = \mu - V_L \equiv$ solution without LTs $\Rightarrow$ market collapse
  $\Rightarrow$ strong efficient price
Thank you 😊