Paper Review: Hawkes Processes in Finance

Section 4: Price Models

Bacry et al. (2015)

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Outline of Presentation

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Overview of Section 4: Price Models
In their paper, Bacry et al. propose an overview of the recent academic literature devoted to the applications of Hawkes processes in finance and review their main empirical applications to address many different problems in high frequency finance. Section 4 deals with price models (mid-price or best limit price), and in particular reviews the following articles:


Recap: Hawkes Process Definition (Section 2.1)

**Definition 1** A Hawkes process is a counting process $N_t$ such that the intensity vector can be written as

$$\lambda_t^i = \mu^i + \sum_{j=1}^{D} \int \phi^{ij}(t - t') dN_{t'}^j$$

where the quantity $\mu = \{\mu^i\}_{i=1}^{D}$ is a vector of exogenous intensities and $\Phi(t) = \{\phi^{ij}(t)\}_{i,j=1}^{D}$ is a matrix kernel which is component-wise positive ($\phi^{ij}(t) \geq 0$ for each $1 \leq i, j \leq D$), component-wise causal (if $t < 0$, $\phi^{ij}(t) = 0$ for each $1 \leq i, j \leq D$) and each component $\phi^{ij}(t)$ belongs to the space of $L^1$-integrable functions.
Recap: Hawkes Process Stationarity (Section 2.1)

**Proposition 1** The process $N_t$ has asymptotically stationary increments and $\lambda_t$ is asymptotically stationary if the kernel satisfies the following stability condition:

The matrix $||\Phi|| = \{||\phi^{ij}||\}_{i,j=1}^D$ has spectral radius smaller than 1.
Recap: Hawkes Process Properties (Section 2.3)
Assuming the stationarity condition (1), we can write explicitly the first- and second-order properties of the Hawkes model in terms of the Laplace transform of the kernel $\Phi$. Therefore, we have to introduce the Kernel inversion:

**Definition 2**  
Consider a Hawkes process $N_t$ with stationary increments. We define $\Psi(t)$ as the causal solution of the equation

$$\Phi(t) + \Psi(t) \star \Phi(t) = \Psi(t)$$

As 1 holds, $\Psi(t)$ exists and can be expressed as

$$\Psi(t) = \Phi(t) + \Phi(t) \star \Phi(t) + \Phi(t) \star \Phi(t) \star \Phi(t) + \ldots$$

Applying component-wise Laplace transforms, we can write in the Laplace domain:

$$\hat{\Psi}(z) = (\mathbb{I} - \hat{\Phi}(z))^{-1} - \mathbb{I}$$
Recap: Hawkes Process Properties (Section 2.3)

**Proposition 2** For a Hawkes process $N_t$ with stationary increments, the following propositions hold:

The average intensity $\Lambda = \frac{E[dN_t]}{dt}$ is equal to

$$\Lambda = (I + \hat{\Psi}(0))\mu$$

The Laplace transforms of the linear correlation matrix

$$c(t - t') = \frac{E[dN_tdN^T_{t'}] - E[dN_t]E[dN^T_{t'}]}{dtdt'}$$

is equal to

$$\hat{c}(z) = (I + \hat{\Psi}(-z))\Sigma(I + \hat{\Psi}^T(z))$$

where $\Sigma$ is a diagonal matrix with non-zero elements equal to $\Sigma^{ii} = \Lambda^i$. 
Recap: Diffusion Limits (Section 2.3.6)
Hawkes processes, under appropriate hypotheses and after a suitable rescaling, behave at large times as linear combinations of Wiener processes. In particular:

**Theorem 1 (Law of Large Numbers)** Consider a Hawkes process as in Definition (1) satisfying the stationarity assumption Proposition (1). Then

$$\sup_{u \in [0,1]} \left| \frac{1}{T} N_u T - u\Lambda \right| \xrightarrow{T \to \infty} 0$$

(2) almost surely and in \(L^2\)-norm.
Recap: Diffusion Limits (Section 2.3.6)

**Theorem 2 (Central Limit Theorem)** Suppose that for all $i, j \leq D$ the kernel $\Phi(t)$ satisfies

$$\int_0^\infty t^{1/2} \phi_{ij}(t) dt < \infty$$

Then for $u \in [0, 1]$ one has the following convergence in law for the Skorokhod topology:

$$T^{1/2}(T^{-1}N_{uT} - u\Lambda) \xrightarrow{T \to \infty} (I + ||\Psi||)\Sigma^{1/2}W_u$$ \hspace{1cm} (3)

where $W_t$ denotes a standard $D$-dimensional Brownian motion.
Bacry et al. (2013a, b): Hawkes Model for Mid-Price

Central issue of financial econometrics: Describe price fluctuations at finest time scales $\rightarrow$ improve volatility and covariance estimations.

However, at very small scales, it is known that there is an additional noise superimposed to the standard diffusion, called microstructure noise.

Thus for small scales (i.e. $\tau \rightarrow 0$), there is a strong increase in the quadratic variation of the mid-price, given by the signature plot:

$$C(\tau) = \frac{1}{T} \frac{T}{\tau} \sum_{i=0}^{T/\tau-1} [P(i+1)\tau - P_i\tau]^2$$

Another phenomenon is the Epps Effect, namely that the covariation between pairs of assets vanishes for $\tau \rightarrow 0$. 
Bacry et al. (2013a,b): Hawkes Model for Mid-Price

In their work, Bacry et al. propose a model that directly accounts for the discrete nature of price variations by describing the tick-by-tick variation of the mid price using a Hawkes process model. They set the mid-price $P_t$ at time $t > 0$ as

$$P_t = P_0 + N_t^1 - N_t^2$$  \hspace{1cm} (4)

where the couple $(N_t^1, N_t^2)$ is a 2-dimensional Hawkes model corresponding to the arrival times of upward and downward price changes respectively.
Bacry et al. (2013a, b): Hawkes Model for Mid-Price

As a first approximation, they assume the dynamics of upward and downward price movements to be identical and thus consider a matrix kernel

$$\Phi(t) = \begin{pmatrix} \phi(s)(t) & \phi(c)(t) \\ \phi(c)(t) & \phi(s)(t) \end{pmatrix}$$

with equal diagonal terms $\phi(s)(t)$ describing self-excitation and anti-diagonal terms $\phi(c)(t)$ describing cross-excitation of the exponential form $\phi^{s/c}(t) = \alpha^{s/c} \beta^{s/c} e^{-\beta^{s/c} t} 1_{t>0}$. 
Bacry et al. (2013a,b): Hawkes Model for Mid-Price

As it is known that the price at microstructure level is displaying mean-reverting behaviour, they consider a purely mean-reverting scenario, that is $\phi^{(s)}(t) = 0$ and $\phi^{(c)}(t)$ exponential as above. They derive a closed-form expression for the signature plot and also show that the model is able to replicate the scale behaviour of the signature plot of Euro-Bund and Euro-Bobl future data (where they use MLE or GMM for parameter estimation).
Bacry et al. (2013a,b): Hawkes Model for Mid-Price

Figure 7: Reproducing the mid-price behavior at the microstructural level using Bacry et al. 2-dimensional Hawkes model. The plots represent the Euro-Bund empirical signature plot as compared to its fit within the model of Bacry et al.
Bacry et al. (2013a,b): Hawkes Model for Mid-Price

As an extension of their model, they consider a 4-dimensional Hawkes model to describe the joint mid-price dynamics of a pair of assets and reproduce the Epps Effect. As they had previously shown that the empirical covariation of a multivariate Hawkes process converges towards its expected value which can be expressed in terms of the Hawkes covariance matrix $c(t)$ as given by (Proposition 2) earlier in the paper.

Using this result allows to provide analytical expressions for the signature plot, lead-lag behavior and the Epps effect in terms of Hawkes matrix kernel $\Phi$. 
Bacry et al. (2013a,b): Hawkes Model for Mid-Price
Recall that Da Fonseca and Zaatour (2014) considered the opposite, purely trend-following scenario, that is $\phi^c(t) = 0$ and $\phi^s(t)$ exponential, and compared this with the purely mean-reverting scenario when used for daily volatility estimation. They showed that the mean-reverting (resp. trend-following) scenario underestimates (resp. overestimates) the volatility and concluded that a complete model with both cross- and self-terms is more realistic and should lead to better volatility estimation (for more details see seminar presentation on 13 June 2018 about Da Fonseca/Zaatour. Hawkes process: Fast calibration, application to trade clustering, and diffusive limit.).

The need for both diagonal and anti-diagonal terms was confirmed by non-parametric estimations performed by Bacry et al. (2012),(2014).
Jaisson and Rosenbaum (2014): Framework for power-law kernels

As shown in many works and discussed previously in this paper, empirical evidence suggests power-law kernels should be used instead of exponential kernels.

Thus Jaisson and Rosenbaum (2014) develop an alternative diffusive framework described in Section 2.3.6. In particular, they take a sequence of rescaled univariate Hawkes processes

\[ Z_u(T) = \frac{1 - a_T}{T} N_{uT}(T) \]  

indexed by a time-scale parameter \( T \) and where \( a_T \in [0;1] \) and prove convergence (for \( T \to \infty \)) towards an integrated Cox-Ingersoll-Ross process under some general conditions on the Hawkes kernel \( \Phi \).
**Jaisson and Rosenbaum (2014): Framework for power-law kernels**

They use their framework to build a version of the 2-dimensional **Hawkes model** (introduced in (4)) that converges at large scales towards the **Heston price model** which displays volatility clustering.

Their work is very important as it can be seen as the first step towards an "across scales" unified model that would fit both **microstructure facts** of price (i.e., point process with strong mean reversion) and "diffusive" facts (volatility clustering and multifractality).
Zheng et al. (2014): Hawkes Model for Coupled Dynamics of Bid-/Ask-Price

As a generalization of the model introduced in (4), Zheng et al. consider a model for the coupled dynamics of best bid and ask prices. They code each best price using the model in (4) leading to a 4-dimensional price model.

In order to ensure that the ask price must lie strictly above the bid price, they introduce a spread point process whose dynamics is coupled with the dynamics of both best prices. It measures the distance (in ticks) between the ask price and the bid price: the spread is increased (resp. decreased) by 1 each time either the ask (resp. bid) component jumps upward (resp. downward) or the bid (resp. ask) component jumps downward (resp. upward).
Zheng et al. (2014): Hawkes Model for Coupled Dynamics of Bid-/Ask-Price

They introduce a non-linear term which sets the intensities of the downward (resp. upward) jumps of the ask (resp. bid) price to 0 as soon as the spread process is equal to 1.

They develop a whole new rigorous framework for this constrained, non-linear Hawkes model in which they were able to establish several properties (including a diffusive limits).

They perform maximum likelihood estimation on real data (using exponential kernels) and show that they were able to reproduce rather well the signature plot.
Fauth and Tudor (2012): Marked Multivariate Hawkes Model

Fauth and Tudor describe bid and ask prices of an asset (or a couple of assets) within the framework of marked multivariate Hawkes model.

Motivated by the empirical observation of FX rates data that, as the transaction volumes increase, the inter-trade durations decrease, they consider an exponential Hawkes kernel with a multiplicative mark corresponding to a power-law function of the volumes. Their model describes the events corresponding to an increase/decrease of the bid/ask as a four dimensional Hawkes process marked by transaction volumes. They calibrate the model on FX rates data by a MLE approach and show that their model is consistent with empirical data by reproducing the signature plots of the considered assets and the behavior of the high-frequency pair correlation function (Epps effect).
References

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References

The End

Thank You!

Q&A time!