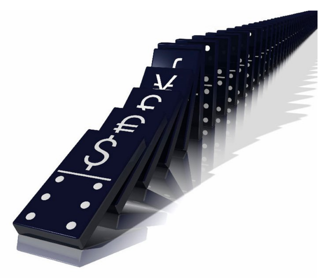
Semi-Markov Model for the Price Dynamics in Limit Order Markets



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Abstract

We introduce a semi-Markov model for the price dynamics in the limit order markets. This model is based on Markov renewal process. Our results generalize the previous results in the literature for the price dynamics in a Markovian limit order markets, and our assumptions are much milder than in the existing literature. Also, the numerical results show justification for using semi-Markov model, and for considering more general model than exponential (as in most existing literature) for the order arrival times, such as Weibull or Gamma. Our data is based on five stocks Amazon, Apple, Google, Intel and Microsoft on 21st 2012and was taken from the website: https://lobster.wiwi.hu-berlin.de/info/DataSamples.php.

1. Introduction: Quick Recap of the Model Presented in [1]

Let q_t^a and q_t^b the respective sizes of the ask and bid queues at time t (these are integer valued). Market buy (resp., sell) orders arrive at independent, exponential times with rate $\mu;$ limit buy (resp., sell) orders arrive at independent, exponential rate $\lambda;$ cancellations occur at independent, exponential times with rate $\theta;$ these events are mutually independent and orders are equal in size (in fact 1). Values T_k^a (resp., T_k^b), the durations between two consecutive queue changes at the ask (resp. the bid), and V_k^a (resp., V_k^b), the size of the associated change in queue size, satisfy the following conditions: T_k^a are independent, exponentially distributed with intensity $\lambda+\mu+\theta;$ T_k^b are independent, exponentially distributed with intensity $\lambda+\mu+\theta;$ $P[V_k^a=1]=\frac{\lambda}{\lambda+\mu+\theta}$, $P[V_k^a=-1]=\frac{\mu+\theta}{\lambda+\mu+\theta};$ $P[V_k^b=1]=\frac{\lambda}{\lambda+\mu+\theta}$, $P[V_k^b=-1]=\frac{\mu+\theta}{\lambda+\mu+\theta};$ all the latter sequences are independent from each other. The bid and ask queues are assumed to be independent. Only the current bid and ask queues are modeled.

2. The Semi-Markov Model of Limit Order Book Dynamics I: $T_k^a, T_k^b, V_k^a, V_k^b$

We generalize the latter assumptions on $T_k^a, T_k^b, V_k^a, V_k^b$: we will assume that between two price changes: (V_k^a, T_k^a) and (V_k^b, T_k^b) are two independent Markov renewal processes of respective kernels Q^a, Q^b (see below) The state space of the embedded Markov chains $\{V_k^a\}$, $\{V_k^b\}$ is $\{-1,1\}$. V_0^a, V_0^b are independent random variables with distributions v_0^a, v_0^b (on the space $\{-1,1\}$). The Markov renewal processes driving distinct intervals between prices changes are independent. In particular, the latter assumptions imply that between two given price changes: V_{k+1}^a can depend on the previous queue change V_k^a: $\{V_k^a\}$ is a Markov chain with arbitrary transition probability matrix (the same observation goes for V_k^b). $\{T_k^a\}$ can have arbitrary distributions. Further, they are not strictly independent anymore but they are independent conditionally on the Markov chain $\{V_k^a\}$ (the same observation goes for T_k^b). The bid and ask queues are still assumed to be independent.

3. The Semi-Markov Model of Limit Order Book Dynamics II

Let s_t be the price process, $X_n : -s_{T_n} - s_{T_n-}$ be the price increments, $\tau_n := T_n - T_{n-1}$, then (X_n, τ_n) is a Markov renewal process, and $s_t = \sum_{k=1}^{N_t} X_k$, $N_t := \sup\{n : T_n \leq t\}$.

$P[V_{k+1}^a = j, T_{k+1}^a \leq t | V_k^a] = Q^a(V_k^a, j, t), \;\; j \in \{-1,1\}$

$P[V_{k+1}^b = j, T_{k+1}^b \leq t | V_k^b] = Q^b(V_k^b, j, t), \;\; j \in \{-1,1\}$

We use the following notations for the ask (for the bid, they are defined similarly): $P^a(i,j) := P[V_{k+1}^a = j | V_k^a = i], \;\; i,j \in \{-1,1\}$

$F^a(i,t) := P[T_{k+1}^a \leq t | V_k^a = i], \;\; i \in \{-1,1\}$

$H^a(i,j,t) := P[T_{k+1}^a \leq t | V_k^a = i, V_{k+1}^a = j], \;\; i,j \in \{-1,1\}$

4. Main Probabilistic Results

Distribution of Duration Until Next Price Move; Probability of Price Increase; Characterization of Markov Renewal Process (X_n, τ_n); Diffusion Limit of the Semi-Markov Price Process (based on [2]).

5. Numerical Results: Real Data and Motivation

In [1], it is assumed that the queue changes V_k^b, V_k^a do not depend on their previous values V_{k-1}^b, V_{k-1}^a. We challenge this assumption by estimating and comparing the probabilities $P(-1,1)$ Vs. $P(1,1)$ on the one side and $P(-1,-1)$ Vs. $P(1,-1)$ on the other side to check whether or not they are approximately equal to each other, for both the ask and the bid. We also give - for both the bid and ask - the estimated probabilities $P[V_k = 1]$, $P[V_k = -1]$ that we call respectively $P(1)$, $P(-1)$, to check whether or not they are approximately equal to 1/2 as in [1]. The results below correspond to the 5 stocks Amazon, Apple, Google, Intel, Microsoft on June 21^{st} 2012 (The data was taken from the webpage: https://lobster.wiwi.hu-berlin.de/info/DataSamples.php). The probabilities are estimated using the strong law of large numbers.

6. Numerical Results: Tables

	Amazon		Apple		Google		Intel		Microsoft	
	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
Avg time btw. orders (ms)	910	873	464	425	1123	1126	116	133	130	113
Avg nb. of stocks per order	100	82	90	82	84	71	502	463	587	565

Average time between orders (ms) & Average number of stocks per order. June 21^{st} 2012.

	Amazon		Apple		Google		Intel		Microsoft	
	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
$P(1,1)$	0.48	0.57	0.50	0.55	0.48	0.53	0.55	0.61	0.63	0.60
$P(-1,1)$	0.46	0.42	0.40	0.42	0.46	0.49	0.44	0.40	0.36	0.41
$P(-1,-1)$	0.54	0.58	0.60	0.58	0.54	0.51	0.56	0.60	0.64	0.59
$P(1,-1)$	0.52	0.43	0.50	0.45	0.52	0.47	0.45	0.39	0.37	0.40
$P(1)$	0.47	0.497	0.44	0.48	0.47	0.51	0.495	0.505	0.49	0.508
$P(-1)$	0.53	0.503	0.56	0.52	0.53	0.49	0.505	0.495	0.51	0.492

Estimated transition probabilities of the Markov Chains V_k^b, V_k^a. June 21^{st} 2012.

7. Numerical Results: Findings and Justification I

We notice that the probabilities $P(-1,1)$, $P(1,1)$ can be significantly different from each other - and similarly for the probabilities $P(-1,-1)$, $P(1,-1)$ - which justifies the use of a Markov Chain structure for the random variables $\{V_k^b\}, \{V_k^a\}$. This phenomenon is particularly visible for example on Microsoft (Bid+Ask), Intel (Bid+Ask), Apple (Bid+Ask) or Amazon Ask. Further, regarding the comparison of $P(1,1)$ and $P(-1,-1)$, it turns out that they are often very smilar, except in the cases Amazon Bid, Apple Bid, Google Bid.

8. Numerical Results: Findings and Justification II

The second assumption of [1] that we would like to challenge is the assumed exponential distribution of the order arrival times T_k^a, T_k^b. To this end, on the same data set as used to estimate the transition probabilities $P^a(i,j)$, $P^b(i,j)$, we calibrate the empirical c.d.f.'s $H^a(i,j,dt)$, $H^b(i,j,dt)$ to the Gamma and Weibull distributions (which are generalizations of the exponential distribution). We recall that the p.d.f.'s of these distributions are given by: $f_{Gamma}(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}} 1_{x>0}, f_{Weibull}(x) = \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1} e^{-\left(\frac{x}{\theta}\right)^k} 1_{x>0}$. Here, $k>0$ and $\theta>0$ represent respectively the shape and the scale parameter. The variable k is dimensionless, whereas θ will be expressed in ms^{-1}. We perform a maximum likelihood estimation of the Weibull and Gamma parameters for each one of the empirical distributions $H^a(i,j,\cdot)$, $H^b(i,j,\cdot)$ (together with a 95% confidence interval for the parameters). As we can see on the tables below, the shape parameter k is always significantly different than 1 (~ 0.1 to 0.3), which indicates that the exponential distribution is not rich enough to fit our observations. To illustrate this, we present below the empirical c.d.f. of $H(1,-1)$ in the case of Google Bid, and we see that Gamma and Weibull allow to fit the empirical c.d.f. in a much better way than Exponential.

9. Numerical Example: Figure ($H(1,-1)$-Google Bid-June21st 2012)

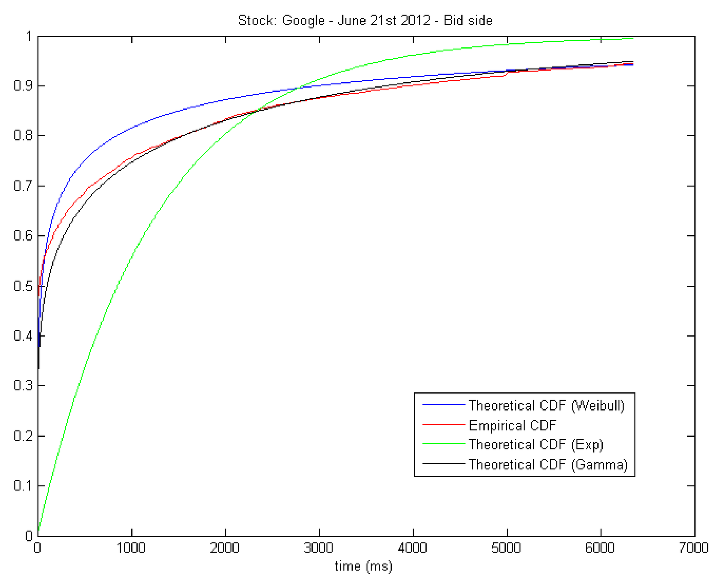

Figure 1: $H(1,-1)$*-Google Bid-June21st 2012*

10. Numerical Results: Calibration (Google Bid and Ask, June 21st 2012)

Google Bid	$H(1,1)$	$H(1,-1)$	$H(-1,-1)$	$H(-1,1)$
Weibull θ	113.9	158.5	67.9	56.8
	(102.8-126.2)	(143.4-175.3)	(60.6-76.0)	(50.5-63.8)
Weibull k	0.276	0.284	0.261	0.246
	(0.270-0.282)	(0.278-0.290)	(0.255-0.266)	(0.241-0.251)
Gamma θ	6720	6647	6381	7025
	(6263-7210)	(6204-7122)	(5913-6886)	(6517-7571)
Gamma k	0.174	0.185	0.160	0.151
	(0.169-0.179)	(0.180-0.191)	(0.155-0.165)	(0.147-0.156)

Google Bid: Fitted Weibull and Gamma parameters. 95 % confidence intervals in brackets. June 21^{st} 2012.

Google Ask	$H(1,1)$	$H(1,-1)$	$H(-1,-1)$	$H(-1,1)$
Weibull θ	196.7	271.6	38.1	57.0
	(180.6-214.2)	(248.5-296.8)	(33.8-43.0)	(51.3-63.3)
Weibull k	0.290	0.310	0.258	0.263
	(0.285-0.295)	(0.303-0.316)	(0.253-0.264)	(0.258-0.268)
Gamma θ	6081	6571	4304	4698
	(5734-6450)	(6165-7003)	(3971-4664)	(4380-5040)
Gamma k	0.195	0.209	0.156	0.164
	(0.190-0.200)	(0.203-0.215)	(0.151-0.161)	(0.159-0.168)

Google Ask: Fitted Weibull and Gamma parameters. 95 % confidence intervals in brackets. June 21^{st} 2012.

11. References

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