Jump Processes

Generalizing Stochastic Integrals with Jumps

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Overview

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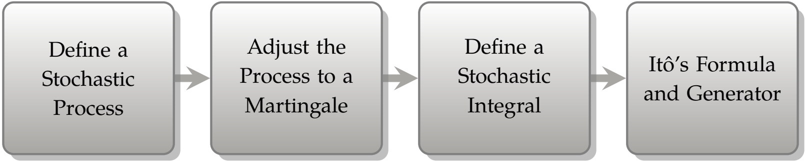
General Method

General Method



Poisson Processes

Definition: Poisson Process

A **Poisson process** $N = \{N_t\}_{0 \le t \le T} \in \mathbb{Z}^+$, with intensity λ, is a stochastic process with the following properties

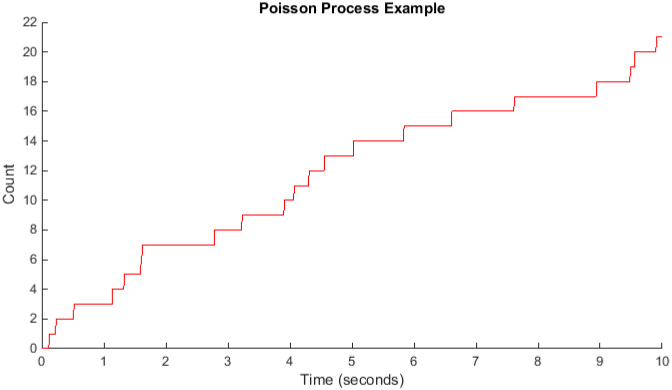
(i) $N_0 = 0$ almost surely,

(ii) $N_t - N_0$ has a Poisson distribution with parameter λt.

(iii) N has independent increments, so $(s, t) \cap (v, u) = \emptyset$ implies $N_t - N_s$ is independent of $N_v - N_u$.

(iv) N has stationary increments, so $N_{s+t} - N_s$ follows the same distribution as N_t for all $s, t > 0$.

Poisson Process Example



Properties

(i) $\mathbb{E}[N_t] = \lambda t$

(ii) $\text{Var}[N_t] = \lambda t$

(iii) The time between jumps of N are independent and follow an exponential distribution.

Compensated Poisson Process

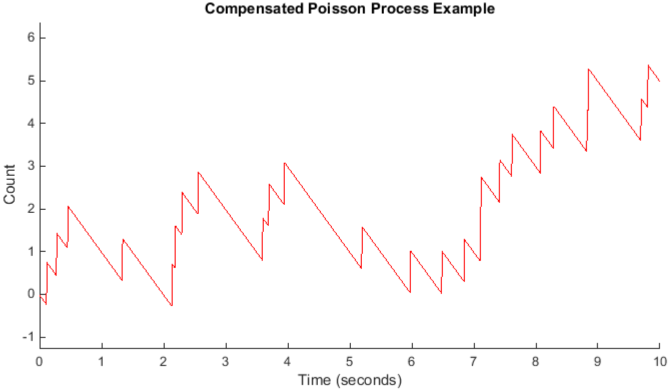
Proposition: Compensated Poisson Process

The compensated Poisson process $\widehat{N} = \left\{\widehat{N}_t\right\}_{0\leq t\leq T}$ where $\widehat{N}_t = N_t - \lambda t$ is a martingale with respect to it's generated filtration $\mathscr{F}$.

Proof.

$$\begin{aligned}\mathbb{E}\left[N_{t+s} - \lambda(t+s)|\mathscr{F}_t\right] &= \mathbb{E}\left[N_{t+s} - N_s + N_s - \lambda(t+s)|\mathscr{F}_t\right] \\ &= \mathbb{E}\left[N_t - \lambda t + N_s - \lambda s|\mathscr{F}_t\right] \\ &= N_t - \lambda t\end{aligned}$$

Compensated Poisson Process Example



Definition: Stochastic Integral with respect to a Compensated Poisson Process

Let g be an $\mathscr{F}_t$-adapted process, where $\mathscr{F}_t$ is the natural filtration generated by Poisson process N. Define stochastic integral $Y = \{Y_t\}_{0 \le t \le T}$ of g with respect to $\widehat{N}$ as

$$Y_t = \int_0^t g_{s^-} d\widehat{N}_s = \sum_{k=1}^{N_t} g_{\tau_k^-} - \int_0^t g_s \lambda ds$$

where $\{\tau_1, \tau_2, \ldots\}$ is the collection of times when N jumps.

Theorem: Itô's Formula for Poisson Processes

Suppose Y is the stochastic integral given previously. Let $Z = \{Z_t\}_{0 \le t \le T}$ with $Z_t = f(t, Y_t)$ for some function f, once differentiable in t. Then

$$\begin{aligned} dZ_t &= (\partial_t f(t,Y_t) - \lambda g_t \partial_y f(t,Y_t))dt \\ &\quad + \left[f(t, Y_{t^-} + g_{t^-}) - f(t, Y_{t^-}) \right] dN_t \\ &= \{\partial_t f(t,Y_t) + \lambda([f(t,Y_{t^-} + g_{t^-}) - f(t,Y_{t^-})] \\ &\quad - g_t \partial_y f(t,Y_t))\} dt \\ &\quad + [f(t,Y_{t^-} + g_{t^-}) - f(t,Y_{t^-})] d\widehat{N}_t \end{aligned}$$

Infinitesimal Generator

Recall that the generator $\mathscr{L}_t$ of a process X_t acts on twice differentiable functions f as

$$\mathscr{L}_t f(x) = \lim_{h \downarrow 0} \frac{\mathbb{E}[f(X_{t+h}|X_t = x)] - f(x)}{h}$$

which is a generalization of a derivative of a function which can be applied to stochastic processes.

The generator of stochastic integral Y from a Poisson process acts as

$$\mathscr{L}_t^Y f(y) = \lambda \left([f(y + g_t) - f(y)] - g_t \partial_y f(y) \right)$$

Diffusion and Single Jumps

Sum of Stochastic Integrals

Using the framework developed previously for Stochastic Integrals with respect to diffusion and jumps, we sum these two as follows.

$$Y_t = \int_0^t f_s ds + \int_0^t g_s dW_s + \int_0^t h_{s^-} d\widehat{N}_s,$$

where f, g, h are $\mathscr{F}_t$ adapted processes, and filtration $\mathscr{F}$ is the natural one generated by both the Brownian motion W and Poisson process N, which are mutually independent.

Theorem: Itô's Formula for Single Jumps and Diffusion

Suppose Y is the stochastic integral given previously. Let $Z = \{Z_t\}_{0\le t\le T}$ with $Z_t = l(t, Y_t)$ for some function l, once differentiable in t and twice differentiable in y. Then

$$
\begin{aligned}
dZ_t &= (\partial_t + f_t\partial_y + \tfrac{1}{2}g_t^2\partial_{yy} - \lambda h_t\partial_y)l(t, Y_t))dt \\
&\quad + g_t\partial_y l(t, Y_t)dW_t + [l(t, Y_{t^-} + h_{t^-}) - l(t, Y_{t^-})]\, dN_t \\
&= \left\{\left(\partial_t + f_t\partial_y + \tfrac{1}{2}g_t^2\partial_{yy}\right) l(t, Y_t)\right. \\
&\quad \left. +\lambda([l(t, Y_{t^-} + h_{t^-}) - l(t, Y_{t^-})] - h_t\partial_y l(t, Y_t))\right\} dt \\
&\quad + g_t\partial_y l(t, Y_t)dW_t + [l(t, Y_{t^-} + h_{t^-}) - l(t, Y_{t^-})]d\widehat{N}_t
\end{aligned}
$$

Generator

The generator of Y acts as

$$\mathscr{L}_t^Y l(y) = f_t \partial_y l(y) + \frac{1}{2} g_t^2 \partial_{yy} l(y) + \lambda \left([l(y + h_t) - l(y)] - h_t \partial_y l(y) \right)$$

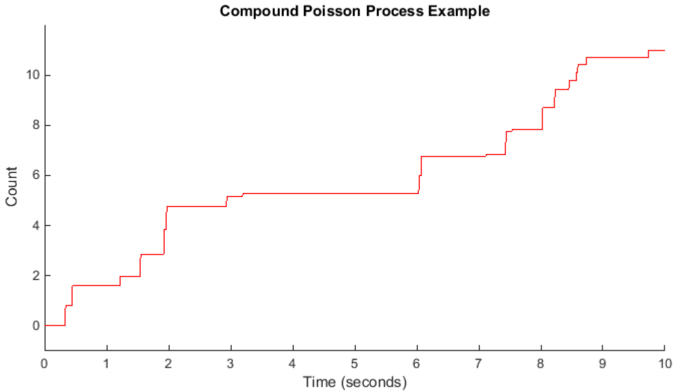
Compound Poisson Process

Definition: Compound Poisson Processes

Let N be a Poisson process with intensity λ and $\{\varepsilon_1, \varepsilon_2, \ldots\}$ be a set of independent identically distributed random variables with distribution function F and $\mathbb{E}[\varepsilon] < +\infty$. A **compound Poisson process** $J = \{J_t\}_{0 \le t \le T}$ is given by

$$J_t = \sum_{k=1}^{N_t} \varepsilon_k, \quad t \ge 0$$

Compound Poisson Process Example



Compound Poisson Process: Properties

Properties

(i) $\mathbb{E}[J_t] = \lambda t \mathbb{E}[\varepsilon]$

(ii) $\mathrm{Var}[J_t] = \lambda t \mathbb{E}[\varepsilon^2]$

(iii) As with the standard Poisson process, the inter-arrival times are independent and exponentially distributed.

Compensated Compound Poisson Process

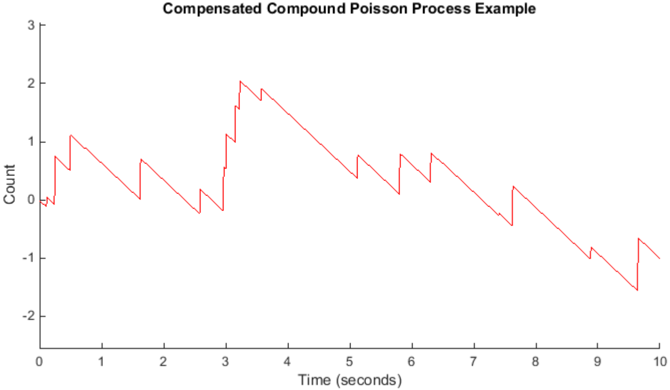
Proposition:

The compensated compound Poisson process $\widehat{J} = \left\{\widehat{J}_t\right\}_{0\le t\le T}$ where $\widehat{J}_t = J_t - \mathbb{E}[\varepsilon]\lambda t$ is a martingale.

Proof.

$$\begin{aligned}
\mathbb{E}\left[\widehat{J}_{t+s}|\mathscr{F}_t\right] &= \mathbb{E}\left[\Sigma_{k=1}^{N_{t+s}}\varepsilon_k - \lambda(t+s)\mathbb{E}[\varepsilon]|\mathscr{F}_t\right] \\
&= \mathbb{E}\left[\Sigma_{k=1}^{N_t}\varepsilon_k + \Sigma_{k=N_t+1}^{N_{t+s}} - \lambda(t+s)\mathbb{E}[\varepsilon]|\mathscr{F}_t\right] \\
&= \Sigma_{k=1}^{N_t} - \lambda t\mathbb{E}[\varepsilon]
\end{aligned}$$

Compensated Compound Poisson Process



Corresponding Stochastic Integral

Let $\mathscr{F}$ be the natural filtration generated by $\widehat{J}$. We define the stochastic integral $Y = \{Y_t\}_{0\leq t\leq T}$ of an $\mathscr{F}$-adapted process g with respect to the compensated compound Poisson process $\widehat{J}$ as

$$Y_t = \int_0^t g_{s^-} d\widehat{J}_s = \sum_{s\leq t} g_{s^-} \Delta J_s - \int_0^t g_s \lambda \mathbb{E}[\varepsilon] ds$$

where $\Delta J_s = J_s - J_{s^-}$

Jump-Diffusion

Sum of Stochastic Integral

Let f, g, and h be $\mathscr{F}$-adapted stochastic processes where $\mathscr{F}$ is the natural filtration generated by an independent Brownian motion W and $\widehat{J}$. We define the stochastic integral Y as

$$Y_t = \int_0^t f_s ds + \int_0^t g_s dW_s + \int_0^t h_{s^-} d\widehat{J}_s$$

Theorem: Itô's Formula for Jump-Diffusion

Suppose Y is the stochastic integral given previously. Let $Z = \{Z_t\}_{0 \le t \le T}$ with $Z_t = l(t, Y_t)$ for some function l, once differentiable in t and twice differentiable in y. Then

$$
\begin{aligned}
dZ_t &= (\partial_t + f_t \partial_y + \tfrac{1}{2} g_t^2 \partial_{yy} - \lambda \mathbb{E}[\varepsilon] h_t \partial_y) l(t, Y_t)) dt \\
&\quad + g_t \partial_y l(t, Y_t) dW_t + \left[l(t, Y_{t^-} + \varepsilon_{N_t} h_{t^-}) - l(t, Y_{t^-}) \right] dN_t \\
&= \Big\{ \left(\partial_t + f_t \partial_y + \tfrac{1}{2} g_t^2 \partial_{yy} \right) l(t, Y_t) \\
&\quad + \lambda (\mathbb{E}[l(t, Y_{t^-} + h_{t^-}) - l(t, Y_{t^-})] - \mathbb{E}[\varepsilon_t] h_t \partial_y l(t, Y_t)) \Big\} dt \\
&\quad + g_t \partial_y l(t, Y_t) dW_t + [l(t, Y_{t^-} + \varepsilon_{N_t} h_{t^-}) - l(t, Y_{t^-})] d\hat{N}_t
\end{aligned}
$$

The generator of Y acts as

$$\begin{aligned}\mathscr{L}_t^Y l(y) = {} & f_t \partial_y l(y) + \tfrac{1}{2} g_t^2 \partial_{yy} l(y) \\ & + \lambda \left(\mathbb{E}[l(t, y + \varepsilon h_t) - l(t, y)] - \mathbb{E}[\varepsilon] h_t \partial_y l(t, Y_t) \right)\end{aligned}$$

References

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Thank you!