Integration: Reduction Formulas
Any positive integer power of \( \sin x \) can be integrated by using a reduction formula.

Example
Prove that for any integer \( n \geq 2 \),
\[
\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.
\]

Solution. We use integration by parts.

- Let \( u = \sin^{n-1} x \) and \( dv = \sin x \, dx \). Then, \( du = (n-1) \sin^{n-2} x \cos x \, dx \) and we can use \( v = -\cos x \).
- So,
\[
\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx \\
= \sin^{n-1} x (-\cos x) - \int (-\cos x)(n-1) \sin^{n-2} x \cos x \, dx \\
= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \\
= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx \\
= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx.
\]
- Re-arranging, we get \( n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx \).
- Dividing both sides by \( n \), we get \( \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \).
Example

Use the reduction formula to find the integrals of $\sin^2 x$, $\sin^3 x$, $\sin^4 x$.

Solution. We recall the reduction formula proved above. For $n \geq 2$,

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$ 

- With $n = 2$,
  $$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int 1 \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + C.$$ 

- With $n = 3$,
  $$\int \sin^3 x \, dx = -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C.$$ 

- With $n = 4$,
  $$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left(-\frac{1}{2} \sin x \cos x + \frac{1}{2} x \right) + C = -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C.$$
Example

Express $\sin^3 x \cos^4 x$ as a sum of constant multiples of $\sin x$. Hence, or otherwise, find the integral of $\sin^3 x \cos^4 x$.

**Solution.** Since $\cos^2 x = 1 - \sin^2 x$,

- $\cos^6 x = (\cos^2 x)^3 = (1 - \sin^2 x)^3 = 1 - 2 \sin^2 x + \sin^4 x$.
- So, $\sin^3 x \cos^6 x = \sin^3 (1 - 2 \sin^2 x + \sin^4 x) = \sin^3 x - 2 \sin^5 x + \sin^7 x$.

For the sake of simplicity, we will denote $\int \sin^n x \, dx$ by $I_n$.

- The reduction formula reads $I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$.
- By using the reduction formula, we get

\[
\begin{align*}
I_1 &= -\cos x + C \\
I_3 &= -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C \\
I_5 &= -\frac{1}{5} \sin^4 x \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + C \\
I_7 &= -\frac{1}{7} \sin^6 x \cos x - \frac{6}{35} \sin^4 x \cos x - \frac{16}{35} \sin^2 x \cos x - \frac{16}{35} \cos x + C
\end{align*}
\]

- Integrating both sides of the identity

$$\sin^3 x \cos^4 x = \sin^3 x - 2 \sin^5 x + \sin^7 x,$$

we get

$$\int \sin^3 x \cos^4 x \, dx = I_3 - 2I_5 + I_7 = -\frac{1}{7} \sin^6 x \cos x + \frac{8}{35} \sin^4 x \cos x - \frac{1}{35} \sin^2 x \cos x - \frac{2}{35} \cos x + C.$$
Remarks.

- One can use integration by parts to derive a reduction formula for integrals of powers of cosine:

\[ \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx. \]

One can integrate all positive integer powers of \( \cos x \).

- By using the identity \( \sin^2 x = 1 - \cos^2 x \), one can express \( \sin^m x \cos^n x \) as a sum of constant multiples of powers of \( \cos x \) if \( m \) is even.

- In view of this and our previous examples, we can integrate \( \sin^m x \cos^n x \) as long as \( m \) and/or \( n \) is even.

- If both \( m \) and \( n \) are odd, then we need a different approach. It turns out that either of the substitutions \( u = \sin x \) and \( u = \cos x \) will work.

Example

- If we use \( u = \sin x \), then \( du = \cos x \, dx \), and since \( \cos^4 x = (1 - \sin^2 x)^2 = (1 - u^2)^2 \),

\[ \int \sin^3 x \cos^5 x \, dx = \int \sin^3 \cos^4 x \, dx = \int u^3 (1 - u^2)^2 \, du = \int (u^3 - 2u^5 + u^7) \, du = \ldots \]

- If we use \( u = \cos x \), then \( du = -\sin x \, dx \), and since \( \cos^2 x = 1 - \sin^2 x = 1 - u^2 \),

\[ \int \sin^3 x \cos^5 x \, dx = \int (-\sin^2 x) \cos^5 x (-\sin x) \, dx = \int -(1 - u^2)u^5 \, du = \int (-u^5 + u^7) \, du = \ldots \]

- Convince yourself this: If \( m \) is odd, then \( u = \cos x \) will work (even if \( n \) is even). If \( n \) is odd, then \( u = \sin x \) will work (even if \( m \) is even).

- All functions of the form \( \sin^m x \cos^n x \) can be integrated.
Let’s consider integrals of the form \( \int \sec^m x \tan^n x \, dx \).

**Example**

Prove that for any integer \( n \geq 2 \),

\[
\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx.
\]

Use this and the fact that \( \int \tan x \, dx = \ln |\sec x| + C \) to find the integrals of \( \tan^2 x, \tan^3 x, \tan^4 x \) and \( \tan^5 x \).

- Using the identity \( \tan^2 x = \sec^2 x - 1 \),
  \[
  \int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx = \int \tan^{n-2} x (\sec^2 x - 1) \, dx = \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx.
  \]

- Using the substitution \( u = \tan x \), we have \( du = \sec^2 x \, dx \). So,
  \[
  \int \tan^{n-2} x \sec^2 x \, dx = \int u^{n-2} du = \frac{1}{n-1} u^{n-1} + C = \frac{1}{n-1} \tan^{n-1} x + C.
  \]
  The reduction formula is proved.

- With \( n = 2 \), we have \( \int \tan^2 x \, dx = \tan x - \int 1 \, dx = \tan x - x + C \).

- With \( n = 3 \), we have \( \int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx = \frac{1}{2} \tan^2 x - \ln |\sec x| + C \).

- With \( n = 4 \), we have \( \int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \int \tan^2 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C \).

- With \( n = 5 \), we have \( \int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \ln |\sec x| + C \).
Remarks.

- One can integrate all positive integer powers of $\tan x$.
- One can derive a reduction formula for $\sec x$ by integration by parts.
- Using the reduction formula and the fact $\int \sec x \, dx = \ln | \sec x + \tan x | + C$, we can integrate all positive integer powers of $\sec x$.
- Similar strategies used for $\sin^m x \cos^n x$ can be formulated to integrate all functions of the form $\sec^m x \tan^n x$. 
Further Examples and Exercises

1. Prove the reduction formula for integrals of powers of \( \cos x \):
\[
\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.
\]
Use it to find the integrals of \( \cos^2 x, \cos^3 x, \cos^4 x, \cos^5 x, \cos^6 x \).

2. Express \( \sin^4 x \cos^6 x \) as a sum of constant multiples of \( \cos x \). Hence, or otherwise, find the integral of \( \sin^4 x \cos^6 x \).

3. Use integration by parts to find a reduction formula for integrals of positive integer powers of \( \sec x \).

4. Find the following integrals.
\[
\int \sin^5 x \cos^2 x \, dx, \quad \int \cos^4 x \sin^2 x \, dx, \quad \int \sin^4 x \cos^4 x \, dx
\]

5. Find the following integrals.
\[
\int \tan^3 x \, dx, \quad \int \sec^5 x \, dx, \quad \int \tan^4 x \, dx, \quad \int \sec^3 x \tan^2 x \, dx, \quad \int \sec^4 x \tan^3 x \, dx
\]