Pricing Variance Swaps for Stochastic Volatilities with Delay and Jumps

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Outline

• Stochastic Volatilities (SV) with Delay
• Multi-Factor SV with Delay
• SV with Delay and Jumps
• Swaps
• Numerical Examples
• Conclusions
Volatility

• **Volatility** is the standard deviation of the change in value of a financial instrument with specific time horizon

• It is often used to quantify the **risk** of the instrument over that time period

• **The higher volatility, the riskier the security**
Types of Volatilities

• **Historical V**: standard deviation (uses historical (daily, weekly, monthly, quarterly, yearly)) price data to empirically measure the volatility of a market or instrument in the past

• **Implied V**: volatility implied by the market price of the option based on an option pricing model (smile and skew-varying volatility by strike)
Volatility Smile

• The models by Black & Scholes (continuous-time (B,S)-security market, 1973) and Cox & Rubinstein (discrete-time (B,S)-security market (binomial tree), 1979) are unable to explain the negative skewness and leptokurticity (fat tail) commonly observed in the stock markets.

• The famous implied-volatility smile would not exist under their assumptions.
Coffee Options

Coffee Call Option

• CSCE May 2001 coffee call option implied volatilities as of March 12, 2001
Implied Volatility: Volatility Smile

- Graph indicates implied volatilities at various strikes for the May 2001 calls based upon their March 12, 2001 settlement prices. The pattern of implied volatilities form a "smile" shape, which is called a volatility smile.
Implied Volatility: Volatility Skew

- Most derivatives markets exhibit persistent patterns of volatilities varying by strike. In some markets, those patterns form a smile. In others, such as equity index options markets, it is more of a skewed curve. This has motivated the name volatility skew. In practice, either the term "volatility smile" or "volatility skew" (or simply skew) may be used to refer to the general phenomena of volatilities varying by strike.
Implied Volatility: Volatility Surface

• Another dimension to the problem of volatility skew is that of volatilities varying by expiration. This is illustrated for CSCE coffee options. It indicates what is known as a volatility surface.
Types of Volatilities II

- **Level-Dependent Volatility** (CEV or Firm Model) - function of the spot price alone
- **Local Volatility** - function of the spot price and time (Dupire formulae, 1994)
- **Stochastic Volatility**: volatility is not constant, but a stochastic process (explains smile and skew)
Two Approaches to Introduce SV

• One approach-to change the clock time \( t \) to a random time \( T(t) \) (change of time)

• Another approach-change constant volatility into a positive stochastic process

\[
\sigma W(t) \Rightarrow W(T(t))
\]

\[
\sigma = \sigma(t), \quad \int_0^t \sigma^2(s)ds < +\infty
\]
Stochastic Volatility: Some Models

- ARCH model (Engle (1982))

- **Discrete SV: GARCH model** (Bollerslev (1986))

- **Heston SV model** (1993)

- **Mean-Reverting SV model** (Wilmott, Haug, Javaheri (2000))

- **Elliott and Chan' Option Pricing with Stochastic Volatility Driven by a Fractional Ornstein-Ohlenbeck Process'.**

\[
\ln(S_t/S_{t-1}) = \sigma_t \xi_t, \quad \xi_t \approx i.i.d. N(0, 1) \\
\sigma_n^2 = \gamma V + \frac{\alpha}{t} \ln^2(S_{n-1}/S_{n-1-t}) + (1 - \alpha - \gamma)\sigma_{n-1}^2 \\
d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dw_t \\
d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t^2 dw_t
\]
SV with Delay

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t) \]

Kazmerchuk, Swishchuk & Wu (2002)
One-Factor SV with Delay

The underlying asset \( S(t) \) follows the process

\[
dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t)
\]

\[ S_t := S(t - \tau) \]

\[ S(t) = \phi(t), \quad t \in [-\tau, 0], \quad \tau > 0. \]

The asset volatility is defined as the solution of the following equation

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s)dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t).
\]

Why this equation?
From GARCH to SV with Delay

\[ \sigma^2_n = \gamma V + \alpha \ln^2(S_{n-1}/S_{n-2}) + (1 - \alpha - \gamma)\sigma^2_{n-1} \]

- discrete-time GARCH(1,1)

\[ \sigma^2_n = \gamma V + \frac{\alpha}{l} \ln^2(S_{n-1}/S_{n-1-l}) + (1 - \alpha - \gamma)\sigma^2_{n-1} \]

- discrete-time GARCH(1,1) \((l=1)\)

\[ \frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau} \ln^2 \left( \frac{S(t)}{S(t-\tau)} \right) - (\alpha + \gamma)\sigma^2(t) \]

- continuous-time GARCH (expectation of log-returns is zero)

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t) \]

- continuous-time GARCH (non-zero expectation of log-return)
**Comparison with GARCH (1,1)**

**GARCH (1,1)**

\[
\ln(S_n/S_{n-1}) = m + \sigma_n \xi_n, \quad \{\xi_n\} \sim \text{i.i.d. } N(0,1), \\
\sigma_n^2 = \gamma V + \alpha (\sigma_{n-1} \xi_{n-1})^2 + (1 - \alpha - \gamma)\sigma_{n-1}^2 \\
= \gamma V + \alpha (\ln(S_{n-1}/S_{n-2} - m)^2 + (1 - \alpha - \gamma)\sigma_{n-1}^2
\]

\[
\ln \frac{S(t)}{S(t-\tau)} = \int_{t-\tau}^{t} (r - \frac{1}{2}\sigma^2(u, S(u)))du + \int_{t-\tau}^{t} \sigma(u, S(u))dW(u)
\]

**Log-returns for S(t) (Ito formula)**

**Continuous-Time GARCH for SV with Delay**

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s)dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]
Main Features of 1-Factor SV with Delay

1) continuous-time analogue of discrete-time GARCH model

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t).
\]

2) Mean-reversion

\[
\frac{d\sigma^2_t}{dt} = (\alpha + \gamma)\left( \frac{\gamma}{\alpha + \gamma} V - \sigma^2_t \right) dt + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma_s dw_s \right]^2
\]

3) Market is complete (W is the same as for the stock price)

\[
dS(t) = \mu S(t) \, dt + \sigma(t, S_t) S_t \, dW(t)
\]

4) Incorporate the expectation of log-returns

\[
\ln \frac{S(t)}{S(t-\tau)} = \int_{t-\tau}^{t} \left( r - \frac{1}{2} \sigma^2(u, S(u)) \right) du + \int_{t-\tau}^{t} \sigma(u, S(u)) dW(u)
\]
Equation for the Expectation of Variance

Equation for the Variance

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]

Equation for the Expectation

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t)
\]

\[
v(t) := E_{\mathcal{P}^*} [\sigma^2(t, S_t)]
\]
Solution of the Equation for the Expectation of Variance (1FSV)

Equation to be solved

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t)
\]

Stationary solution

\[v(t) \equiv X = V + \frac{\alpha \tau (\mu - r)^2}{\gamma}\]

General solution

\[v(t) \approx X + Ce^{-\gamma t}\]

\[C = v(0) - X = \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma}\]
General Solution

Integro-differential equation with delay

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma) v(t)
\]

General solution

\[
v(t) = E_{P^*} [\sigma^2(t, S_t)] \\
\approx V + \frac{\alpha \tau (\mu - r)^2}{\gamma} + \left( \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \right) e^{-\gamma t}
\]
Multi-Factor SV Models

• One-Factor SV Models (*all above-mentioned*):  
  1) incorporate the leverage between returns and volatility and  
  2) reproduce the ‘skew’ of implied volatility

• However, it *fails to match either the high conditional kurtosis of returns* (*Chernov et. al. (2003)*) *or the full term structure of implied volatility surface* (*Cont & Tankov (2004)*)

• Adding jump components in returns and/or volatility process, or considering *multi-factor SV models* are two primary generalizations of one-factor SV models
Multi-Factor SV Models

- **Chernov et al. (2003):** used efficient method of moments to obtain comparable empirical-of-fit from affine jump-diffusion models & two-factor SV family models
- **Molina et al. (2003):** used a Markov Chain Monte Carlo method to find strong evidence of two-factor SV models with well-separated time scales in foreign exchange data
- **Cont & Tankov (2004):** found that jump-diffusion models have a fairly good fit to the implied volatility surface
- **Fouque et al. (2000):** found that two-factor SV models provide a better fit to the term structure of implied volatility than one-factor SV models by capturing the behaviour at short and long maturities
- **Swishchuk (2006):** introduced two-factor and three-factor SV models with delay (incorporating mean-reverting level as a random process (GBM, OU, Pilipovich or continuous-time GARCH(1,1) model))
Advantages and Disadvantages of Multi-Factor SV Models

• Multi-Factor SV models do not admit in general explicit solutions for option prices
• But have direct implications on hedges
• Comparison: class of **jump-diffusion models (Bates (1996))** enjoys **closed-form solutions** for option prices **but the complexity of hedging strategies increases due to jumps**
• There is no strong empirical evidence to judge the overwhelming position between jump-diffusion models and multi-factor SV models
Multi-Factor SV with Delay

One-Factor SV with Delay

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]

- Multi-Factor-Mean SV with Delay-mean-reversion level \( V \) is a stochastic process

\( V \rightarrow V(t) \)-stochastic process

- \( V(t) \) - geometric Brownian motion (GBM) (two-factor)
- \( V(t) \) - Ornstein-Uhlenbeck (UE) process (two-factor)
- \( V(t) \) - Pilipovich one-factor (two-factor)
- \( V(t) \) - Pilipovich two-factor process (three-factor)
2-Factor SV with Delay: GBM
Mean-Reversion (GBMMR)

\[
\begin{aligned}
\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 \\
&\quad - (\alpha + \gamma)\sigma^2(t, S_t), \\
\frac{dV_t}{V_t} &= \xi \, dt + \beta \, dW_1(t).
\end{aligned}
\]
2-Factor SV with Delay (GBMMR): Equation for the E and Solution

Equation for the E

\[
\frac{dv(t)}{dt} = \gamma V_0 e^{(\xi - \lambda \beta) t} + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) \, ds - (\alpha + \gamma) v(t).
\]

where \( v(t) := E_{P^*} \sigma^2(t, S_t) \).

Solution

\[
v(t) \approx X + C e^{-\gamma t} + (\xi - \lambda \beta) \gamma V_0 \times \left[ \frac{X}{\xi - \lambda \beta} \left( e^{(\xi - \lambda \beta) t} - 1 \right) + \frac{C}{\xi - \lambda \beta - \gamma} \left( e^{(\xi - \lambda \beta) t} - e^{\gamma t} \right) \right]
\]
2-Factor SV with Delay: OU Mean-Reversion (OUMR)

\[
\begin{aligned}
\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 \\
&\quad - (\alpha + \gamma)\sigma^2(t, S_t) \\
&\quad dV_t = \xi (L - V_t) dt + \beta dW_1(t).
\end{aligned}
\]
2-Factor SV with Delay (OUMR): Equation for the E and Solution

Equation for E

\[
\frac{dv(t)}{dt} = \gamma \left( e^{-\xi t} \left( V_0 - \left( L - \frac{\lambda \beta}{\xi} \right) \right) + \left( L - \frac{\lambda \beta}{\xi} \right) \right) \\
+ \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma) v(t)
\]

Solution

\[
v(t) \approx X + Ce^{-\gamma t} + \xi \gamma \left( V_0 - \left( L - \frac{\lambda \beta}{\xi} \right) \right) \\
\times \left[ \frac{X}{\xi} (e^{-\xi t} - 1) + \frac{C}{\xi + \gamma} (e^{-\xi t} - e^{\gamma t}) \right]
\]
2-Factor SV with Delay: Pilipovich 1-Factor Mean-Reversion (OFMR)

\[
\begin{align*}
\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) \, dW(s) \right]^2 \\
- (\alpha + \gamma)\sigma^2(t, S_t), \\
\end{align*}
\]

\[
dV_t = \xi(L - V_t) \, dt + \beta V_t \, dW_1(t).
\]
2-Factor SV with Delay (Pilipovich 1FMR): Equation for the E and Solution

Equation for E

\[
\frac{dv(t)}{dt} = \gamma \left( e^{-(\xi + \lambda\beta)t} \left( V_0 - L \frac{\xi}{\xi + \lambda\beta} \right) + L \frac{\xi}{\xi + \lambda\beta} \right) + \alpha \tau (\mu - r)^2 \\
+ \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) \, ds - (\alpha + \gamma)v(t)
\]

Solution

\[
v(t) \approx X + Ce^{-\gamma t} + \frac{\gamma \xi}{\xi + \lambda\beta} \left[ \left( X \left( \frac{V_0(\xi + \lambda\beta)}{\xi} - L \right) \right) \right. \\
\left. \times (1 - e^{-(\xi + \lambda\beta)t}) + XLt \\
+ \frac{C(\frac{V_0(\xi + \lambda\beta)}{\xi} - L)}{\xi + \lambda\beta + \gamma} + \frac{CL}{\gamma}(e^{\gamma t} - 1) \right].
\]
3-Factor SV with Delay: Pilipovich 2-Factor Mean-Reversion (2FMR)

\[
\begin{aligned}
\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) \, dW(s) \right]^2 \\
&\quad - (\alpha + \gamma)\sigma^2(t, S_t), \\
\end{aligned}
\]

\[
\begin{aligned}
dV_t &= \xi (L_t - V_t) \, dt + \beta V_t \, dW_1(t), \\
\end{aligned}
\]

\[
\begin{aligned}
\quad dL_t &= \beta_1 L_t \, dt + \eta L_t \, dW_2(t). \\
\end{aligned}
\]
3-Factor SV with Delay (Pilipovich 2FMR):
Equation for the E and Solution

Equation for E

\[
\frac{dv(t)}{dt} = \gamma (e^{-\xi \beta t} V_0 + \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \beta_1} L_0 (e^{\beta_1 - \lambda_1 \eta} t - e^{-\xi \beta t}))
+ \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma) v(t)
\]

Solution

\[
v(t) \approx X + C e^{-\gamma t} - (\xi + \lambda \beta) \gamma V_0 \left[ \frac{X}{\xi + \lambda \beta} (1 - e^{-(\xi + \lambda \beta) t}) \right.
+ \frac{C}{\xi + \lambda \beta + \gamma} (e^{\gamma t} - e^{-(\xi + \lambda \beta) t}) \left. \right]
+ L_0 \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \beta_1} \\
\times \left[ X (e^{(\beta_1 - \lambda_1 \eta) t} - e^{-(\xi + \lambda \beta) t}) + \frac{C (\beta_1 - \lambda_1 \eta)}{(\beta_1 - \lambda_1 \eta - \gamma)} \right.
\left. \times (e^{(\beta_1 - \lambda_1 \eta) t} - e^{\gamma t}) + \frac{C (\xi + \lambda \beta)}{(\xi + \lambda \beta + \gamma)} (e^{\gamma t} - e^{-(\xi + \lambda \beta) t}) \right]
\]
Main Features of All the Solutions for MFSVD

- 1) Contains solution of one-factor SV with Delay
- 2) Contains additional terms due to the new parameters (more factors - more parameters)
- Solution (MFSVD) = Solution (1FSVD) + Additional Terms (Due to the stochastic mean-reversion)
Variance Swaps

**Forward contract** - *an agreement to buy or sell something at a future date for a set price* (forward price)

**Variance swaps** are forward contract on future realized stock variance
Realized Continuous Variance

Realized (or Observed) Continuous Variance:

\[ \sigma^2_R(S) := \frac{1}{T} \int_0^T \sigma^2(s) ds, \]

where \( \sigma(t) \) is a stock volatility,

\( T \) is expiration date or maturity.
Why Trade Volatility (Variance)?

- Volatility Swaps allow investors to **profit** from the risks of an increase or decrease in future volatility of an index of securities or to **hedge** against these risks.

- If you think current volatility is low, for the right price you might want to take a position that profits if volatility increases.
How does the Volatility Swap Work?

Fixed leg = strike price
Floating leg = realized volatility

SCENARIOS

A – The volatility increases:

B – The volatility decreases:
Payoff of Variance Swaps

A Variance Swap is a forward contract on realized variance.

Its payoff at expiration is equal to

\[ N \left( \sigma^2_R(S) - K_{var} \right) \]

N is a notional amount ($/variance);
K_{var} is a strike price
Valuing of Variance Swap for Stochastic Volatility with Delay

Value of Variance Swap (present value):

\[ P = e^{-rT} E_{P^*} \left[ \sigma^2_R(S) - K_{var} \right] \]

where \( E_{P^*} \) is an expectation (or mean value), \( r \) is interest rate.

To calculate variance swap we need only \( E_{P^*} \left[ \sigma^2(t, S_t) \right] \),

\[ \sigma^2_R(S) := \frac{1}{T} \int_0^T \sigma^2(u, S(u - \tau)) du. \]
Valuing of Variance Swap for One-Factor SV with Delay in Stationary Regime

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t)
\]

\[v(t) = E_{P^*}[\sigma^2(t)] = V + \alpha \tau (\mu - r)^2 / \gamma.\]

\[
E_{P^*}[\text{Var}(S)] = \frac{1}{T} \int_{0}^{T} E_{P^*}[\sigma^2(t)] dt = V + \alpha \tau (\mu - r)^2 / \gamma.
\]

\[P^* = e^{-rT}[V - K + \alpha \tau (\mu - r)^2 / \gamma].\]
Valuing of Variance Swap for One-Factor SV with Delay in General Case

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma) v(t)
\]

\[v(t) \approx X + Ce^{-\gamma t} = V + \alpha \tau (\mu - r)^2 / \gamma + Ce^{-\gamma t}\]

\[C = v(0) - X = \sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma.
\]

In this way

\[v(t) = E_{\mathbb{P}^*} [\sigma^2(t)] \approx V + \alpha \tau (\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma)e^{-\gamma t}.
\]
Valuing of Variance Swap for One-Factor SV with Delay in General Case

We need to find \( E_P^*[\text{Var}(S)] \):

\[
E_{P^*}[\text{Var}(S)] = \frac{1}{T} \int_0^T E_{P^*}[\sigma^2(t)] \, dt
\approx \frac{1}{T} \int_0^T \left[ V + \alpha \tau (\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma) e^{-\gamma t} \right] \, dt
= V + \alpha \tau (\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma) \frac{1 - e^{-\gamma T}}{T \gamma}.
\]

\[
P^* = e^{-rT} \left[ V - K + \alpha \tau (\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma) \frac{1 - e^{-\gamma T}}{T \gamma} \right]
\]
Comparison of SV in Heston Model with SV with Delay

Heston Model (1993)

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sigma_t d\omega_1^t \\
\frac{d\sigma_t^2}{\sigma_t^2} &= k(\theta^2 - \sigma_t^2) dt + \gamma \sigma_t d\omega_2^t,
\end{align*}
\]
When \( \tau = 0 \) (the same expression as above):

\[
E\{V\} = \frac{1 - e^{-kT}}{kT} (\sigma_0^2 - \theta^2) + \theta^2,
\]

Swap for SV in Heston Model

\[
E\{V\} \approx \frac{1 - e^{-\gamma T}}{\gamma T} (\sigma^2(0, \phi(-\tau)) - V - \alpha \tau (\mu - r)^2 / \gamma) + [V + \alpha \tau (\mu - r)^2 / \gamma]
\]

Swap for SV with Delay

Table 1

<table>
<thead>
<tr>
<th>Statistics on Log Returns S&amp;P60 Canada Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Series:</td>
</tr>
<tr>
<td>Sample:</td>
</tr>
<tr>
<td>Observations:</td>
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<tr>
<td>Mean</td>
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<tr>
<td>Median</td>
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<tr>
<td>Maximum</td>
</tr>
<tr>
<td>Minimum</td>
</tr>
<tr>
<td>Std. Dev.</td>
</tr>
<tr>
<td>Skewness</td>
</tr>
<tr>
<td>Kurtosis</td>
</tr>
</tbody>
</table>

\[
E\{Var(S)\} = V + \frac{1-\gamma}{\gamma} \alpha \tau (\mu - r)^2 + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2/\gamma) \frac{1-e^{-\gamma T}}{T \gamma} \\
= 0.0002 + ((1 - 0.0124)/0.0124) \times 0.0604 \times (0.0002 - 0.02)^2 \\
+ (0.0001 - 0.0002 - 0.0604 \times (0.0002 - 0.02)^2/0.0124) \times \frac{1-e^{-0.0124}}{0.0124} \\
= 0.000102.
\]
Dependence of Variance (Realized) Swap for One-Factor SV with Delay on Maturity (S&P60 Canada Index)
Variance (Realized) Swap for One-Factor SV with Delay (S&P60 Canada Index)

**Table 3**

<table>
<thead>
<tr>
<th>Statistics on Log Returns S&amp;P500 Index</th>
<th>LOG RETURN S&amp;P500 INDEX</th>
</tr>
</thead>
<tbody>
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<td>Series:</td>
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<td><strong>Observations:</strong></td>
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<td><strong>Median</strong></td>
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<tr>
<td><strong>Maximum</strong></td>
<td>0.034025839</td>
</tr>
<tr>
<td><strong>Minimum</strong></td>
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<tr>
<td><strong>Sample Variance</strong></td>
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<tr>
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</tr>
<tr>
<td><strong>Kurtosis</strong></td>
<td>3.296144083</td>
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</table>

\[
E\{\text{Var}(S)\} = V + \frac{1-\gamma}{\gamma} \alpha (\mu - r)^2 + (\sigma_0^2 - V - \alpha (\mu - r)^2 / \gamma) \frac{1-e^{-\gamma r}}{T \gamma} \\
= 0.004038144 + ((1 - 0.511)/0.511) \times 0.3828 \times 14 \times (0.000263 - 0.02)^2 \\
+ (0.000063 - 0.04038144 - 0.3828 \times 14 \times (0.000263 - 0.02)^2/0.511) \\
\times \frac{1-e^{-0.611}}{0.511} \\
= 0.00774376584.
\]
Dependence of Variance (Realized) Swap for One-Factor SV with Delay on Maturity (S&P500)
Variance (Realized) Swap for One-Factor SV with Delay (S&P500 Index)
Numerical Example 1: S&P60 Canada Index

\[ X = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} = 0.0002 \]

\[ C = \sigma_0^2 - V - \alpha \tau (\mu - r)^2 / \gamma = 0.007. \]
Expectation of Variance and the Price of Variance Swap for 1-Factor SV with Delay

\[ v(t) \approx X + Ce^{-\gamma t} = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} + C e^{-\gamma t} \]

\[ \mathcal{P}^* \approx e^{-rT} \left[ V - K + \frac{\alpha \tau (\mu - r)^2}{\gamma} \right. \\
\left. + \left( \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \right) \frac{1 - e^{-\gamma T}}{T \gamma} \right] \]

**FIGURE 1:** Variance of one-factor SV with delay (formula (10)).

**FIGURE 2:** The price of variance swap for one-factor SV with delay (formula (13)).
E(Variance) and The Price of Variance Swap for SV with Delay (GBMMR)

\[ v(t) \approx X + Ce^{-\gamma t} + (\xi - \lambda \beta) \gamma V_0 \times \left[ \frac{X}{\xi - \lambda \beta} (e^{(\xi - \lambda \beta)T} - 1) + \frac{C}{\xi - \lambda \beta - \gamma} (e^{(\xi - \lambda \beta)T} - e^{\gamma T}) \right] \]

\[ P^* \approx e^{-rT} \left\{ \left[ X - K + C \frac{1 - e^{-\gamma T}}{T \gamma} \right] \right. \\
+ \left. \frac{(\xi - \lambda \beta) \gamma V_0}{T} \left[ \frac{X}{(\xi - \lambda \beta)} \left( \frac{e^{(\xi - \lambda \beta)T} - 1}{(\xi - \lambda \beta)} - T \right) \right. \\
+ \left. \frac{C(e^{(\xi - \lambda \beta)T} - 1)}{(\xi - \lambda \beta)(\xi - \lambda \beta - \gamma)} - \frac{C(e^{\gamma T} - 1)}{\gamma(\xi - \lambda \beta - \gamma)} \right] \right\} \]

FIGURE 3: Variance of two-factor SV with delay and with GBM mean-reversion (formula (17)).

FIGURE 4: Price of variance swap for two-factor SV with delay and with GBM mean-reversion (formula (19)).
Variance and The Price of Variance Swap SV with Delay (OUMR)

\[ v(t) \approx X + C e^{-\gamma t} + \xi \gamma \left( V_0 - \left( L - \frac{\lambda \beta}{\xi} \right) \right) \times \left[ \frac{X}{\xi} (e^{-\xi t} - 1) + \frac{C}{\xi + \gamma} (e^{-\xi t} - e^{-\gamma t}) \right] \]

\[ P^* \approx e^{-rT} \left\{ \left[ X - K + C \frac{1 - e^{-\gamma T}}{T\gamma} \right] + \frac{\xi \gamma (V_0 - (L - \frac{\lambda \beta}{\xi}))}{T} \right\} \times \left[ \frac{X}{\xi} \left( \frac{e^{-\xi T} - 1}{\xi} + T \right) + \frac{C(e^{-\xi T} - 1)}{\xi(\xi + \gamma)} + \frac{C(e^{-\gamma T} - 1)}{\gamma(\gamma + \xi)} \right] \]

FIGURE 5: Variance of two-factor SV with delay and with OU mean-reversion (formula (22)).

FIGURE 6: Price of variance swap for two-factor SV with delay and with OU mean-reversion (formula (24)).
Variance and The Price of Variance Swap for SV with Delay (Pilipovich 1FMR)

\[
v(t) \approx X + Ce^{-\gamma t} + \frac{\gamma\xi}{\xi + \lambda\beta} \left[ X \left( \frac{V_0(\xi + \lambda\beta)}{\xi} - L \right) \right.
\times \left( 1 - e^{-(\xi + \lambda\beta)t} \right) + XLTt
\]
\[
+ \frac{C(V_0(\xi + \lambda\beta) - L)}{\xi + \lambda\beta + \gamma} + \frac{CL}{\gamma} \left( e^{\gamma t} - 1 \right) \left. \right] .
\]

\[
P^* \approx e^{-rT} \left\{ X - K + C \frac{1 - e^{-\gamma T}}{T\gamma} \right. \right.
\times \left[ X \left( \frac{V_0(\xi + \lambda\beta)}{\xi} - L \right) \left( \frac{e^{-(\xi + \lambda\beta)} - 1}{\xi + \lambda\beta} - T \right) \right)
\]
\[
+ \frac{XLT^2}{2} + \frac{C(V_0(\xi + \lambda\beta) - L)}{\xi + \lambda\beta + \gamma}
\times \left( \frac{e^{\gamma T} - 1}{\gamma} + \frac{e^{-(\xi + \lambda\beta)T} - 1}{\xi + \lambda\beta} \right) + \frac{CL}{\gamma} \left( \frac{e^{\gamma T} - 1}{\gamma} - T \right) \}
\]
Variance and The Price of Variance Swap for SV with Delay (Pilipovich 2FMR)

\[ v(t) \approx X + C e^{-\gamma t} - (\xi + \lambda \beta) \gamma V_0 \left[ \frac{X}{\xi + \lambda \beta} (1 - e^{-(\xi + \lambda \beta) t}) \right. \\
+ \left. \frac{C}{\xi + \lambda \beta + \gamma} (e^{\gamma t} - e^{-(\xi + \lambda \beta) t}) \right] + L_0 \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \beta_1} \\
\times \left[ X(e^{(\beta_1 - \lambda_1) t} - e^{-(\xi + \lambda \beta) t}) + \frac{C(\beta_1 - \lambda_1 \eta)}{(\beta_1 - \lambda_1 \eta - \gamma)} \\
\times (e^{(\beta_1 - \lambda_1) t} - e^{\gamma t}) + \frac{C(\xi + \lambda \beta)}{(\xi + \lambda \beta + \gamma)} (e^{\gamma t} - e^{-(\xi + \lambda \beta) t}) \right] \]

\[ P^* \approx e^{-rT} \left\{ \left[ X - K + C \frac{1 - e^{-\gamma T}}{T} \right] - (\xi + \lambda \beta) \gamma V_0 \right. \\
\times \left[ \frac{X}{(\xi + \lambda \beta)} \left( \frac{e^{-(\xi + \lambda \beta) T} - 1}{(\xi + \lambda \beta)} + T \right) \\
+ \frac{C(e^{-(\xi + \lambda \beta) T} - 1)}{(\xi + \lambda \beta) (\xi + \lambda \beta + \gamma)} + \frac{C(e^{\gamma T} - 1)}{\gamma (\gamma + \xi + \lambda \beta)} \right] \\
+ \frac{(\xi + \lambda \beta)L_0}{(\xi + \lambda \beta + \beta_1) T} \left[ X(e^{(\beta_1 - \lambda_1) T} - 1 - (\beta_1 - \lambda_1 \eta) T) \right. \\
\left. \left. + \frac{X(e^{-(\xi + \lambda \beta) T} - 1 + (\xi + \lambda \beta) T)}{(\xi + \lambda \beta)} \right) \\
+ C \left( \frac{\beta_1 - \lambda_1 \eta}{\beta_1 - \lambda_1 \eta - \gamma} \left( \frac{e^{(\beta_1 - \lambda_1 \eta) T} - 1}{\beta_1 - \lambda_1 \eta} \right) - e^{\gamma T} - 1 \right) \\
\left. + \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \gamma} \left( e^{-(\xi + \lambda \beta) T} - 1 \right) \right] \right\}. \]

FIGURE 9: Variance of three-factor SV with delay and with Pilipovich two-factor mean-reversion (formula (31)).

FIGURE 10: Price of variance swap for three-factor SV with delay and with Pilipovich two-factor mean-reversion (formula (33)).
Comparison

One-Factor

2-F(GBMMR)

2-F(OUMR)

2-F(Pilipovich 1FMR)

3-F(Pilipovich 2FMR)
Conclusion I

- There is no big difference between One-Factor SV with Delay and Multi-Factor SV with Delay
- One-Factor SV with Delay catches almost all the features of Multi-Factor SV with Delay
- One-Factor SV with Delay is Similar to the SV in Heston Model (at least for variance swaps)
**SV with Delay and Jumps**

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) + \int_{t-\tau}^{t} y_s dN(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]

**SV with Delay without Jumps**

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t).
\]
SV with Delay and Jumps
(Simple Poisson Process Case)

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) + \int_{t-\tau}^{t} dN(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]
Equation for The Mean of Variance

\[ v(t) = \mathbb{E}^*[\sigma^2(t, S_t)] \]

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \lambda + \alpha \lambda^2 \tau - 2\alpha \lambda \tau (\mu - r) + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t)
\]
Stationary Solution

\[ v(t) \equiv X = V + \frac{\alpha \lambda + \alpha \lambda^2 \tau - 2 \alpha \lambda \tau (\mu - r) + \alpha \tau (\mu - r)^2}{\gamma} \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda + \tau (\lambda - \mu + r)^2 \right] \]

Stationary Solution without Jumps

\[ v(t) \equiv X = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} \]
Price of Var Swap in Stationary Case

\[ P = e^{-r(T-t)} \left\{ V - K + \frac{\alpha}{\gamma} \left[ \lambda + \tau(\lambda - \mu + r)^2 \right] \right\} \]
General Solution

\[ v(t) \approx X + Ce^{-\gamma t} \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda + \tau(\lambda - \mu + r)^2 \right] + Ce^{-\gamma t} \]

\[ C = \sigma_0^2 - V - \frac{\alpha}{\gamma} \left[ \lambda + \tau(\lambda - \mu + r)^2 \right] \]
Price of Var Swap in General Case

\[ P \approx e^{-r(T-t)}[X - K + (\sigma^2_0 - X)\frac{1 - e^{-\gamma T}}{\gamma T}] \]

\[ X = V + \left[ a\lambda + a\lambda^2\tau - 2a\lambda\tau(\mu - r) + a\tau(\mu - r)^2 \right] / \gamma \]
\[ = V + \frac{\alpha}{\gamma} \left[ \lambda + \tau(\lambda - \mu + r)^2 \right] \]
Compound Poisson Process
Case

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW^* (s) + \int_{t-\tau}^{t} y_s dN(s) - (\mu - r) \tau \right]^2
\]

\[-(\alpha + \gamma)\sigma^2(t, S_t)\]
Equation for the Mean of Variance

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \lambda + a \lambda^2 \tau - 2a \lambda \tau (\mu - r) + a \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds \\
-(\alpha + \gamma)v(t)
\]
Stationary Solution

\[ v(t) \equiv X = V + \left[ \alpha \lambda (\xi^2 + \eta) + \alpha \lambda^2 \tau \xi^2 - 2 \alpha \lambda \tau \xi (\mu - r) + \alpha \tau (\mu - r)^2 \right] / \gamma \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \]
Price of Swap in Stationary Case

\[ P = e^{-r(T-t)} \{ V - K + \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \} \]
$v(t) \approx X + Ce^{-\gamma t}$

$$= V + \frac{\alpha}{\gamma} \left[ \lambda(\xi^2 + \eta) + \tau(\lambda \xi - \mu + r)^2 \right] + Ce^{-\gamma t}$$

$C = \sigma_0^2 - V - \frac{\alpha}{\gamma} \left[ \lambda(\xi^2 + \eta) + \tau(\lambda \xi - \mu + r)^2 \right]$
Price of Swap in General Case

\[ P \approx e^{-r(T-t)} [X - K + (\sigma_0^2 - X) \frac{1 - e^{-\gamma T}}{\gamma T}] \]

\[ X = V + \left[ \alpha \lambda (\xi^2 + \eta) + \alpha \lambda^2 \tau \xi^2 - 2\alpha \lambda \tau \xi (\mu - r) + \alpha \tau (\mu - r)^2 \right] / \gamma \]

\[ = V + \frac{\alpha}{\gamma} \left[ \lambda (\xi^2 + \eta) + \tau (\lambda \xi - \mu + r)^2 \right] \]
More General case

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW^*(s) + \int_{t-\tau}^{t} y_s dN(s) - (\mu - r)\tau \right]^2 \\
- (\alpha + \gamma)\sigma^2(t, S_t)
\]

(38)

where \(W^*(t)\) is a Brownian motion, \(N(t)\) is a Poisson process with intensity \(\lambda\) and \(y_t\) is the jump size at time \(t\). We assume that \(\mathbb{E}[y_t] = A(t)\), \(\mathbb{E}[y_s y_t] = C(s, t), s < t\) and \(\mathbb{E}[y_t^2] = B(t) = C(t, t)\), where \(A(t), B(t), C(s, t)\) are all deterministic functions. Note that the change of measure do not change the Poisson intensity \(\lambda\) and the distribution of jump size \(y_t\), since they are independent to the Brownian motion.
Equation for the Mean of Variance

$$\frac{dv(t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} v(s) ds + \lambda \int_{t-\tau}^{t} B(s) ds + \lambda^2 (K(t, \tau) + G) \right]$$

$$+ (\mu - r)^2 \tau^2 - 2\lambda \tau (\mu - r) \int_{t-\tau}^{t} A(s) ds \] - (\alpha + \gamma)v(t)$$
General Solution

\[ v(t) \approx \frac{1-e^{-\gamma t}}{\gamma} v'(0) + \left[ \frac{\alpha}{\gamma} (1-e^{-\gamma t}) + 1 \right] v(0) - \frac{\alpha}{\gamma \tau} \int_{-\tau}^{0} v(s) [1-e^{-\gamma(t-s-\tau)}] ds \]

\[ + \frac{1}{\gamma} \int_{0}^{t} h(s, \tau) [1-e^{-\gamma(t-s)}] ds + C. \quad (49) \]

\[ C = \frac{\alpha}{\gamma \tau} \int_{-\tau}^{0} v(s) [1-e^{\gamma(s+\tau)}] ds. \]
## Numerical Example

<table>
<thead>
<tr>
<th>Statistics on Log Returns</th>
<th>S&amp;P60 CANADA Index</th>
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<tr>
<td><strong>Series:</strong></td>
<td>LOG RETURNS S&amp;P60 CANADA INDEX</td>
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<td><strong>Observations:</strong></td>
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</tr>
<tr>
<td><strong>Kurtosis</strong></td>
<td>7.787327</td>
</tr>
</tbody>
</table>
Constant $V$ and $E^*[v]$

\[
V + \frac{\alpha}{\gamma} \left[ \lambda(\xi^2 + \eta) + \tau(\lambda \xi - \mu + r)^2 \right] \\
= 0.0002 + 0.0604/0.0124 \times \left[ 0.0115 \times \left[ (-0.003)^2 + 0.0035 \right] \\
+ (0.0115 \times (-0.003) - 0.0002 + 0.0124)^2 \right] \\
= 0.0023.
\]

\[
E^*[v] \approx V + \frac{\alpha}{\gamma} \left[ \lambda(\xi^2 + \eta) + \tau(\lambda \xi - \mu + r)^2 \right] \\
+ \left\{ \sigma_0^2 - V - \frac{\alpha}{\gamma} \left[ \lambda(\xi^2 + \eta) + \tau(\lambda \xi - \mu + r)^2 \right] \right\} \frac{1-e^{-\gamma T}}{\gamma T} \\
= 0.0023 + (0.0001 - 0.0023) \times \frac{1-e^{-0.0124}}{0.0124} \\
= 0.0001136.
\]
Delivery Price and Maturity

Figure 1: Dependence of Delivery Price on Maturity (S&P60 Canada Index).
Figure 2: Dependence of Delivery Price on Delay (S&P60 Canada Index).
Figure 3: Dependence of Delivery Price on Jump Intensity (S&P60 Canada Index).
Figure 4: Dependence of Delivery Price on Delay and Jump Intensity ($S&P_{60}$ Canada Index).
Delivery Price, Delay and Maturity

Figure 5: Dependence of Delivery Price on Delay and Maturity ($S&P60$ Canada Index).
Figure 6: Dependence of Delivery Price on Jump Intensity and Maturity (S&P60 Canada Index).
Comparison (1F&Jumps)

Figure 1: Dependence of Delivery Price on Maturity (S&P500 Canada Index).
Conclusion II

- There is no big difference between One-Factor SV with Delay, Multi-Factor SV with Delay and 1F SV with Delay & Jumps.

- One-Factor SV with Delay catches all the features of Multi-Factor SV with Delay and 1F SV with Delay & Jumps.
Publications

• ‘Continuous-time GARCH model for Stochastic Volatility with Delay’ (Kazmerchuk, Sw, Wu), CAMQ, 2005, v. 3, No. 2
• ‘Modeling and Pricing of Variance Swap for Stochastic Volatility with Delay’ (Sw), Wilmott Magazine, Issue 19, September 2005
• ‘Modeling and Pricing of Variance Swaps for Multi-Factor Stochastic Volatilities with Delay’ (Sw), CAMQ, 2006, v. 14, No. 4
• ‘Pricing Variance Swaps for Stochastic Volatilities with Delay and Jumps’ (Sw, Xu, L.), Quantitative Finance (submitted), 2007
The End

• Thank You for Your Attention!
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