$$M(t) = B(T(t))$$

Stochastic Volatility and Change of Time: Overview

Anatoliy Swishchuk
Mathematical & Computational Finance Lab
Dept of Math & Stat, University of Calgary, Calgary, AB, Canada

North/South Dialogue Meeting
Financial Mathematics Session
May 2, 2008

The research is supported by NSERC

Outline

- Volatility: Types
- Stochastic Volatility (SV):1-& Multi-Factor
- Change of Time (CT)
- Relationship: SV & CT
- Numerical Example
- Problems

Volatility

- **Volatility** is the standard deviation of the change in value of a financial instrument with specific time horizon
- It is often used to quantify the **risk** of the instrument over that time period
- The higher volatility, the riskier the security

Types of Volatilities

- **Historical V**: standard deviation (uses historical (daily, weekly, monthly, quarterly, yearly)) price data to empirically measure the volatility of a market or instrument in the past
- **Implied V**: volatility implied by the market price of the option based on an option pricing model (smile and skew-varying volatility by strike)

Volatility Smile

- The models by Black & Scholes (continuous-time (B,S)-security market,1973) and Cox & Rubinstein (discrete-time (B,S)-security market (binomial tree), 1979) are unable to explain the **negative skewness** and **leptokurticity (fat tail)** commonly observed in the stock markets
- The famous **implied-volatility smile** would not exist under their assumptions

Commodity: Coffee

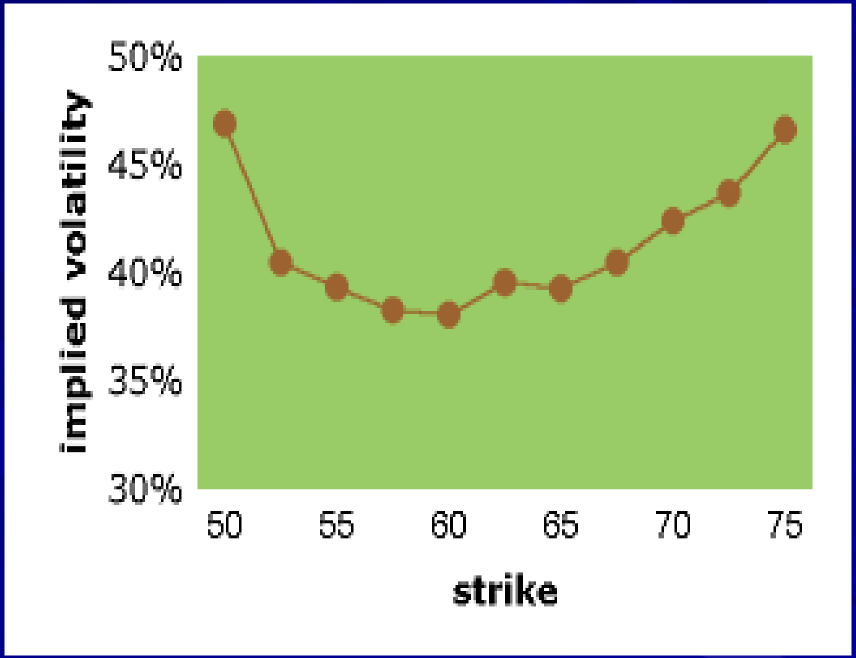
- Coffee options trade on New York's Coffee, Sugar and Cocoa Exchange (**CSCE**).

Coffee Call Option

- CSCE May 2001 coffee call option **implied volatilities** as of March 12, 2001

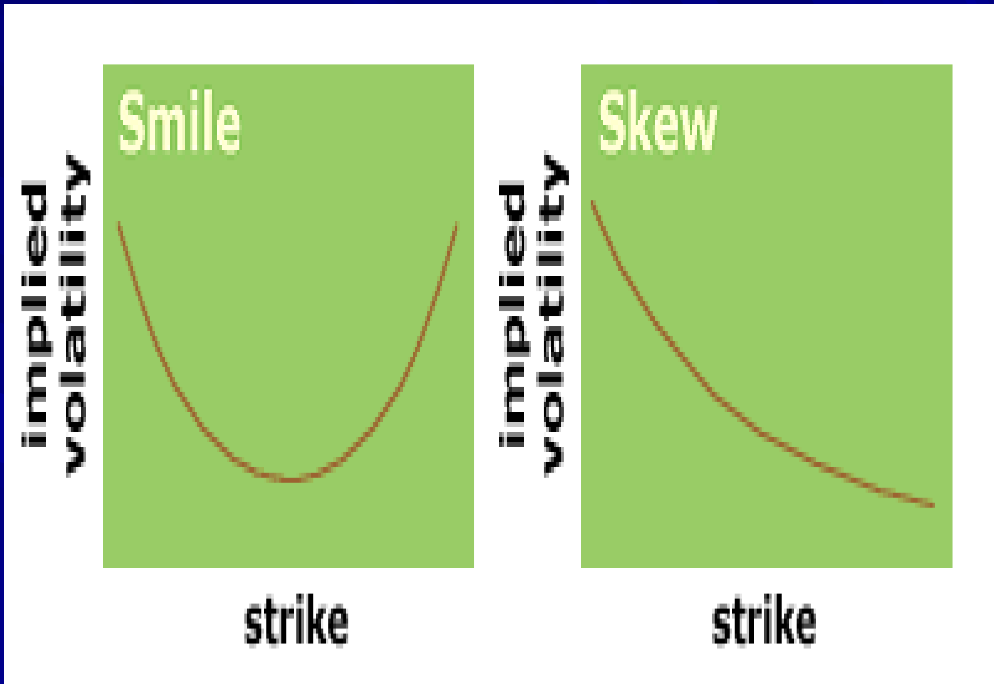
Implied Volatility: Volatility Smile

- Graph indicates implied volatilities at various strikes for the May 2001 calls based upon their March 12, 2001 settlement prices. The pattern of implied volatilities form a "smile" shape, which is called a **volatility smile**.



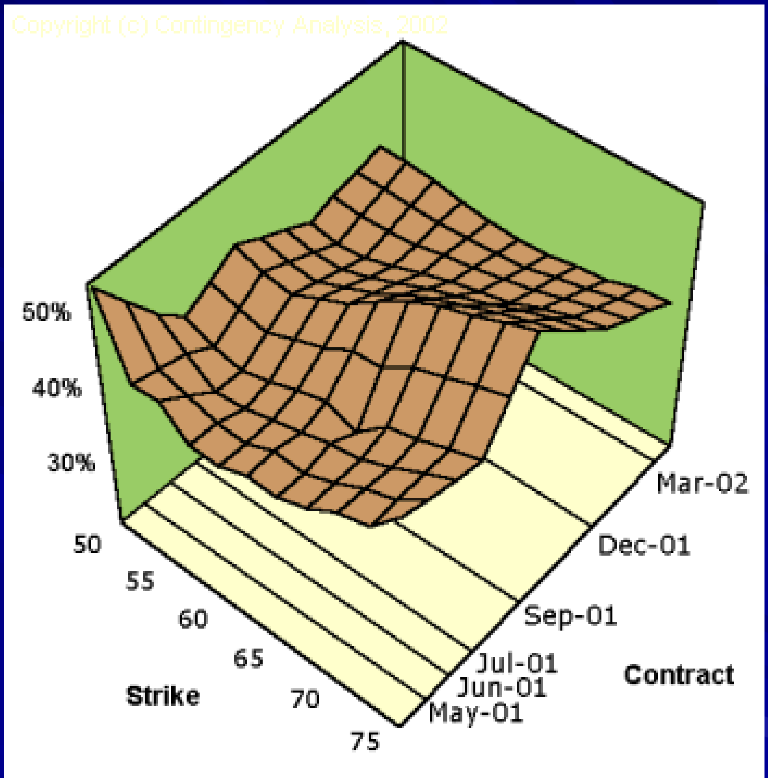
Implied Volatility: Volatility Skew

- Most derivatives markets exhibit persistent patterns of volatilities varying by strike. In some markets, those patterns form a **smile**. In others, such as *equity index options* markets, it is more of a skewed curve. This has motivated the name **volatility skew**. In practice, either the term "volatility smile" or "volatility skew" (or simply **skew**) may be used to refer to the general phenomena of volatilities varying by strike.



Implied Volatility: Volatility Surface

- Another dimension to the problem of volatility skew is that of volatilities varying by expiration. This is illustrated for CSCE coffee options. It indicates what is known as a **volatility surface**



Types of Volaltilites II

- **Jump-Diffusion Volatility**
- **Level-Dependent Volatility** (CEV or Firm Model)-*function of the spot price alone*
- **Local** ***V****-function of the spot price* ***and time*** (Dupire formulae, 1994)
- **Stochastic V**: volatility is not constant, but a stochastic process (explains smile and skew)

Jump-Diffusion

- In addition to the volatility smile observable from the implied volatilities of the options, there is evidence that assumption of a pure diffusion for the stock return is not accurate
- ‘Fat Tails’ have been observed away from the mean of the stock return
- This phenomenon is called leptokurticity and could be explained in many different ways
- One way to explain smile and leptocurticity is to introduce a jump-diffusion process

Jump-Diffusion and Leverage Effect

- Jump-diffusion is not a level-dependent volatility process
- Explains leverage effect
- **Merton (1976)** was first to introduce jumps in the stock distribution
- **Kou (2000)** used the same idea to explain both existence of fat tails and the volatility smile

Level-Dependent Volatility

- Level-dependent V (LDV)-function of spot price alone
- Constant Elasticity Variance (CEV): Cox (1976, 1996)
- Important feature of the level-dependent volatility: represents the negative correlation between the stock price and the volatility (**leverage effect**)
- LDV by Firm structure model: Bensoussan, Crouhy & Galai (1995)

$$\sigma = \sigma(S)$$

$$\sigma(t, S) = CS_t^{\gamma}$$

Local Volatility

- Local Volatility- ***V****-function of the* *<u>spot price</u>* *<u>and time</u>*
- Volatility smile was retrieved from the option prices

$$\sigma = \sigma(t, S)$$

- **Dupire (1994)-**local volatility formula (V-call price)

$$\sigma^2(K,T) = \frac{\frac{\partial V}{\partial T} + rK\frac{\partial V}{\partial K}}{\frac{1}{2}K^2\frac{{}^{2}V}{\partial K^2}}$$

- **Derman & Kani (1994)-**used the binomial (or trinomial tree) framework instead of the continuous one

Local Volatility: Drawbacks

- The LV models are very elegant and theoretically sound
- However, they present in practice many **stability issues**
- They are **ill-posed inversion problems** and are extremely **sensitive to the input data**
- This might introduce **arbitrage opportunities and in some cases negative probabilities or variances**

Stochastic Volatility (SV)

- **SV** is the main concept used in the fields of financial economics and mathematical finance to deal with the endemic time-varying volatility and co-dependence found in financial markets
- Such dependence has been known for a long time, early comments include **Mandelbrot (1963)** and **Officer (1973)**

Stochastic Volatility

- The aim with a stochastic volatility model: *volatility appears not to be constant* and indeed varies, at least in part, randomly. The idea is to make the **volatility itself a stochastic process**.
- **Stochastic volatility** models are useful because they explain in a self-consistent way why it is that options with different strikes and expirations have different Black-Scholes implied volatilities (the **volatility smile**)

Two Approaches to Introduce SV

- One approach-to change the clock time t to a random time T(t) (change of time)

$$\sigma W(t) \Rightarrow W(T(t))$$

- Another approach-change constant volatility into a positive stochastic process

$$\sigma \equiv \sigma(t), \quad \int_0^t \sigma^2(s)ds < +\infty$$

Stochastic Volatilities: Continuous- Time Models

- **Ornstein-Uhlenbeck Process**

$$d\sigma_t = -\alpha\sigma_t dt + \beta dZ_t$$

- **Hull & White (1987)**

(GBM, positive)

$$\frac{d\sigma}{\sigma} = \mu dt + \xi dW_2, \quad \rho = 0$$

- **Wiggins (1987)**

(GBM, positive)

$$\frac{d\sigma}{\sigma} = \mu dt + \xi dW_2, \quad \rho \neq 0$$

- **Scott (1989)**

(OU, mean-reverting, positive)

$$d\ln(\sigma^2) = (w - \zeta\ln(\sigma^2))dt + \xi dW_2, \quad \rho \neq 0$$

- **Stein & Stein (1991)**

(OU, mean-reverting, negative)

$$d\sigma = (w - \zeta\sigma)dt + \xi dW_2, \quad \rho = 0$$

- **Heston (1993)**

(mean-reverting, semi-analytical pricing formulae)

$$d\sigma^2 = (w - \zeta\sigma^2)dt + \xi\sigma dW_2, \quad \rho \neq 0$$

Stochastic Volatilities: Continuous- Time Models II

- **Heston & Nandi (1997)-** showed that OU process corresponds to a special case of the GARCH model
- Another popular process is the continuous-time GARCH(1,1) process, developed by Engle (1982) and Bollerslev (1986) in discrete framework

$$d\sigma = \theta(w - \sigma)dt + \xi \sigma dW$$

Stochastic Volatilities: Discrete-Time Models

- Even though continuous time models provide the natural framework for an analysis of option pricing, **discrete time models** are ideal for the **statistical and descriptive analysis** of the patterns of daily price changes
- **Volatility Clustering**: there are periods of high and low variance ('large changes tend to be followed by small changes' (**Mandelbrot**))-led to use of GARCH models

Discrete-Time SV Models: Two Main Classes

- The first class, the **autoregressive random variance (ARV) or stochastic variance models**, is a discrete time approximation to the continuous time diffusion models that we outlined above
- The second class is the **autoregressive conditional heteroskedastic (ARCH)** models introduced by Engle (1982), and its descendents (GARCH (Bolerslev (1986)), NARCH, NGARCH (Duan,1996), LGARCH, EGARCH, GJR-GARCH, etc.)

SV With Delay: Continuous-Time GARCH Model with Delay

(Kazmerchuk, Swishchuk, Wu (2002))

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(s,S_s)dW(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t).$$

$$\sigma_n^2 = \gamma V + \frac{\alpha}{l}\ln^2(S_{n-1}/S_{n-1-l}) + (1-\alpha-\gamma)\sigma_{n-1}^2$$

General Class of SV Models

(C.-O. Ewald, R. Poulsen, K.R. Schenk-Hoppe (2006))

$$\frac{dS}{S} = \mu dt + S^{\gamma} f(\sigma)[\sqrt{1-\rho^2}dW_1 + \rho dW_2]$$

$$\frac{d\sigma}{\sigma} = \beta(\sigma)dt + g(\sigma)dW_2$$

Specification of General SV Models

$$\frac{dS}{S} = \mu dt + S^{\gamma} f(\sigma)[\sqrt{1-\rho^2}dW_1 + \rho dW_2]$$
$$\frac{d\sigma}{\sigma} = \beta(\sigma)dt + g(\sigma)dW_2$$

Authors & year	Specification	Remarks
Hull-White 1987	$f(v) = v$, $\beta(v) = 0$, $g(v) = \sigma$, $\rho = 0, \gamma = 0$	Local variance: Geometric Brownian motion. Options priced by mixing.
Wiggins 1987	$f(v) = e^{v/2}$, $\beta(v) = \kappa(\theta - v)/v$, $g(v) = \sigma$, $\rho = 0, \gamma = 0$	Local volatility: Ornstein-Uhlenbeck in logarithms.
Stein-Stein 1991	$f(v) = \|v\|$, $\beta(v) = \kappa(\theta - v)/v$, $g(v) = \sigma/v$, $\rho = 0, \gamma = 0$	Local volatility: Reflected Ornstein-Uhlenbeck.
Heston 1993	$f(v) = \sqrt{v}$, $\beta(v) = \kappa(\theta - v)/v$, $g(v) = \sigma/\sqrt{v}$, $\rho \in [-1,1], \gamma = 0$	Local variance: CIR process. First model with correlation. Options priced by inversion of characteristic fct.
Romano-Touzi 1997	$f(v) = \sqrt{v}$, β and g are free, $\rho \in [-1,1], \gamma = 0$	Extension of mixing to correlation.
SABR 2002	$f(v) = v$ $\beta(v) = 0$, $g(v) = \sigma$, $\rho \in [-1,1], \gamma \in [-1,0]$	Level dependence in volatility. Options priced perturbation tech.

Heston-Like Model

(J. Gatheral (2005), Merrill Lynch)

$$\frac{dS}{S} = \mu_t dt + \sqrt{\sigma}\, dW_1$$

$$d\sigma = \alpha(S,\sigma)dt + \eta\beta(S,\sigma)\sqrt{\sigma}\, dW_2$$

Multi-Factor SV Models

- **One-Factor SV Models** (*all above-mentioned*): 1) incorporate the leverage between returns and volatility and

  2) reproduce the 'skew' of implied volatility
- However, it *fails to match either the high conditional kurtosis of returns* (**Chernov *et. al*. (2003)**) or *the full term structure of implied volatility surface* **(Cont&Tankov (2004))**
- *Adding jump components* in returns and/or volatility process, or considering **multi-factor SV models** are two primary generalizations of one-factor SV models

Multi-Factor SV Model

(J.-P. Fouque, C.-H. Han (2005))

$$
\begin{aligned}
\frac{dS}{S} &= \mu dt + \sigma dW_1 \\
\sigma &= f(Y, Z) \\
dY &= \alpha c_1(Y) dt + \sqrt{\alpha} g_1(Z) dW_2 \\
dZ &= \delta c_2(Z) dt + \sqrt{\delta} g_2(Z) dW_3
\end{aligned}
$$

Multi-Factor SV Models

- **Chernov et al. (2003):** used efficient method of moments to obtain comparable empirical-of-fit from affine jump-diffusion mousiondels & two-factor SV family models
- **Molina et al. (2003):** used a Markov Chain Monte Carlo method to find strong evidence of two-factor SV models with well-separated time scales in foreign exchange data
- **Cont &Tankov (2004):** found that jump-diffusion models have a fairly good fit to the implied volatility surface
- **Fouque et al. (2000):** found that two-factor SV models provide a better fit to the term structure of implied volatility than one-factor SV models by capturing the behavior at short and long maturities
- **Swishchuk (2006):** introduced two-factor and three-factor SV models with delay (incorporating mean-reverting level as a random process (GBM, OU, Pilipovich or continuous-time GARCH(1,1) model))

Advantages and Disadvantages of Multi-Factor SV Models

- Multi-Factor SV models do not admit in general explicit solutions for option prices
- But have direct implications on hedges
- Comparison: class of **jump-diffusion models (Bates (1996))** enjoys <u>closed-form solutions</u> for option prices *but the complexity of hedging strategies increases due to jumps*
- There is no strong empirical evidence to judge the overwhelming position between jump-diffusion models and multi-factor SV models

Other Generalization of SVM

- Allow jumps into the volatility SDE (**Bates (1996), Barndorff-Nielsen & Shephard (2001**), **Eraker, Johannes & Polson (2003), Nicolato & Venardos (2003)-affine class (Duffie, Pan & Singleton (2000)))**
- Discrete and continuous-time long memory SV (**Breidt, Crato & Lima (1998), Harvey (1998), Comte & Renault (1998), Compte, Coutin & Renault (2003), Barndorff-Nielsen (2001**))
- Multivariate models: introducing volatility clustering into traditional factor models (**Diebold & Nerlove (1989))**

Change of Time: Definition and Examples

- **Change of Time-**change time from t to a non-negative process T(t) with non-decreasing sample paths
- *Example1* (**Subordinator**): X(t) and T(t)>0 are some processes, then X(T(t)) is subordinated to X(t); T(t) is change of time
- *Example 2* (**Time-Changed Brownian Motion**): M(t)=B(T(t)), B(t)-Brownian motion
- *Example 3* (**Product Process**):

$$M_t = \int_0^t \sigma_s dW_s$$

$$T(t) = [M]_t = \int_0^t \sigma_s^2 ds$$

Interpretation of CT

- If M(t) is a **martingale** (another name- **fair game** process)
- Then **M(t)=B(T(t))** (Dambis-Dubins-Schwartz Theorem)
- Time-change is the **quadratic variation** process T(t)=[M(t)]
- Then M(t) can be written as a **SV process** (martingale representation theorem, Doob (1953))
- This implies that *time-changed BMs are* ***canonical in continuous sample path price processes*** and ***SVMs are special cases of this class***

$$T(t) = [M]_t = \int_0^t \sigma_s^2 ds$$

$$M_t = \int_0^t \sigma_s dW_s$$

Time-Changed Brownian Motion by Bochner

- **Bochner** (1949) ('Diffusion Equation and Stochastic Process', Proc. N.A.S. USA, v. 35)-*introduced the notion of change of time (CT) (time-changed Brownian motion)*
- **Bochner** (1955) ('Harmonic Analysis and the Theory of Probability', UCLA Press, 176)-*further development of CT*

Change of Time: First Intro into Financial Economics

- **Clark (1973)** (*'A 'Subordinated Stochastic Process Model with Fixed Variance for Speculative Prices*', Econometrica, 41, 135-156)-introduced Bochner's (1949) time-changed Brownian motion into financial economics*:*

- He wrote down a model for the log-price *M as*

  M(t)=B(T(t)),

- where B(t) is Brownian motion, T(t) is time-change (B and T are independent)

Change of Time: Short History. I.

- **Feller (1966)** (‘*An Introduction to Probability Theory*’, vol. II, NY: Wiley)-introduced subordinated processes X(T(t)) with Markov process X(t) and T(t) as a process with independent increments (i.e., Poisson process); T(t) was called *randomized operational time*
- **Johnson (1979)** (‘Option Pricing When the Variance Rate is Changing’, working paper, UCLA)-*introduced time-changed SVM in continuous time*
- **Johnson & Shanno (1987)** (‘Option Pricing When the Variance is Changing’, J. of Finan. & Quantit. Analysis, 22, 143-151)-*studied the pricing of options using time-changing SVM*

Change of Time: Short History. II.

- **Ikeda & Watanabe** (1981) ('SDEs and Diffusion Processes', North-Holland Publ. Co)-*introduced and studied CTM for the solution of SDEs*
- ***Barndorff-Nielsen, Nicolato & Shephard*** *(2003) ('Some recent development in stochastic volatility modelling')-review and put in context some of their recent work on stochastic volatility (SV) modelling, including the relationship between subordination and SV (random time-chronometer)*
- **Carr, Geman, Madan & Yor (2003)** ('SV for Levy Processes', mathematical Finance, vol.13)-used subordinated processes to construct SV for Levy Processes (T(t)-business time)

Time-Changed Models and SVMs

- The probability literature has demonstrated that *SVMs and their time-changed BM relatives and time-changed models are* ***fundamentals***

- **Shephard (2005):** Stochastic Volatility, working paper, *University of Oxford*
- **Shephard (2005):** Stochastic Volatility: Selected Readings, Oxford, *Oxford University Press*

Change of Time: Simplest (Martingale) Case

$$M(t) - martingale, \quad \lim_{t\to+\infty}[M](t) = +\infty$$

$$\phi_t := \inf\{u : [M](u) > t\}$$

$$W(t) := M(\phi_t) - Brownian \quad motion$$

$$M(t) = W([M](t)) - martingale$$

$$\phi = [M]^{-1}(t), \quad \phi_t^{-1} = [M](t)$$

Change of Time: General (Ito Integral) Case

$$M(t) = \int_0^t \sigma(s)dW(s) - martingale, \quad [M] = \int_0^t \sigma^2(s)ds \to +\infty$$

$$\phi_t := \inf\{u : [M](u) > t\}$$

$$W(t) := M(\phi_t) - Brownian \quad motion$$

$$M(t) = W([M](t)) - martingale$$

$$\phi_t = [M]^{-1}(t), \quad \phi_t^{-1} = [M](t) = \int_0^t \sigma^2(s)ds$$

Change of Time: SDE's Case

$$dX(t) = \alpha(t, X(t))dW(t)$$

$$V(t) = X(0) + \tilde{W}(t)$$

$$\phi_t = \int_0^t \alpha^{-2}(\phi_s, X(0) + \tilde{W}(s))ds.$$

$$X(t) := V(\phi_t^{-1}) = X(0) + \tilde{W}(\phi_t^{-1})$$

Geometric Brownian Motion SVM

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Change of Time Method

$$V(t) = e^{\mu t} S(t) \Rightarrow V(t) = S(0) + \tilde{W}(\phi_t^{-1}),$$

where

$$\phi_t^{-1} = \sigma^2 \int_0^t (S(0) + \tilde{W}(\phi_s^{-1}))^2 ds,$$

and $\tilde{W}(t)$ is one-dimensional Wiener process.

Solution for GBM Equation Using Change of Time

$$S(t) = e^{\mu t}(S(0) + \tilde{W}(\phi_t^{-1}))$$

Explicit Expression for $\tilde{W}(\phi_t^{-1})$

$$dV(t) = \sigma V(t)dW(t) \Rightarrow V(t) = V(0)e^{\sigma W(t) - \sigma^2 t/2}$$

$$V(t) = V(0) + \tilde{W}(\phi_t^{-1}) \Rightarrow W(\phi_t^{-1}) = S(0)\left(e^{\sigma W(t) - \sigma^2 t/2} - 1\right).$$

Mean-Reverting SV Model

$$dS_t = a(L - S_t)dt + \sigma S_t dW_t,$$

where W is a standard Wiener process, $\sigma > 0$ is the volatility, the constant L is called the 'long-term mean' of the process, to which it reverts over time, and $a > 0$ measures the 'strength' of mean reversion.

Solution of MRM by CTM

$$S_t = e^{-at}\left[S_0 - L + \tilde{W}(\phi_t^{-1})\right] + L, \quad (4)$$

where $\tilde{W}(t)$ is an $\mathcal{F}_t$-measurable standard one-dimensional Wiener process, $\phi_t^{-1}$is an inverse function to ϕ_t :

$$\phi_t = \sigma^{-2}\int_0^t (S_0 - L + \tilde{W}(s) + e^{a\phi_s}L)^{-2}ds. \quad (5)$$

We note that

$$\phi_t^{-1} = \sigma^2\int_0^t (S_0 - L + \tilde{W}(\phi_t^{-1}) + e^{as}L)^2 ds, \quad (6)$$

Explicit Expression for $\tilde{W}(\phi_t^{-1})$

$$\tilde{W}(\phi_t^{-1}) = S(0)(e^{\sigma W(t)-\frac{\sigma^2 t}{2}}-1)+L(1-e^{at})+aLe^{\sigma W(t)-\frac{\sigma^2 t}{2}}\int_0^t e^{as}e^{-\sigma W(s)+\frac{\sigma^2 s}{2}}ds.$$

Explicit Expression for $\tilde{W}(\phi_t^{-1})$

$$\tilde{W}(\phi_t^{-1}) = m_1(t) + Lm_2(t),$$

where

$$m_1(t) := S(0)(e^{\sigma W(t) - \frac{\sigma^2 t}{2}} - 1)$$

and

$$m_2(t) = (1 - e^{at}) + ae^{\sigma W(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds.$$

Comparison: Solution of GBM & MRM)

$$S(t) = e^{\mu t}(S(0) + \tilde{W}(\phi_t^{-1}))$$

$$S_t = e^{-at}[S_0 - L + \tilde{W}(\phi_t^{-1})] + L$$

Explicit Expression for S(t)

$$
\begin{aligned}
S(t) &= e^{-at}\left[S_0 - L + \tilde{W}\left(\phi_t^{-1}\right)\right] + L \\
&= e^{-at}\left[S_0 - L + m_1(t) + L m_2(t)\right] + L \\
&= S(0)e^{-at}e^{\sigma W(t) - \frac{\sigma^2 t}{2}} + aLe^{-at}e^{\sigma W(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds,
\end{aligned}
$$

Heston Model (1993)

$$\begin{cases} \frac{dS_t}{S_t} &= r_t dt + \sigma_t dw_t^1 \\ d\sigma_t^2 &= k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dw_t^2 \end{cases}$$

Explicit Solution for CIR Process: CTM

$$d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dw_t^2$$

$$\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1})) + \theta^2$$

$$\phi_t = \gamma^{-2}\int_0^t \{e^{k\phi_s}(\sigma_0^2 - \theta^2 + \tilde{w}^2(t)) + \theta^2 e^{2k\phi_s}\}^{-1} ds$$

Comparison: Solutions to the Three Models

$$S(t) = e^{\mu t}(S(0) + \tilde{W}(\phi_t^{-1}))$$

-GBM

$$S_t = e^{-at}\left[S_0 - L + \tilde{W}(\phi_t^{-1})\right] + L$$

-MRM

$$\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1})) + \theta^2$$

-Heston model

Volatility Swap for Heston Model. I.

A stock *volatility swap* is a forward contract on the annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R(S) - K_{vol}),$$

where $\sigma_R(S)$ is the realized stock volatility (quoted in annual terms) over the life of contract,

$$\sigma_R(S) := \sqrt{\frac{1}{T}\int_0^T \sigma_s^2 ds},$$

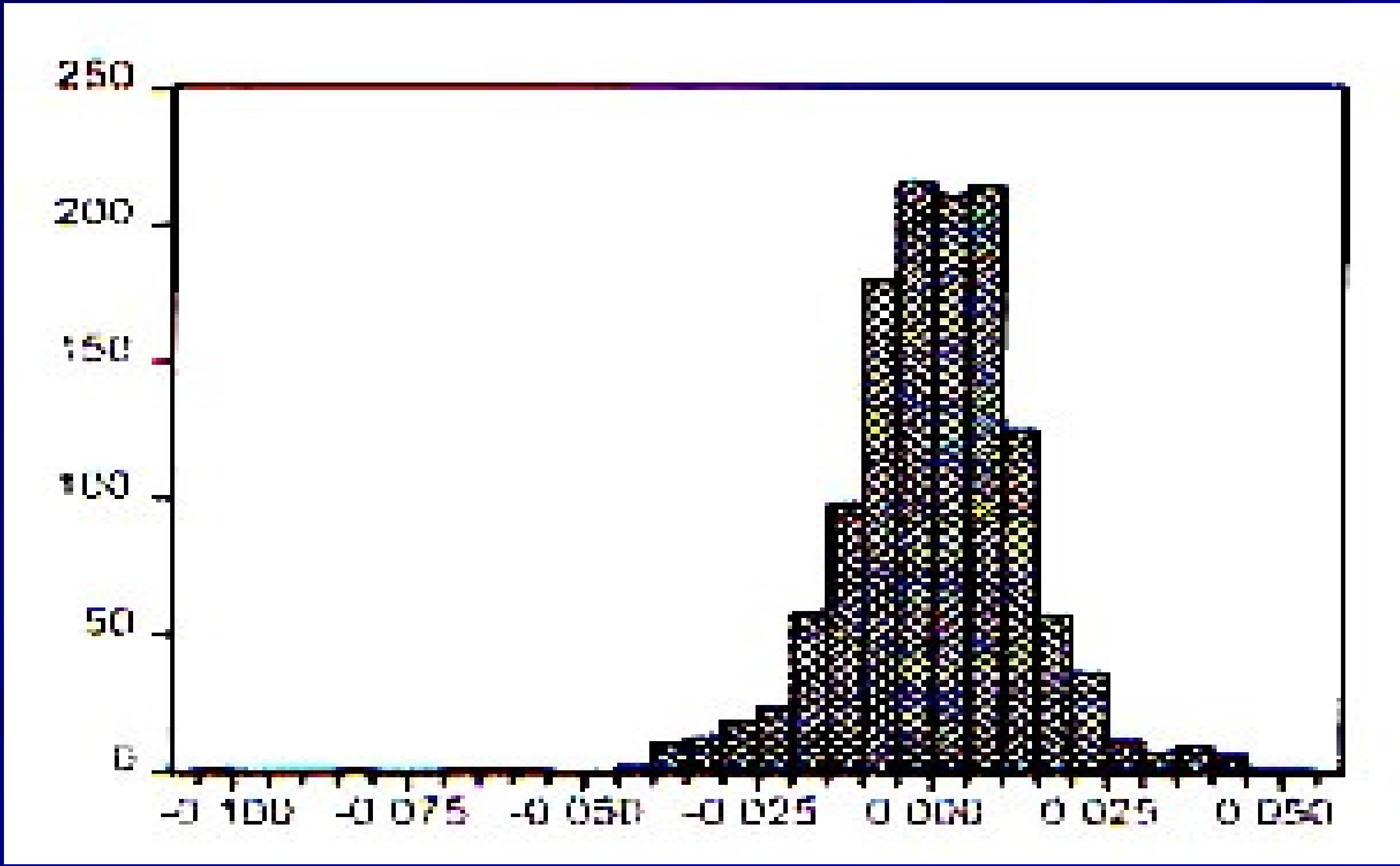
Why Trade Volatility (Variance)?

- Volatility Swaps allow investors to **profit** from the risks of an increase or decrease in future volatility of an index of securities or to **hedge** against these risks.
- If you think current volatility is low, for the right price you might want to take a position that profits if volatility increase.

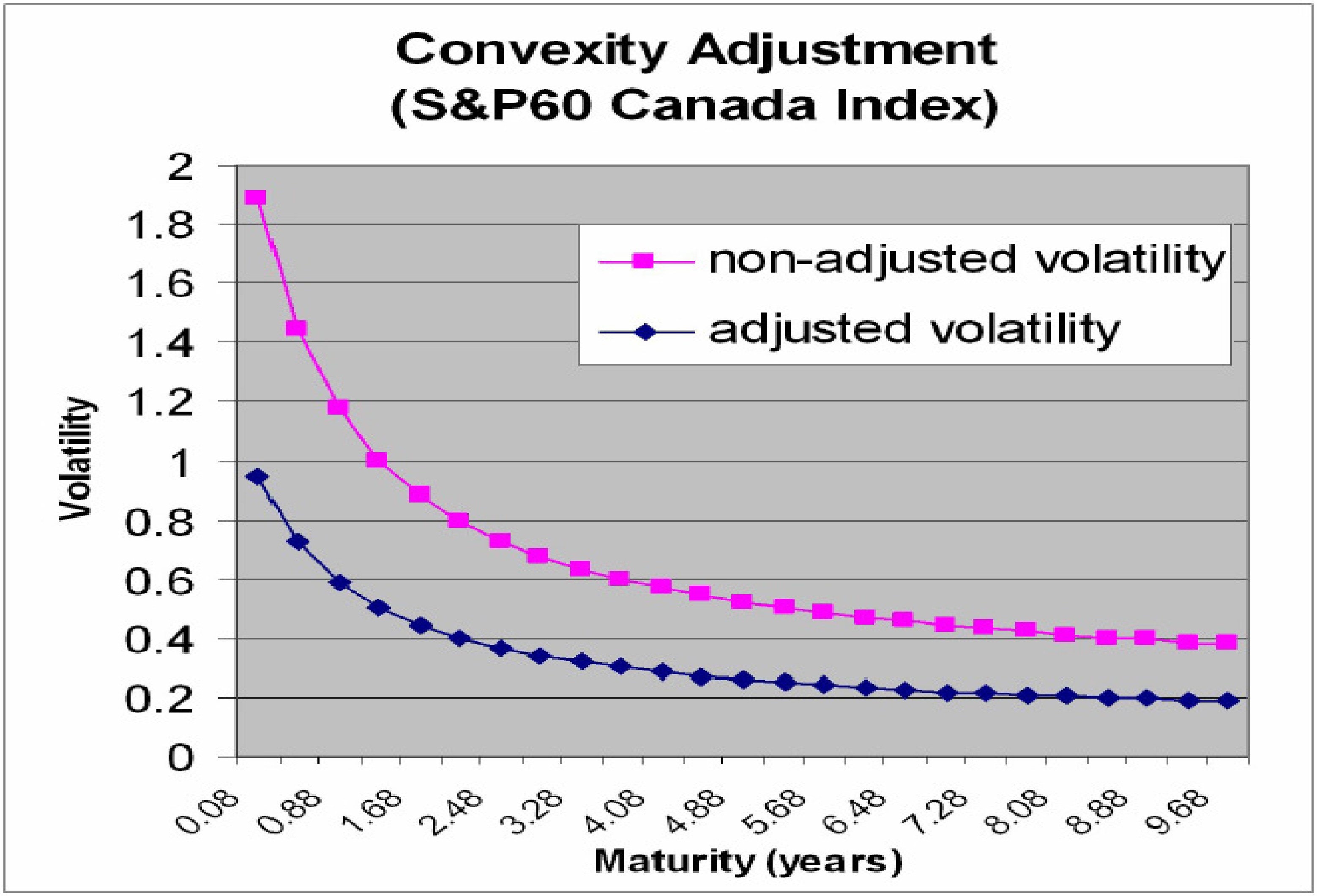
Statistics on Log Returns of S&P Canada Index (Jan 1997-Feb 2002)

Statistics on Log Returns $S\&P60$ Canada Index	
Series:	LOG RETURNS $S\&P60$ CANADA INDEX
Sample:	1 1300
Observations:	1300
Mean	0.000235
Median	0.000593
Maximum	0.051983
Minimum	-0.101108
Std. Dev.	0.013567
Skewness	-0.665741
Kurtosis	7.787327

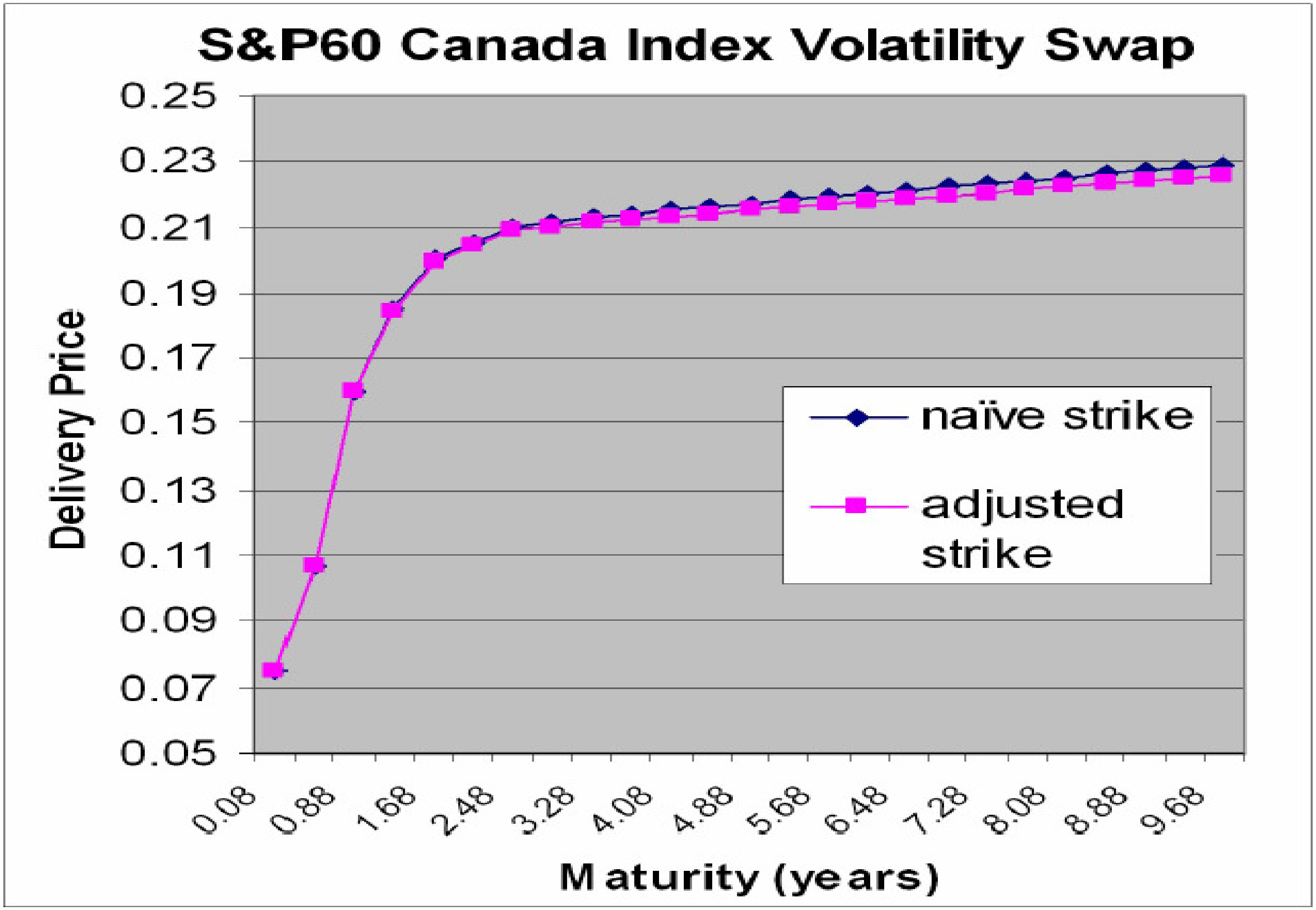
Histograms of Log-Returns for S&P60 Canada Index



Convexity Adjustment



S&P60 Canada Index Volatility Swap



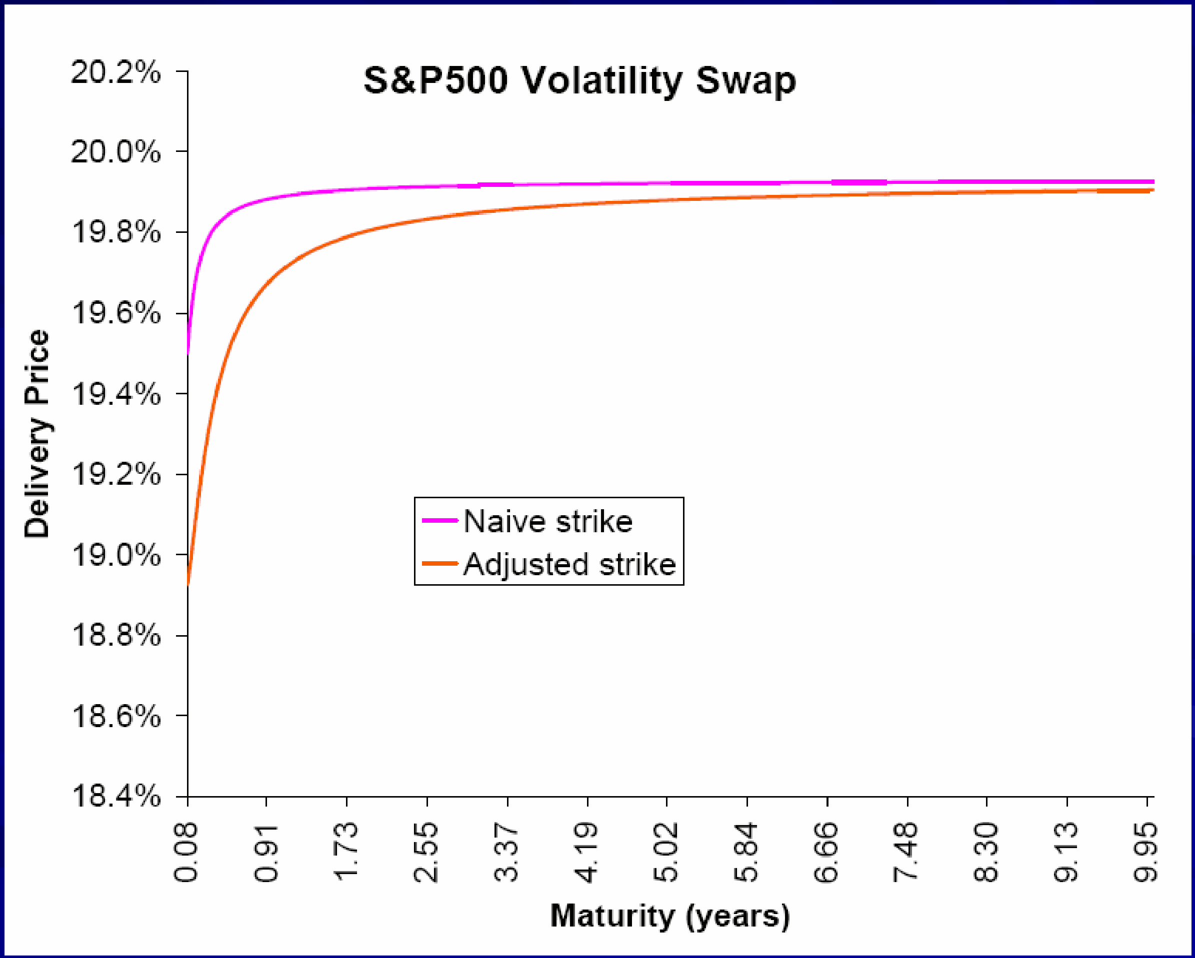
Wilmott, Javaheri & Haug (2002) Model

- **Wilmott, Javaheri & Haug** (GARCH and Volatility Swaps,Wilmott Magazine, 2002) Result

$$d\sigma = \theta(w - \sigma)dt + \xi\sigma dW$$

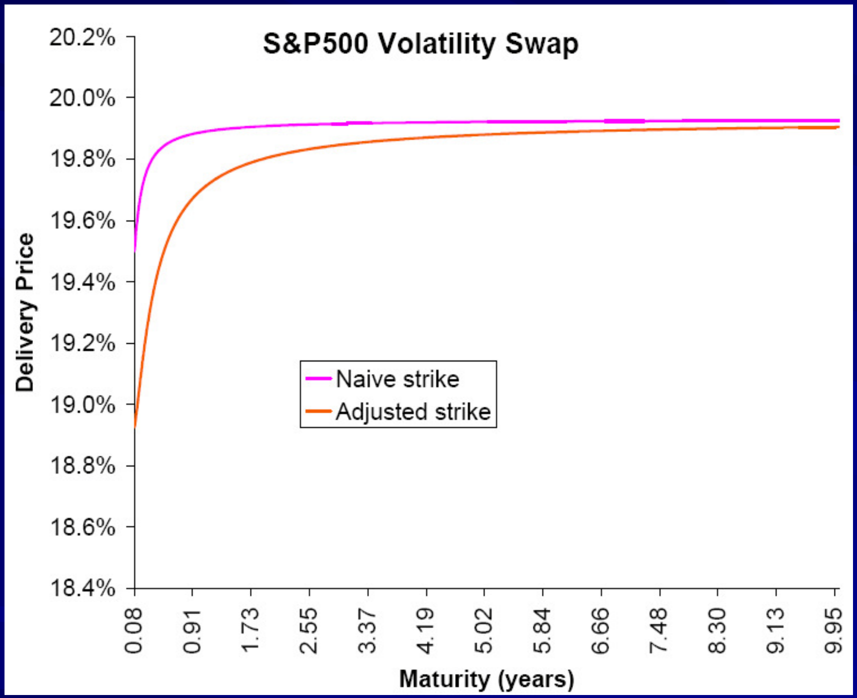
-**continuous-time GARCH(1,1) model**

Wilmott, Javaheri & Haug (2002) Volatility Swap

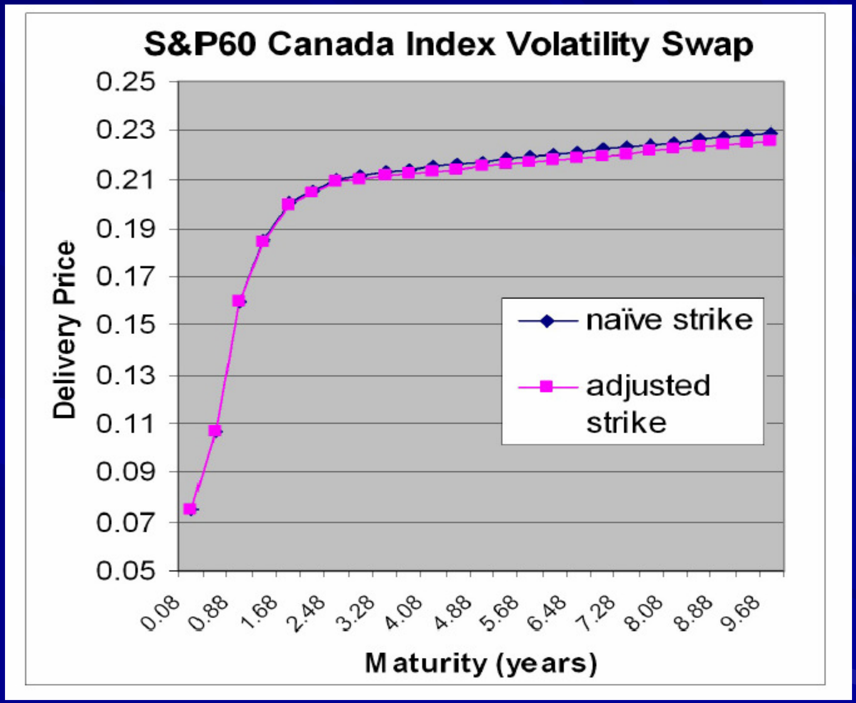


Comparison

Wilmott, Javaheri&Haug (2002)
Continuous-time GARCH(1,1)
S&P500

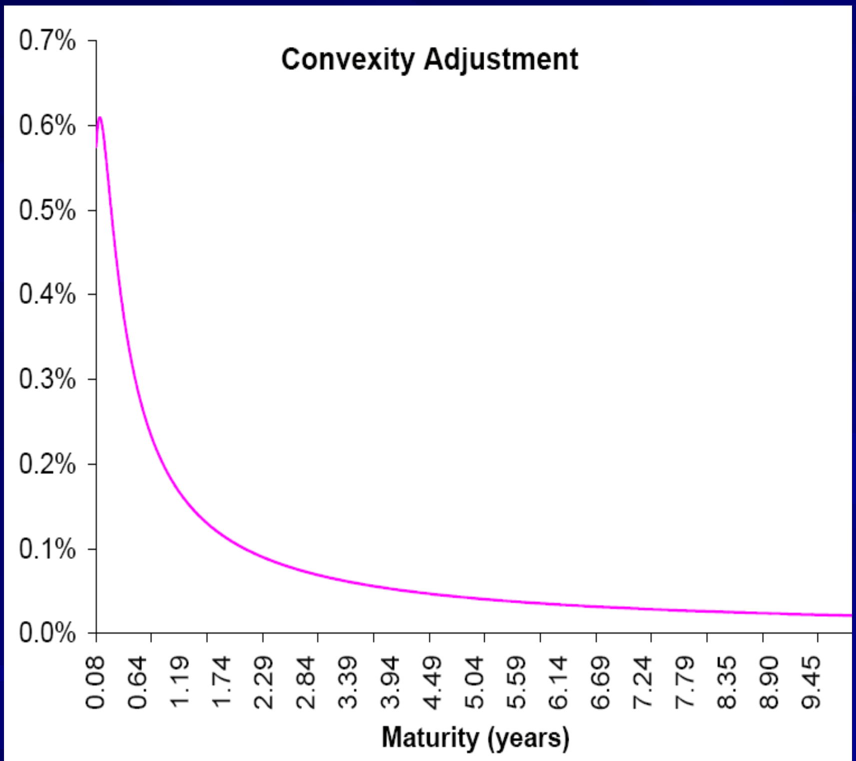


Sw (2004), Heston model,
S&P60 Canada Index

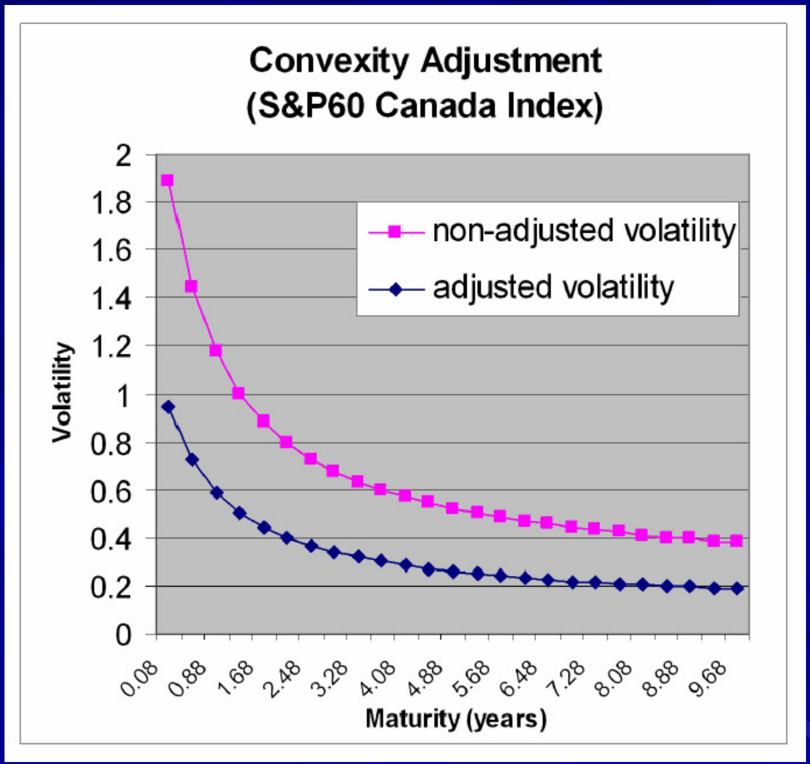


Comparison

Wilmott, Javaheri&Haug (2002) Convexity adjustment, S&P500



Sw (2004), Convexity adjustment, S&P60 Canada Index



Summary (SV and CT)

GBM Model

1. $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$

$$S(t) = e^{\mu t}(S(0) + \tilde{W}(\phi_t^{-1}))$$

$$\tilde{W}(\phi_t^{-1}) = S(0)(e^{\sigma W(t) - \frac{\sigma^2}{2}t} - 1)$$

-martingale

Mean-Reverting Model

2. $dS_t = a(L - S_t)dt + \sigma S_t dW_t$

$$S_t = e^{-at}[S_0 - L + \tilde{W}(\phi_t^{-1})] + L$$

$$\tilde{W}(\phi_t^{-1}) = S(0)(e^{\sigma W(t) - \frac{\sigma^2 t}{2}} - 1) + L(1 - e^{at}) + aLe^{\sigma W(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds$$

-sum of two martingales

Heston Model

3. $d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dw_t^2$

$$\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1})) + \theta^2$$

$\tilde{w}^2(\phi_t^{-1})$ -martingale

Problems. I.

$$d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dw_t^2$$

$$\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1})) + \theta^2$$

$\tilde{w}^2(\phi_t^{-1})$ -explicit expression ?

To calculate an option price for Heston model, for example

We know all the moments at this moment, though

Problems. II

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(s,S_s)dW(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t).$$

- Continuous-Time SV Model with Delay
- Solution by Change of Time Method?

The End

Thank you for your time and attention!

Home page:

http://www.math.ucalgary.ca/~aswish/

E-mails:

aswish@ucalgary.ca