Lévy-Based Interest Rate Derivatives: Change of Time Method and PIDEs

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Abstract: In this paper, we show how to calculate the price of zero-coupon bonds for many Gaussian and Lévy one-factor and multi-factor models of $r(t)$ using change of time method. These models include, in particular, Ornstein-Uhlenbeck (1930), Vasicek (1977), Cox-Ingersoll-Ross (1985), continuous-time GARCH, Ho-Lee (1986), Hull-White (1990) and Heath-Jarrow-Morton (1992) models and their various combinations. We also derive partial integro-differential equations (PIDEs) for the values of swaps, caps, floors and options on them, swaptions, captions and floortions, respectively. We apply the change of time method to price the interest rate derivatives for the interest rates $r(t)$ described by various stochastic differential equations driven by α -stable Lévy processes.

1 Introduction

Interest rate models are mainly used to price and hedge bonds and bond options. One of the biggest assumptions is that interest rates r are constant or, at least, known functions of time $r \equiv r(t)$. In reality this is far from the case. Many other securities that are influenced by interest rates have long duration. Their analysis in the presence of unpredictable interest rates is of crucial practical importance.

In this paper we will look at models for describing the stochastic behaviour of interest rates. Typically, the classical models are often based on assumptions that returns in the bond market are approximately normally distributed (Gaussian). For comprehensive introduction to interest rate modelling we refer to Bjork (1998), Brigo and Mercurio (2001), Filipović (2001), James and Webber (2000), Bingham and Kiesel (1998) and Rebonato (1996).

Empirical studies show that the Gaussian assumptions of the classical models do not hold in general. We will describe the Lévy-based models first introduced by Eberlein and Raible (1999). See also Raible (2000).

The main drawback of the Vasicek, the Cox-Ingersoll-Ross and other one-factor models lies in the fact that prices are explicit functions of the instantaneous 'spot' interest rate so that these models are unable to take the whole yield curve observed on the market into account in the price structure. Some authors have resorted to a two-dimensional analysis to improve the models in terms of discrepancies between short and long rates. See, for example, Brennan and Schwartz (1979), Schaefer and Schwartz (1984), Courtadon (1982).

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These more complex models do not lead to explicit formulae and require the solution of partial differential equations. The general concept of multifactor affine-yield models is developed in Duffie and Kan (1994a,1994b). The reduction of affine-yield models to canonical versions is due to Dai and Singleton (2000). Maghsoodi (1996) provides a detailed study of the one-dimensional CIR equation when the parameters are time-varying. Although affine-yield models have simple bond price formulas, the prices for fixed income derivatives are more complicated. However, numerical solution of partial differential equations can be avoided by Fourier transform analysis (see Duffie, Pan and Singleton (2000)). There are many other common short rate models, such as Black, Derman and Toy (1990), Black and Karasinski (1991), and Longstaff and Schwartz (2001). An empirical comparison of various short rate models is provided by Chan et al. (1992).

Ho and Lee (1986) have proposed a discrete-time model describing the behaviour of the whole yield curve. The continuous-time model is based on the same idea and has been introduced by Heath, Jarrow and Morton (HJM) (1987). They considered forward interest rates. Filipovich (2001) examines the issue of making the yield curves generated by the HJM model consistent with the scheme used to generate the initial yield curve. Jara (2000) considers an HJM-type model but for interest rate futures rather than forward interest rates.

The use of a log-normal simple interest rate to price caps and floors was worked out by Miltersen, Sandman and Sondermann (1997). This idea was embedded in a full forward LIBOR term-structure model by Brace, Gatarek and Musiela (1997). Recently, a variation on forward LIBOR models has been developed for swaps markets (see Jamshidian (1990)). Term-structure models with jumps have been studied by Bjork, Kabanov and Runggaldier (1997), Das (1999), Das and Foresi (1996), Glasserman and Kou (2003), Galsserman and Merener (2003) and Shirakawa (1991). Three relatively recent books by authors with practical experience in term-structure modeling are Pelsser (2000), Brigo and Mercurio (2001) and Rebonato (2002).

In this paper, we show how to calculate the price of zero-coupon bonds for many models of $r(t)$ using change of time method, Swishchuk (2004, 2007). These models include, in particular, Ornstein-Uhlenbeck (1930), Vasicek (1977), Cox-Ingersoll-Ross (1985), continuous-time GARCH, Ho-Lee (1986), Hull-White (1990) and Heath-Jarrow-Morton (1992) models. In the same way, we can calculate the values of swaps, caps, floors and options on them, swaptions, captions and floortions, respectively. We also apply the change of time method to price the interest rate derivatives for the interest rates $r(t)$ described by various stochastic differential equations driven by α -stable Lévy processes, see Sato (1999), Applebaum (2003), Schoutens (2003). The change of time method is convenient for presentation of a solution of a SDE. Sometimes one can not explicitly present the solution and uses numerical approaches. For example, there was no explicit solution's presentation for the CIR SDE (7). In Swishchuk (2004) the solution for this CIR SDE (7) was

presented for the first time using change of time method. Applying change of time method we can use the presentation for the solution to calculate many financial derivatives, for example variance and volatility swaps, options for mean-reverting models, etc. (see Swishchuk (2004, 2007)). This paper incorporates the change of time method for pricing of interest rate derivatives.

The rest of the paper is organized as follows. In section 2 we give an overview of the basics of bond pricing. One-factor and multi-factor stochastic interest rate models, both Gaussian and Lévy, are introduced in section 3. The change of time method in Gaussian and Lévy settings is described in section 4. Solutions to all models considered in section 3 using change of time method are stated in section 5. Bond pricing in Gaussian and Lévy setting using both change of time and partial integro-differential equations is studied in section 6. Various classes of interest rate derivatives, such as bond options, swaps, caps and floors, and options on them, swaptions, cvaptions and floortions, respectively, are introduced in section 7. Pricing of Gaussian and Lévy bond options is studied in section 8. Pricing of swaps, caps and floors for Gaussian and Lévy interest rate models is considered in section 9. Pricing of swaptions, captions and floortions in Gaussian and Lévy setting is studied in section 10. Section 11 concludes the paper. Appendix A contains one-factor and multi-factor Gaussian interest rate models and Appendix B gives the solutions to these models using change of time method.

2 Basics of Bond Pricing

2.1 Zero-Coupon Bonds

We begin with the subject of bond pricing. We do this under the assumption of a deterministic interest rate. This simplification allows us to discuss the effect of coupons on the prices of bonds and the appearance of the yield curve. A bond is a contract, paid for up-front, that yields a known amount on a known date in the future, the maturity date, $t = T$. The bond may also pay a known cash dividend (the coupon) at fixed times during the life of the contract. If there is no coupon the bond is known as a zero-coupon bond. In this way, we call zero-coupon bond a security paying \$1 at a maturity date T and we note $P(t, T)$ the value of this security at time t . We have $P(T, T) = 1$ and in the world where the future is certain $P(t, T) = \exp(-\int_t^T r(s)ds)$, where $r(t)$ is the instantaneous interest rate.

2.2 Bond Pricing with Known Interest Rates

Let $V(t, T)$ be the value of the bond contract, $t < T$. If the interest rate $r(t)$ and coupon payment $K(t)$ are known functions of time, the bond price is also a function of time only: $V = V(t, T)$. If this bond pays the owner Z at time $t = T$ then we know that $V(T, T) = Z$. We now derive an equation for

the value of the bond at a time before maturity, $t < T$.

Suppose we hold one bond. Then arbitrage considerations lead us to the following equation:

$$\frac{dV}{dt} + K(t) = r(t)V. \quad (1)$$

The right-hand side is the return we would have received had we converted our bond into cash at time t .

2.3 Yield Curve

Interest rates are not deterministic. For short-dated derivatives products such as options the errors associated with assuming a deterministic, or even constant, rate are small, typically 2%. In dealing with products with a longer lifespan we must address the problem of random interest rate. The first step is to decide on a suitable measure for future values of interest rates, one that enables traders to communicate effectively about the same quantity. In the previous section we have seen a definition, (4), that gives an interest rate from bond price data but this relies on bond prices being differentiable with respect to the maturity date.

The yield curve is another measure of future values of interest rates. With the value of zero-coupon bonds $V(t, T)$ taken from real data, define

$$Y(t, T) = -\frac{\log(V(t, T)/V(T, T))}{T - t}, \quad (2)$$

where t is the current time.

The yield curve is the plot of Y against time to maturity $T - t$. The dependence of the yield curve on the time to maturity is called the *term structure of interest rate*.

2.4 The Short Rate

The short rate $r(t) := Y(t, t) = \lim_{T \rightarrow t} Y(t, T)$ is the rate on instantaneous borrowing and lending. Historically, it was the short rate which was modelled as the basic process. In practice, this rate is stochastic and can fluctuate over time. Note that the short rate is actually a theoretical entity which does not exist in real life and can not be directly observed. A sum of 1 invested in the short rate at time zero and continuously rolled over, i.e., instantaneously reinvested, is called the money-market account. Its value $S_0(t) = \exp[\int_0^t r(s)ds]$. If r is deterministic and constant, $S_0(t)$ reduces to the classical bank account: $S_0(t) = B(t) = \exp[rt]$.

2.5 The Instantaneous Forward Rate

We define the instantaneous forward rate to be

$$f(t, T) := -\frac{\partial \log P(t, T)}{\partial T}. \quad (3)$$

The function $f(t, T)$ corresponds to the rate we can contract for at time t on a riskless loan that begins at time T and is returned an instant later. Since

$$P(t, T) = \exp\left[-\int_t^T f(t, s)ds\right], \quad (4)$$

zero-coupon bond prices and forward rates represent equivalent information. Note that the short rate $r(t)$ is contained in this forward rate structure since $r(t) = f(t, t)$.

2.6 Stochastic Interest Rates

For an uncertain future, one must think of the interest rate $r(t)$ in terms of a random or stochastic process (short rate). We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ and filtration $\mathcal{F}_t$ is the natural filtration of a standard Brownian motion $(W_t)_{0 \leq t \leq T}$ and $\mathcal{F}_T = \mathcal{F}$.

In view of our uncertainty about the future of the interest rate, it is natural to model it as a random variable. We should specify that r is the interest rate received by the shortest possible deposit. The interest rate for the shortest possible deposit is commonly called the spot rate.

We consider a model for the interest rate r that is governed by a SDE of the form

$$dr(t) = a(r, t)dt + b(r, t)dW(t), \quad (5)$$

where W is a standard Wiener process,
or

$$dr(t) = a(r, t)dt + b(r, t)dL(t), \quad (6)$$

where L is a Lévy process, and a and b are ‘good’ functions.

A basic model for the behaviour of the stochastic process r is the CIR model (Cox-Ingersoll-Ross (1985)). It is based on the CIR process:

$$dr(t) = k(\theta - r(t))dt + \gamma\sqrt{r}dW(t). \quad (7)$$

Under this model, the interest rate is mean reverting: it fluctuate around a long-term mean θ . In the literature, this model was generalized in different ways. Well-known alternatives are the Vasicek and the Ho-Lee models, together with their extensions. We refer to Bjork (1998).

We consider a ‘riskless’ asset $S_t^0 = \exp(\int_0^t r(s)ds)$, where $r(t)$ is an adapted process, such that $\int_0^T |r(s)|ds < +\infty$, almost surely. The risky assets are the zero-coupon bonds with maturity less or equal to the horizon T . For each $u \leq T$, we define adapted process $P(t, u)$, $0 \leq t \leq u$, satisfying $P(u, u) = 1$, giving the price of the zero-coupon bond with maturity u as a function of time. We note that $P(t, u) = E^*[\exp(-\int_t^u r(s)ds)|\mathcal{F}_t]$, where E^* is an expectation with respect to the risk-neutral probability P^* .

We note, that for each maturity u , there is an adapted process $\sigma(t, u), 0 \leq t \leq u$, such that, on $[0, u]$,

$$\frac{dP(t, u)}{P(t, u)} = (r(t) - \sigma(t, u)q(t))dt + \sigma(t, u)dW(t), \quad (8)$$

where $q(t)$ is an adapted process corresponding to 'risk premium' and $W(t)$ is a Wiener process under physical measure P . For example, for the Vasicek model $q(t) = -\lambda, \lambda \in R$, for the CIR model $q(t) = -\alpha\sqrt{r(t)}, \alpha \in R$.

The last formula is to be related with the equality $dS_0(t) = r(t)S_0(t)dt$, satisfied by the so-called riskless asset. It is the term in $dW(t)$ which makes the bond riskier.

Under risk-neutral probability P^* , the process $W^*(t)$ defined by $W^*(t) := W(t) - \int_0^t q(s)ds$ is a standard Brownian motion (Girsanov theorem), and we have

$$\frac{dP(t, u)}{P(t, u)} = r(t)dt + \sigma(t, u)dW^*(t). \quad (9)$$

(See Lamperton, Lapeyre (1993))

3 Lévy Stochastic Interest Rate Models (SIRMs)

Throughout the paper, we set $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in R_+}, P)$ for a filtered probability space, Jacod and Shiryaev (1987).

In this section we give an overview on one-factor and multi-factor Lévy interest rate models.

3.1 One-Factor Lévy SIRMs

DEFINITION 1. By **LÉVY PROCESS** we define a process with stationary and independent increments, Sato (1999), Applebaum (2003), Schoutens (2003).

DEFINITION 2. Let $\alpha \in (0, 2]$. An α -STABLE LÉVY PROCESS L such that L_1 (or equivalently any L_t) has a strictly α -stable distribution (i.e., $L_1 \equiv S_\alpha(\sigma, \beta, \nu)$) for some $\alpha \in (0, 2] \setminus \{1\}$, $\sigma \in R_+$, $\beta \in [-1, 1]$, $\nu = 0$ or $\alpha = 1$, $\sigma \in R_+$, $\beta = 0$, $\nu \in R$). (See Sato (1999)). We call L a **SYMMETRIC α -STABLE LÉVY PROCESS** if the distribution of L_1 is even symmetric α -stable (i.e., $L_1 \equiv S_\alpha(\sigma, 0, 0)$ for some $\alpha \in (0, 2]$, $\sigma \in R_+$.) A process L is called $(T_t)_{t \in R_+}$ -adapted if L is constant on $[T_{t-}, T_t]$ for any $t \in R_+$.

$L(t)$ below is a symmetric α -stable Lévy process with triplet $(\gamma, \sigma, \nu(dy))$. Similar to section 3.1 definitions of various processes defined via SDE driven by Brownian motion, we define below various processes via SDE driven by α -stable Lévy process.

1. GEOMETRIC α -STABLE LÉVY MOTION. $dr(t) = \mu r(t)dt + \sigma r(t)dL(t)$.
2. ORNSTEIN-UHLENBECK PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = -\mu r(t)dt + \sigma dL(t)$.
3. VASIČEK PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = \mu(b - r(t))dt + \sigma dL(t)$.
4. CONTINUOUS-TIME GARCH PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = \mu(b - r(t))dt + \sigma r(t)dL(t)$.
5. COX-INGERSOLL-ROSS PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = k(\theta - r(t))dt + \gamma\sqrt{r(t)}dL(t)$.
6. HO AND LEE PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = \theta(t)dt + \sigma dL(t)$.
7. HULL AND WHITE PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = (a(t) - b(t)r(t))dt + \sigma(t)dL(t)$.
8. HEATH, JARROW AND MORTON PROCESS DRIVEN BY α -STABLE LÉVY MOTION. Define the forward interest rate $f(t, s)$, for $t \leq s$, characterized by the following equality $P(t, u) = \exp[-\int_t^u f(t, s)ds]$ for any maturity u . $f(t, s)$ represents the instantaneous interest rate at time s as ‘anticipated’ by the market at time t . It is natural to set $f(t, t) = r(t)$. The process $f(t, u)_{0 \leq t \leq u}$ satisfies an equation $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dL(v)$, where the processes a and b are continuous.

3.2 Multi-Factor Lévy SIRM

Multi-factor models driven by α -stable Lévy motions can be obtained using various combinations of above-mentioned processes, see subsection 3.3. We give one example of two-factor continuous-time GARCH model driven by α -stable Lévy motions.

$$\begin{cases} dS(t) &= \mu(b(t) - S(t))dt + \sigma S(t)dL^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dL^2(t), \end{cases} \quad (10)$$

where L^1, L^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Also, we can consider various combinations of models, presented above, i.e., mixed models containing Brownian and Lévy motions. For example,

$$\begin{cases} dS(t) &= \mu(b(t) - S(t))dt + \sigma S(t)dL(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW(t), \end{cases} \quad (11)$$

where Brownian motion $W(t)$ and Lévy process $L(t)$ may be correlated.

Remark. Corresponding one-factor and multi-factor Gaussain interest rate models may be found in Appendix A.

4 Change of Time Method

DEFINITION 3. A TIME CHANGE is a right-continuous increasing $[0, +\infty]$ -valued process $(T_t)_{t \in R_+}$ such that T_t is a stopping time for any $t \in R_+$. By

$\hat{\mathcal{F}}_t := \mathcal{F}_{T_t}$ we define the time-changed filtration $(\hat{\mathcal{F}}_t)_{t \in R_+}$. The INVERSE TIME CHANGE $(\hat{T}_t)_{t \in R_+}$ is defined as $\hat{T}_t := \inf\{s \in R_+ : T_s > t\}$. (See Ikeda and Watanabe (1983)).

4.1 Change of Time Method (CTM) for SDE driven by Brownian motion

We consider the following SDE driven by a Brownian motion:

$$dX(t) = a(t, X(t))dW(t), \quad (12)$$

where $W(t)$ is a Brownian motion and $a(t, X)$ is a continuous and measurable by t and X function on $[0, +\infty) \times R$.

The reason to consider this equation is the following one: if we solve the equation, then we can solve and more general equation with a drift $\beta(t, X)$ by drift transformation method or Girsanov transformation (see Ikeda and Watanabe (1981), Chapter 4, Section 4). We use the following result to get the solution of the previous equation using change of time method.

Theorem. (Ikeda and Watanabe (1981), Chapter IV, Theorem 4.3) Let $\hat{W}(t)$ be an one-dimensional $\mathcal{F}_t$ -Wiener process with $\hat{W}(0) = 0$, given on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and let $X(0)$ be an $\mathcal{F}_0$ -adopted random variable. Define a continuous process $V = V(t)$ by the equality

$$V(t) = X(0) + \hat{W}(t). \quad (13)$$

Let T_t be the change of time process (see Section 2.3):

$$T_t = \int_0^t a^{-2}(T_s, X(0) + \hat{W}(s))ds. \quad (14)$$

If

$$X(t) := V(\hat{T}_t) = X(0) + \hat{W}(\hat{T}_t) \quad (15)$$

and $\hat{\mathcal{F}}_t := \mathcal{F}_{\hat{T}_t}$, then there exists $\hat{\mathcal{F}}_t$ -adopted Wiener process $W = W(t)$ such that $(X(t), W(t))$ is a solution of (2.1) on probability space $(\Omega, \mathcal{F}, \hat{\mathcal{F}}_t, P)$, where $\hat{T}_t$ is the inverse to T_t time change.

4.2 Change of Time Method (CTM) for SDE driven by Lévy motion

We denote by $L_{a.s.}^\alpha$ the family of all real measurable $\mathcal{F}_t$ -adapted processes a on $\Omega \times [0, +\infty)$ such that for every $T > 0$, $\int_0^T |a(t, \omega)|^\alpha dt < +\infty$ a.s. We consider the following SDE driven by a Lévy motion:

$$dX(t) = a(t, X(t))dL(t), \quad (16)$$

with the same reason as above for SDE with a Brownian motion as a driven process.

We are going to use the following result to get the solutions of SDE driven by Lévy processes.

Theorem. (Rosinski and Woyczynski (1986), Theorem 3.1., p.277). Let $a \in L_{a.s.}^\alpha$ be such that $T(u) := \int_0^u |a|^\alpha dt \rightarrow +\infty$ a.s. as $u \rightarrow +\infty$. If $\hat{T}(t) := \inf\{u : T(u) > t\}$ and $\hat{\mathcal{F}}_t = \mathcal{F}_{\hat{T}(t)}$, then the time-changed stochastic integral $\hat{L}(t) = \int_0^{\hat{T}(t)} a dL(t)$ is an $\hat{\mathcal{F}}_t - \alpha$ -stable Lévy process, where $L(t)$ is $\mathcal{F}_t$ -adapted and $\mathcal{F}_t$ - α -stable Lévy process. Consequently, a.s. for each $t > 0$ $\int_0^t a dL = \hat{L}(T(t))$, i.e., the stochastic integral with respect to a α -stable Lévy process is nothing but another α -stable Lévy process with randomly changed time scale.

5 Solutions Lévy SIRM using CTM

5.1 Solution of One-Factor Lévy SIRM Using CTM

Below we give the solutions to some one-factor Lévy IRM decribed by SDE driven by α -stable Lévy process. $L(t)$ below is a symmetric α -stable Lévy process, and $\hat{L}$ is a $(\hat{T}_t)_{t \in R_+}$ -adapted symmetric α -stable Levy process on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t \in R_+}, P)$.

1. GEOMETRIC α -STABLE LÉVY MOTION. $dr(t) = \mu r(t)dt + \sigma r(t)dL(t)$. Solution $r(t) = e^{\mu t}[r(0) + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t [r(0) + \hat{L}(\hat{T}_s)]^\alpha ds$.

2. ORNSTEIN-UHLENBECK PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = -\mu r(t)dt + \sigma dL(t)$. Solution $r(t) = e^{-\mu t}[r(0) + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t (e^{\mu s}[r(0) + \hat{L}(\hat{T}_s)])^\alpha ds$.

3. VASIĆEK PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = \mu(b - r(t))dt + \sigma dL(t)$. Solution $r(t) = e^{-\mu t}[r(0) - b + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t (e^{\mu s}[r(0) - b + \hat{L}(\hat{T}_s)] + b)^\alpha ds$.

4. CONTINUOUS-TIME GARCH PROCESS DRIVEN BY α -STABLE LÉVY PROCESS. $dr(t) = \mu(b - r(t))dt + \sigma r(t)dL(t)$. Solution $r(t) = e^{-\mu t}(r(0) - b + \hat{L}(\hat{T}_t)) + b$, where $\hat{T}_t = \sigma^\alpha \int_0^t [r(0) - b + \hat{L}(\hat{T}_s) + e^{\mu s}b]^\alpha ds$.

5. COX-INGERSOLL-ROSS PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = k(\theta^2 - r(t))dt + \gamma\sqrt{r(t)}dL(t)$. Solution $r^2(t) = e^{-kt}[r_0^2 - \theta^2 + \hat{L}(\hat{T}_t)] + \theta^2$, where $T_t = \gamma^{-\alpha} \int_0^t [e^{kT_s}(r_0^2 - \theta^2 + \hat{W}(s)) + \theta^2 e^{2kT_s}]^{-\alpha/2} ds$.

6. HO AND LEE PROCESS DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = \theta(t)dt + \sigma dL(t)$. Solution $r(t) = r(0) + \hat{L}(\sigma^2 t) + \int_0^t \theta(s)ds$.

7. HULL AND WHITE DRIVEN BY α -STABLE LÉVY MOTION. $dr(t) = (a(t) - b(t)r(t))dt + \sigma(t)dL(t)$. Solution $r(t) = \exp[-\int_0^t b(s)ds][r(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \int_0^t \sigma^2(s)[r(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T}_s) + \exp[\int_0^s b(u)du] \frac{a(s)}{b(s)}] ds$.

8. HEATH, JARROW AND MORTON DRIVEN BY α -STABLE LÉVY MOTION. $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dL(v)$. Solution $f(t, u) = f(0, u) + \hat{L}(\hat{T}_t) + \int_0^t a(v, u)dv$, where $\hat{T}_t = \int_0^t b^2(f(0, u) + \hat{L}(\hat{T}_s) + \int_0^s a(v, u)dv)ds$.

5.2 Solution of Multi-Factor Lévy SIRMs Using CTM

Solution of multi-factor models driven by α -stable Lévy motions can be obtained using various combinations of solutions of the above-mentioned processes, see subsection 5.3, and CTM. We give one example of two-factor continuous-time GARCH model driven by α -stable Lévy motions:

$$\begin{cases} dr(t) &= \mu(b(t) - r(t))dt + \sigma r(t)dL^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dL^2(t), \end{cases} \quad (17)$$

where L^1, L^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$. Solution, using CTM for the first and the second equations, subsection 5.3: $r(t) = e^{-\mu t}[r(0) - e^{\xi t}(b(0) + \hat{L}^2(\hat{T}_t^2)) + \hat{L}^1(\hat{T}_t^1)] + e^{\xi t}[b(0) + \hat{L}^2(T_t^2)]$, where $\hat{T}^i$ are defined in 4. ($i = 1$) and 1. ($i = 2$), respectively, subsection 5.3.

Remark. Solutions of one-factor and multi-factor Gaussian SIRMs may be found in Appendix B.

6 Bond Pricing

Pricing a bond is harder than pricing an option even in Gaussian case, since there is no underlying asset with which to hedge: you can not buy an interest rate of, say, 15%. The only alternative in Gaussian case is the case with bonds of different maturity dates. To calculate the value of a European options with maturity θ on the zero-coupon bond with maturity equal to T we could proceed as follows. If it is a call with strike price K , the value of the option at time θ is $\max(P(\theta, T) - K, 0)$ and it seems reasonable to hedge this call with a portfolio of riskless asset S_t^0 and zero-coupon bond $P(\theta, T)$ with maturity T . Using the classical arguments (and using the money-market account as numeraire), the price at time 0 of a European call option on a bond is given by

$$V(\theta, T, r) = E^*\left[\frac{1}{S_0(t)} \max(P(\theta, T) - K, 0)\right], \quad (18)$$

where E^* is the expectation under the risk-neutral measure, $S_0(t) = \exp[\int_0^t r(s)ds]$ and $P(\theta, T)$ is defined in section 2.6.

6.1 Gaussian Bond Pricing for One-Factor SIRMs via CTM

The solution of the SDE (8) or (9) can be written in the form

$$\begin{aligned} P(t, T) &= P(0, T) \exp\left[\int_0^t r(s)ds\right] \frac{\exp[\int_0^t \sigma(s, T)d\hat{W}(s)]}{E\{\exp[\int_0^t \sigma(s, T)d\hat{W}(s)]\}} \\ &= P(0, T) \exp\left[\int_0^t r(s)ds\right] \exp\left[\int_0^t \sigma(s, T)d\hat{W}(s) - \frac{1}{2} \int_0^t \sigma(s, T)^2 ds\right], \end{aligned} \quad (19)$$

where $r(t)$ has one of the representations in section 5.1.

We see that the log returns under risk-neutral measure approximately follow a Normally distributed random variable.

Remark. Empirical studies (see, for example, Raible (2000)) show that the normality assumption does not reflect reality. Empirically observed log returns of bonds turn out to have leptokurtic distribution.

6.2 Gaussian Bond Pricing for One-Factor SIRMs via PDE

Consider $V(t, T, r)$ -bond price at time t , where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dW(t). \quad (20)$$

For example, for GBM, $a = \mu r$ and $b = \sigma r$, for OU process, $a = -\mu r$ and $b = \sigma$, and so on (see section 3.1).

The zero-coupon bond pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 V}{\partial r^2} + (a - \lambda b) \frac{\partial V}{\partial r} - rV = 0, \quad (21)$$

where the function λ is often called the market price of risk. The final condition is

$$V(T, T, r) = Z.$$

This PDE may be solved approximately by standard numerical methods, see, for example, Wilmott, Howison and Dewynne (1995).

Remark. To price options on bonds with coupon, the reader is referred to Jamshidian (1989) and El Karoui and Rochet (1989).

6.3 Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform

We model the zero-coupon bond price with the following process (see Eberlein and Raible (1999), Raible (2000)):

$$P(t, T) = P(0, T) \exp\left[\int_0^t r(s)ds\right] \frac{\exp\left[\int_0^t \sigma(s, T)dL(s)\right]}{E\{\exp\left[\int_0^t \sigma(s, T)dL(s)\right]\}},$$

where $r(t)$ has one of the representations in section 5.3.

Eberlein and Raible (1999) derived the bond price process in the form

$$P(t, T) = P(0, T) \exp\left[\int_0^t r(s)ds\right] \frac{\exp\left[\int_0^t \sigma(s, T)dL(s)\right]}{\exp\left[\int_0^t \theta(\sigma(s, T))dL(s)\right]},$$

where $\theta(u) := \log(E[\exp(uL(1))])$ denotes the logarithm of the moment-generating function of the Lévy process at time 1. For example, in the

classical Gaussian model we choose $\theta(u) = u^2/2$ and $L(s) = W(s)$. We note, that we know the expressions for $r(t)$ in the above formula for many SIRMs, see section 5.3.

Except when $L(t)$ is a Poisson or a Brownian motion, our Lévy market model is an incomplete model. It means that there are many different equivalent martingales measures to choose. In general this leads to many different possible prices for European options or bond options, etc.

One of the way to price bond is to use for the P^* in (19) the Esscher transform equivalent martingale measure. Following Gerber and Shiu (1994), we can by using the so-called Esscher transform find in some cases at least one equivalent martingale measure P^* . Let $f(t, x)$ be the density of our model's (real world, i.e. under P) distribution of $L(t)$. For some real number $\theta \in \{\theta \in R \mid \int_{-\infty}^{+\infty} \exp(\theta y) f(t, y) dy < +\infty\}$ we can define a new density

$$f^{(\theta)}(t, x) = \frac{\exp(\theta x) f(t, x)}{\int_{-\infty}^{+\infty} \exp(\theta y) f(t, y) dy}.$$

In order to assume finiteness of the expectation in the denominator above in the case of general Lévy processes, we assume that

$$\int_{\{|x|>1\}} \exp[vx] \nu(dx) < \infty, \quad \text{for } |v| < (1 + \epsilon)M,$$

where $\epsilon > 0$ and M is such that $0 \leq \sigma(s, T) \leq M$ (a.s.) (see (8) and (9)) for $0 \leq s \leq T$ and $\nu(dx)$ is the Lévy measure of L_1 . Typical choices of this Lévy process are the variance gamma, the normal inverse Gaussian, the generalized hyperbolic, the Meixner or CGMY processes (see Schoutens (2003)). Another way to price bond is to consider characteristic function (or Fourier transform), if it is known, of the risk-neutral log returns (see Carr and Madan (1998)). We note, that if we know the explicit expression for $r(t)$ (see section 5.3), then we can find the characteristic function of the risk-neutral bond price.

6.4 Lévy Bond Pricing via PIDE

One more way to price bond is to consider the solution of a partial differential integral equation with boundary condition, all in terms of the triplet of Lévy characteristics $[\gamma, \sigma, \nu^{P^*}(dy)]$ of the Lévy process under the risk-neutral measure P^* .

Consider $V(t, T, r)$ -bond price at time t , where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dL(t), \tag{22}$$

$L(t)$ is a Lévy process.

For example, for GBM, $a = \mu r$ and $b = \sigma r$, for OU process, $a = -\mu r$ and $b = \sigma$, and so on (see section 3.1).

The zero-coupon bond pricing equation is

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V}{\partial r^2} + (a + \gamma b - \lambda b\sigma)\frac{\partial V}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [V(t, r + by) - V(t, r) - by\frac{\partial V(t, r)}{\partial r}] \nu(dy) - rV = 0, \end{aligned} \quad (23)$$

where λ is the *market price of risk*. The final condition is

$$V(T, T, r) = Z.$$

This PDIE is the analogue of the famous Black-Scholes PDE and follows from the Feynman-Kac formula for Lévy processes.

Since there is no closed solution in general we must resort to approximate methods. We may pose the PIDE in a form that becomes amenable to a solution using the finite difference method. A number of schemes is discussed in Duffy (2005). He considers a) implicit and explicit methods, b) implicit-explicit Runge-Kutta methods, c) operator splitting (method of fractional steps) and d) splitting in conjunction with predictor-corrector methods. See also Cont and Voltchkova (2003, 2005).

Remark. European call option price for the stock price that follows exponential Lévy process was derived by Nualart and Schoutens (2001) and Raible (2000).

7 Interest Rate Derivatives

There is a large number of different interest rate derivative products. Here, we consider options, swaps, caps and floors. Having valued swaps, caps and floors we can value options on these instruments: swaptions, captions, floortions.

A bond option is the option on bond and is identical to an equity option except that the underlying asset is a bond.

Both European and American versions exist.

An interest rate swap is an agreement between two parties to exchange the interest rate payments on a certain amount, the principal, for a certain length of time.

One party A pays the other B a fixed rate of interest rate in return for a variable interest rate payment. For example, A pays 10% of \$1,000,000 p.a. to B and B pays r (variable) of the same amount to A . This agreement is to last for three years, say.

A cap is a loan at the floating interest rate but with the proviso that the interest rate charged is guaranteed not to exceed a specified value, the cap, which is denoted by r^* .

A floor is similar to a cap except that the interest rate does not go below r^* .

Swaption is an option on swap. Caption is an option on cap. Floortion is an option on floor.

8 Pricing of Gaussian and Lévy Bond Options

8.1 Pricing of Gaussian Bond Options

Let interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dW(t), \quad (24)$$

where $W(t)$ is a standard Wiener process.

Consider the European call bond option, with exercise price K and expiry date T , on a zero-coupon bond with maturity date $T_B \geq T$.

To find the value of the call option on bond (to buy a bond) we proceed with the following steps:

1. To find the value of the bond: $V_B(r, t; T_B)$, that satisfies the following PDE:

$$\frac{\partial V_B}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 V_B}{\partial r^2} + (a - \lambda b) \frac{\partial V_B}{\partial r} - rV_B = 0. \quad (25)$$

with the final condition

$$V_B(r, T_B; T_B) = Z.$$

2. Let $C_B(r, t)$ be the value of the call option on this bond. Since C_B also depends on the random walk $r(t)$, it must satisfy equation (1) too:

$$\frac{\partial C_B}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 C_B}{\partial r^2} + (a - \lambda b) \frac{\partial C_B}{\partial r} - rC_B = 0. \quad (26)$$

with the final condition

$$C_B(r, T) = \max(V_B(r, T; T_B) - K, 0).$$

Remark. These PDEs can be solved numerically using standard methods, see Wilmott, Howison and Dewynne (1995).

8.2 Pricing of Lévy Bond Options

Let interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dL(t), \quad (27)$$

where $L(t)$ is a Lévy process.

Consider the European Call Bond Option, with exercise price K and expiry date T , on a zero-coupon bond with maturity date $T_B \geq T$.

To find the value of the call option on bond (to buy a bond) we proceed with the following steps:

1. To find the value of the bond: $V_B(r, t; T_B)$, that satisfies the following PIDE:

$$\begin{aligned} \frac{\partial V_B}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_B}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_B}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [V_B(t, r + by) - V_B(t, r) - by\frac{\partial V_B(t, r)}{\partial r}] \nu(dy) - rV_B = 0. \end{aligned} \quad (28)$$

with the final condition

$$V_B(r, T_B; T_B) = Z.$$

2. Let $C_B(r, t)$ be the value of the call option on this bond. Since C_B also depends on the random walk $r(t)$, it must satisfy equation (27) too:

$$\begin{aligned} \frac{\partial C_B}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_B}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_B}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [C_B(t, r + by) - C_B(t, r) - by\frac{\partial C_B(t, r)}{\partial r}] \nu(dy) - rC_B = 0. \end{aligned} \quad (29)$$

with the final condition

$$C_B(r, T) = \max(V_B(r, T; T_B) - K, 0). \quad (30)$$

Remark. One of the approach to solve this PDIE could be numerical using different finite difference methods, see Duffy (2005).

9 Pricing of Swaps, Caps and Floors

9.1 Pricing of Swaps, Caps and Floors for Gaussian IRMs

Let interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dW(t), \quad (31)$$

where $W(t)$ is a standard Wiener process.

Pricing of Swaps. We consider to value such swaps in general. Suppose that A pays the interest on an amount Z to B at a fixed rate r^* and B pays interest to A at the floating rate r . These payments continue until time T_S . Denote the value of this swap to A by $ZV_S(r, t)$. We note, that in a time-step dt A receives $(r - r^*)Zdt$. If we think of this payment as being similar to a coupon payment on a simple bond then we find that:

$$\frac{\partial V_S}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V_S}{\partial r^2} + (a - \lambda b)\frac{\partial V_S}{\partial r} - rV_S + (r - r^*) = 0. \quad (32)$$

with the final condition

$$V_S(r, T_S) = 0.$$

We note, that r can be greater or less than r^* and so $V_S(r, t)$ need not be positive.

Pricing of Caps. The loan of Z is to be paid back at time T_C . The value of the capped loan, $ZV_C(r, t)$ satisfies the following PDE:

$$\frac{\partial V_C}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 V_C}{\partial r^2} + (a - \lambda b) \frac{\partial V_C}{\partial r} - rV_C + \min(r, r^*) = 0. \quad (33)$$

with the final condition

$$V_C(r, T_C) = 1.$$

Pricing of Floors. The value of the floored loan, $ZV_F(r, t)$, satisfies the following PDE:

$$\frac{\partial V_F}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 V_F}{\partial r^2} + (a - \lambda b) \frac{\partial V_F}{\partial r} - rV_F + \max(r, r^*) = 0. \quad (34)$$

with the final condition

$$V_F(r, T_F) = 1,$$

where T_F is an expiry time for floor.

Remark. These PDE can be solved numerically using standard methods, see Wilmott, Howison and Dewynne (1995).

9.2 Pricing of Swaps, Caps and Floors for Lévy IRMs

Consider $V(r, t)$ -bond price at time t , where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dL(t), \quad (35)$$

$L(t)$ is a Lévy process.

Pricing of Swaps. We consider to value such swaps in general. Suppose that A pays the interest on an amount Z to B at a fixed rate r^* and B pays interest to A at the floating rate r . These payments continue until time T_S . Denote the value of this swap to A by $ZV_S(r, t)$. We note, that in a time-step dt A receives $(r - r^*)Zdt$. If we think of this payment as being similar to a coupon payment on a simple bond then we find that:

$$\begin{aligned} \frac{\partial V_S}{\partial t} &+ \frac{1}{2}b^2\sigma^2 \frac{\partial^2 V_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma) \frac{\partial V_S}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [V_S(t, r + by) - V_S(t, r) - by \frac{\partial V_S(t, r)}{\partial r}] \nu(dy) \\ &- rV_S + (r - r^*) = 0. \end{aligned} \quad (36)$$

with the final condition

$$V_S(r, T_S) = 0.$$

We note, that r can be greater or less than r^* and so $V_S(r, t)$ need not be positive.

Pricing of Caps. The loan of Z is to be paid back at time T_C . The value of the capped loan, $ZV_C(r, t)$ satisfies the following PIDE:

$$\begin{aligned} \frac{\partial V_C}{\partial t} &+ \frac{1}{2}b^2\sigma^2 \frac{\partial^2 V_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma) \frac{\partial V_C}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [V_C(t, r + by) - V_C(t, r) - by \frac{\partial V_C(t, r)}{\partial r}] \nu(dy) \\ &- rV_C + \min(r, r^*) = 0. \end{aligned} \quad (37)$$

with the final condition

$$V_C(r, T_C) = 1.$$

Pricing of Floors. The value of the floored loan, $ZV_F(r, t)$, satisfies the following PDE:

$$\begin{aligned} \frac{\partial V_F}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_F}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_F}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [V_F(t, r + by) - V_F(t, r) - by\frac{\partial V_F(t, r)}{\partial r}] \nu(dy) \\ &- rV_F + \max(r, r^*) = 0. \end{aligned} \quad (38)$$

with the final condition

$$V_F(r, T_F) = 1,$$

where T_F is an expiry time for floor.

Remark. One of the approach to solve thes PDIEs could be numerical using different finite difference methods, see Duffy (2005).

10 Pricing of Swaptions, Captions and Floor-tions

10.1 Pricing of Swaptions, Captions and Floortions for Gaussian IRMs

Let interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dW(t), \quad (39)$$

where $W(t)$ is a standard Wiener process.

Pricing of Swaptions. Consider European Swap Call Option, option to buy this swap (a call swaption) for an amount K at time $T < T_S$, where T_S is an expiry time for swap with value $V_S(r, t)$, $t \leq T_S$. Thus, this value V_S satisfies the following PDE (see (32)):

$$\frac{\partial V_S}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V_S}{\partial r^2} + (a - \lambda b)\frac{\partial V_S}{\partial r} - rV_S + (r - r^*) = 0. \quad (40)$$

with the final condition

$$V_S(r, T_S) = 0.$$

Then the value $C_S(r, t)$ of this call swap option (call swaption) satisfies the following PDE:

$$\frac{\partial C_S}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 C_S}{\partial r^2} + (a - \lambda b)\frac{\partial C_S}{\partial r} - rC_S = 0. \quad (41)$$

with the final condition

$$C_S(r, T) = \max(V_S(r, T) - K, 0). \quad (42)$$

We solve for the value of the swap first and then use this value as the final data for the value of the swaption.

Pricing of Captions. Consider European Cap Call Option, option to buy this cap (a call caption) for an amount K at time $T < T_C$, where T_C is an expiry time for cap with value $V_C(r, t)$, $t \leq T_C$. Thus, this value V_C satisfies the following PDE (see (33)):

$$\frac{\partial V_C}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 V_C}{\partial r^2} + (a - \lambda b) \frac{\partial V_C}{\partial r} - rV_C + \min(r, r^*) = 0. \quad (43)$$

with the final condition

$$V_C(r, T_C) = 1.$$

Then the value $C_C(r, t)$ of this call cap option (call caption) satisfies the following PDE:

$$\frac{\partial C_C}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 C_C}{\partial r^2} + (a - \lambda b) \frac{\partial C_C}{\partial r} - rC_C = 0. \quad (44)$$

with the final condition

$$C(r, T) = \max(V_C(r, T) - K, 0). \quad (45)$$

We solve for the value of the cap first and then use this value as the final data for the value of the caption.

Pricing of Floortions. Consider European Floor Call Option, option to buy this floor (a call floortion) for an amount K at time $T < T_F$, where T_F is an expiry time for floor with value $V_F(r, t)$, $t \leq T_F$. Thus, this value V_F satisfies the following PDE (see (34)):

$$\frac{\partial V_F}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 V_F}{\partial r^2} + (a - \lambda b) \frac{\partial V_F}{\partial r} - rV_F + \min(r, r^*) = 0. \quad (46)$$

with the final condition

$$V_C(r, T_F) = 1.$$

Then the value $C_F(r, t)$ of this call floor option (call floortion) satisfies the following PDE:

$$\frac{\partial C_F}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 C_F}{\partial r^2} + (a - \lambda b) \frac{\partial C_F}{\partial r} - rC_F = 0. \quad (47)$$

with the final condition

$$C_F(r, T) = \max(V_F(r, T) - K, 0). \quad (48)$$

We solve for the value of the floor first and then use this value as the final data for the value of the floortion.

Remark. These PDE can be solved numerically using standard methods, see Wilmott, Howison and Dewynne (1995).

10.2 Pricing of Swaptions, Captions and Floortions for Lévy IRMs

Consider $V(r, t)$ -bond price at time t , where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dL(t), \quad (50)$$

$L(t)$ is a Lévy process.

Pricing of Swaptions. Consider European Swap Call Option, option to buy this swap (a call swaption) for an amount K at time $T < T_S$, where T_S is an expiry time for swap with value $V_S(r, t)$, $t \leq T_S$. Thus, this value V_S satisfies the following PIDE (see (36)):

$$\begin{aligned} \frac{\partial V_S}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_S}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [V_S(t, r + by) - V_S(t, r) - by\frac{\partial V_S(t, r)}{\partial r}]\nu(dy) \\ &- rV_S + (r - r^*) = 0. \end{aligned} \quad (51)$$

with the final condition

$$V_S(r, T_S) = 0.$$

Then the value $C_S(r, t)$ of this call swap option (call swaption) satisfies the following PIDE:

$$\begin{aligned} \frac{\partial C_S}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_S}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [C_S(t, r + by) - C_S(t, r) - by\frac{\partial C_S(t, r)}{\partial r}]\nu(dy) - rC_S = 0. \end{aligned} \quad (52)$$

with the final condition

$$C_S(r, T) = \max(V_S(r, T) - K, 0). \quad (53)$$

We solve for the value of the swap first and then use this value as the final data for the value of the swaption.

Pricing of Captions. Consider European Cap Call Option, option to buy this cap (a call caption) for an amount K at time $T < T_C$, where T_C is an expiry time for cap with value $V_C(r, t)$, $t \leq T_C$. Thus, this value V_C satisfies the following PIDE (see (37)):

$$\begin{aligned} \frac{\partial V_C}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_C}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [V_C(t, r + by) - V_C(t, r) - by\frac{\partial V_C(t, r)}{\partial r}]\nu(dy) \\ &- rV_C + \min(r, r^*) = 0. \end{aligned} \quad (54)$$

with the final condition

$$V_C(r, T_C) = 1.$$

Then the value $C_C(r, t)$ of this call cap option (call caption) satisfies the following PDE:

$$\begin{aligned} \frac{\partial C_C}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_C}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [C_C(t, r + by) - C_C(t, r) - by\frac{\partial C_C(t, r)}{\partial r}]\nu(dy) - rC_C = 0. \end{aligned} \quad (55)$$

with the final condition

$$C(r, T) = \max(V_C(r, T) - K, 0). \quad (56)$$

We solve for the value of the cap first and then use this value as the final data for the value of the caption.

Pricing of Floortions. Consider European Floor Call Option, option to buy this floor (a call floortion) for an amount K at time $T < T_F$, where T_F is an expiry time for floor with value $V_F(r, t)$, $t \leq T_F$. Thus, this value V_F satisfies the following PIDE (see (38)):

$$\begin{aligned} \frac{\partial V_F}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_F}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_F}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [V_F(t, r + by) - V_F(t, r) - by\frac{\partial V_F(t, r)}{\partial r}]\nu(dy) \\ &- rV_F + \min(r, r^*) = 0. \end{aligned} \quad (57)$$

with the final condition

$$V_C(r, T_F) = 1.$$

Then the value $C_F(r, t)$ of this call floor option (call floortion) satisfies the following PIDE:

$$\begin{aligned} \frac{\partial C_F}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_F}{\partial r^2} + (a + \gamma - \lambda b\sigma)\frac{\partial C_F}{\partial r} \\ &+ \int_{-\infty}^{+\infty} [C_F(t, r + y) - C_F(t, r) - y\frac{\partial C_F(t, r)}{\partial r}]\nu(dy) - rC_F = 0. \end{aligned} \quad (58)$$

with the final condition

$$C_F(r, T) = \max(V_F(r, T) - K, 0). \quad (59)$$

We solve for the value of the floor first and then use this value as the final data for the value of the floortion.

Remark. One of the approach to solve these PDIEs could be numerical using different finite difference methods, see Duffy (2005).

11 Conclusion

We discussed how to calculate the price of zero-coupon bonds for many Gaussian and Lévy one-factor and multi-factor models of $r(t)$ using change of time method. These models include, in particular, Ornstein-Uhlenbeck (1930), Vasicek (1977), Cox-Ingersoll-Ross (1985), continuous-time GARCH, Ho-Lee (1986), Hull-White (1990) and Heath-Jarrow-Morton (1992) models and their various combinations. We also derive PDIE for the values of swaps, caps, floors and options on them, swaptions, captions and floortions, respectively. We apply the change of time method to price the interest rate derivatives for the interest rates $r(t)$ described by various stochastic differential equations driven by α -stable Lévy processes. We could also apply the same techniques, i.e., change of time and PIDE, to price many interest rate derivatives for multi-factor Gaussian and Lévy interest rate models. But this discussion is outside the scope of this paper and will be considered in the future research paper, as well as the numerical solutions of PIDEs presented in the paper.

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12 Appendix A: One-Factor and Multi-Factor Gaussian Interest Rate Models

12.1 One-Factor Gaussian SIRMs

1. THE GEOMETRIC BROWNIAN MOTION MODEL (RENDLEMAN AND BARTTER (1980)). $dr(t) = \mu r(t)dt + \sigma r(t)dW(t)$.

2. THE ORNSTEIN-UHLENBECK (1930) MODEL. $dr(t) = -\mu r(t)dt + \sigma dW(t)$,

3. THE VASIČEK (1977) MODEL. $dr(t) = \mu(b - r(t))dt + \sigma dW(t)$.

4. THE CONTINUOUS-TIME GARCH MODEL. $dr(t) = \mu(b - r(t))dt + \sigma r(t)dW(t)$.

5. THE COX-INGERSOLL-ROSS (1985) MODEL. $dr(t) = k(\theta - r(t))dt + \gamma\sqrt{r}dW(t)$.

6. THE HO AND LEE (1986) MODEL. $dr(t) = \theta(t)dt + \sigma dW(t)$.

7. THE HULL AND WHITE (1990) MODEL. $dr(t) = (a(t) - b(t)r(t))dt + \sigma(t)dW(t)$

8. THE HEATH, JARROW AND MORTON (1987) MODEL. Define the forward interest rate $f(t, s)$, for $t \leq s$, characterized by the following equality $P(t, u) = \exp[-\int_t^u f(t, s)ds]$ for any maturity u . $f(t, s)$ represents the instantaneous interest rate at time s as ‘anticipated’ by the market at time t . It is natural to set $f(t, t) = r(t)$. The process $f(t, u)_{0 \leq t \leq u}$ satisfies an equation $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dW(v)$, where the processes a and b are continuous. We note, that the last SDE may be written in the following form: $df(t, u) = b(f(t, u))(\int_t^u b(f(t, s))ds + b(f(t, u))d\hat{W}(t)$, where $\hat{W}(t) = W(t) - \int_0^t q(s)ds$ and $q(t) = \int_t^u b(f(t, s))ds - \frac{a(t, u)}{b(f(t, u))}$.

12.2 Multi-Factor Gaussian SIRMs

Multi-factor models driven by Brownian motions can be obtained using various combinations of above-mentioned processes. We give one example of two-factor continuous-time GARCH SIRM:

$$\begin{cases} dr(t) &= \mu(b(t) - r(t))dt + \sigma r(t)dW^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW^2(t), \end{cases}$$

where W^1, W^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

13 Appendix B: Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models

13.1 Solution of One-Factor Gaussian SIRMs Using CTM

We use change of time method (see Ikeda and Watanabe (1981)) to get the solutions to the following below equations (see Swishchuk (2007)). $W(t)$ below is an standard Brownian motion, and $\hat{W}$ is a $(\hat{T}_t)_{t \in R_+}$ -adapted standard Brownian motion on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t \in R_+}, P)$.

1. GEOMETRIC BROWNIAN MOTION. $dr(t) = \mu r(t)dt + \sigma r(t)dW(t)$. Solution $r(t) = e^{\mu t}[r(0) + \hat{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t [r(0) + \hat{W}(\hat{T}_s)]^2 ds$.

2. ORNSTEIN-UHLENBECK PROCESS. $dr(t) = -\mu r(t)dt + \sigma dW(t)$, Solution $r(t) = e^{-\mu t}[r(0) + \hat{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t (e^{\mu s}[r(0) + \hat{W}(\hat{T}_s)])^2 ds$.

3. VASIĆEK PROCESS. $dr(t) = \mu(b - r(t))dt + \sigma dW(t)$, solution $r(t) = e^{-\mu t}[r(0) - b + \hat{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t (e^{\mu s}[r(0) - b + \hat{W}(\hat{T}_s)] + b)^2 ds$.

4. CONTINUOUS-TIME GARCH PROCESS. $dr(t) = \mu(b - r(t))dt + \sigma r(t)dW(t)$. Solution $r(t) = e^{-\mu t}(r(0) - b + \hat{W}(\hat{T}_t)) + b$, where $\hat{T}_t = \sigma^2 \int_0^t [r(0) - b + \hat{W}(\hat{T}_s) + e^{\mu s}b]^2 ds$.

5. COX-INGERSOLL-ROSS PROCESS. $dr^2(t) = k(\theta - r^2(t))dt + \gamma r(t)dW(t)$, solution $r^2(t) = e^{-kt}[r_0^2 - \theta^2 + \hat{W}(\hat{T}_t)] + \theta^2$, where $\hat{T}_t = \gamma^{-2} \int_0^t [e^{kT_s}(r_0^2 - \theta^2 + \hat{W}(\hat{T}_s)) + \theta^2 e^{2kT_s}]^{-1} ds$.

6. HO AND LEE PROCESS. $dr(t) = \theta(t)dt + \sigma dW(t)$. Solution $r(t) = r(0) + \hat{W}(\sigma^2 t) + \int_0^t \theta(s)ds$.

7. HULL AND WHITE. $dr(t) = (a(t) - b(t)r(t))dt + \sigma(t)dW(t)$. Solution $r(t) = \exp[-\int_0^t b(s)ds][r(0) - \frac{a(s)}{b(s)} + \hat{W}(\hat{T}_t)]$, where $\hat{T}_t = \int_0^t \sigma^2(s)[r(0) - \frac{a(s)}{b(s)} + \hat{W}(\hat{T}_s) + \exp[\int_0^s b(u)du] \frac{a(s)}{b(s)}] ds$.

8. HEATH, JARROW AND MORTON. $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dW(v)$. Solution $f(t, u) = f(0, u) + \hat{W}(\hat{T}_t) + \int_0^t a(v, u)dv$, where $\hat{T}_t = \int_0^t b^\alpha(f(0, u) + \hat{W}(\hat{T}_s) + \int_0^s a(v, u)dv)ds$.

13.2 Solution of Multi-Factor Gaussian SIRMs Using CTM

Solution of multi-factor models driven by Brownian motions can be obtained using various combinations of solutions of the above-mentioned processes, see subsection 5.1, and CTM. We give one example of two-factor Continuous-Time GARCH model driven by Brownian motions:

$$\begin{cases} dr(t) &= \mu(b(t) - r(t))dt + \sigma r(t)dW^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW^2(t), \end{cases}$$

where W^1, W^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Solution, using CTM for the first and the second equations, subsection 13.1: $r(t) = e^{-\mu t}[r(0) - e^{\xi t}(b(0) + \hat{W}^2(\hat{T}_t^2)) + \hat{W}^1(\hat{T}_t^1)] + e^{\xi t}[b(0) + \hat{W}^2(T_t^2)]$, where $\hat{T}^i$ are defined in 4. ($i = 1$) and 1. ($i = 2$), respectively, subsection 13.1. Here, $W^1(t)$ and $W^2(t)$ are independent.

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