Variance and Volatility Swaps in Energy Markets *

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Intro: Aim of the Talk

This talk is devoted to the pricing of variance and volatility swaps in energy market. We found explicit variance swap formula and closed form volatility swap formula (using Brockhaus-Long approximation) for energy asset with stochastic volatility that follows continuous-time GARCH (1,1) model (mean-reverting) (or Pilipović one-factor model). A numerical example is presented as well.

Intro: Energy Commodities

Commodities are emerging as an asset class in their own. The range of products offered to investors range from exchange traded funds (ETFs) to sophisticated products including principal protected structured notes on individual commodities or baskets of commodities and commodity range-accrual or varinace swap.

More and more institutional investors are including commodities in their asset allocation mix and hedge funds are also increasingly active players in commodities. Example: Amaranth Advisors lost USD 6 billion during September 2006 from trading natural gas futures contracts, leading to the fund's demise.

Intro: Energy Commodities

Concurrent with these developments, a number of recent papers have examined the risk and return characteristics of investments in individual commodity futures or commodity indices composed of baskets of commodity futures. See, e.g., Erb and Harvey (2006), Gorton and Rouwenhorst (2006), Ibbotson (2006), Kan and Oomen (2007).

However, since all but the most plain-vanilla investments contain an exposure to volatility, it is equally important for investors to understand the risk and return characteristics of commodity volatilities.

Intro: Energy Commodities

Our focus on energy commodities derives from two resons:

1) energy is the most important commodity sector, and crude oil and natural gas constitute the largest components of the two most widely tracked commodity indices: the Standard & Poors Goldman Sachs Commodity Index (S&P GSCI) and the Dow Jones-AIG Commodity Index (DJ-AIGCI).

2) existence of a liquid options market: crude oil and natural gas indeed have the deepest and most liquid options marketss among all commodities.

The idea is to use variance (or volatility) swaps on futures contracts.

Intro: Energy Commodities

At maturity, a variance swap pays off the difference between the realized variance of the futures contract over the life of the swap and the fixed variance swap rate.

And since a variance swap has zero net market value at initiation, absence of arbitrage implies that the fixed variance swap rate equals to conditional risk-neutral expectation of the realized variance over the life of swap.

Therefore, e.g., the time-series average of the payoff and/or excess return on a variance swap is a measure of the variance risk premium.

Intro: Energy Commodities

Variance risk premia in energy commodities, crude oil and natural gas, has been considered by A. Trolle and E. Schwartz (2009).

The same methodology as in Trolle & Schwartz (2009) was used by Carr & Wu (2009) in their study of equity variance risk premia. The idea was to use variance swaps on futures contracts.

Intro: Energy Commodities

The study in Trolle&Schwartz (2009) is based on daily data from January 2, 1996 until November 30, 2006-a total of 2750 business days. The source of the data is NYMEX.

Trolle & Schwartz (2009) found that:

Intro: Energy Commodities

1) the average variance risk premia are negative for both energy commodities but more strongly statistically significant for crude oil than for natural gas;

2) the natural gas variance risk premium (defined in dollars terms or in return terms) is higher during the cold months of the year (seasonality and peaks for natural gas variance during the cold months of the year);

3) energy risk premia in dollar terms are time-varying and correlated with the level of the variance swap rate. In contrast, energy variance risk premia in return terms, paerticularly in the case of natural gas, are much less correlated with the varinace swap rate.

Intro: Energy Commodities

The S&P GSCI is comprised of 24 commodities with the weight of each commodity determined by their relative levels of world production over the past five years.

The DJ-AIGCI is comprised of 19 commodities with the weight of each component detrmined by liquidity and world production values, with liquidity being the dominant factor.

Crude oil and natural gas are the largest components in both indices. In 2007, their weight were 51.30% and 6.71%, respectively, in the S&P GSCI and 13.88% and 11.03%, respectively, in the DJ-AIGCI.

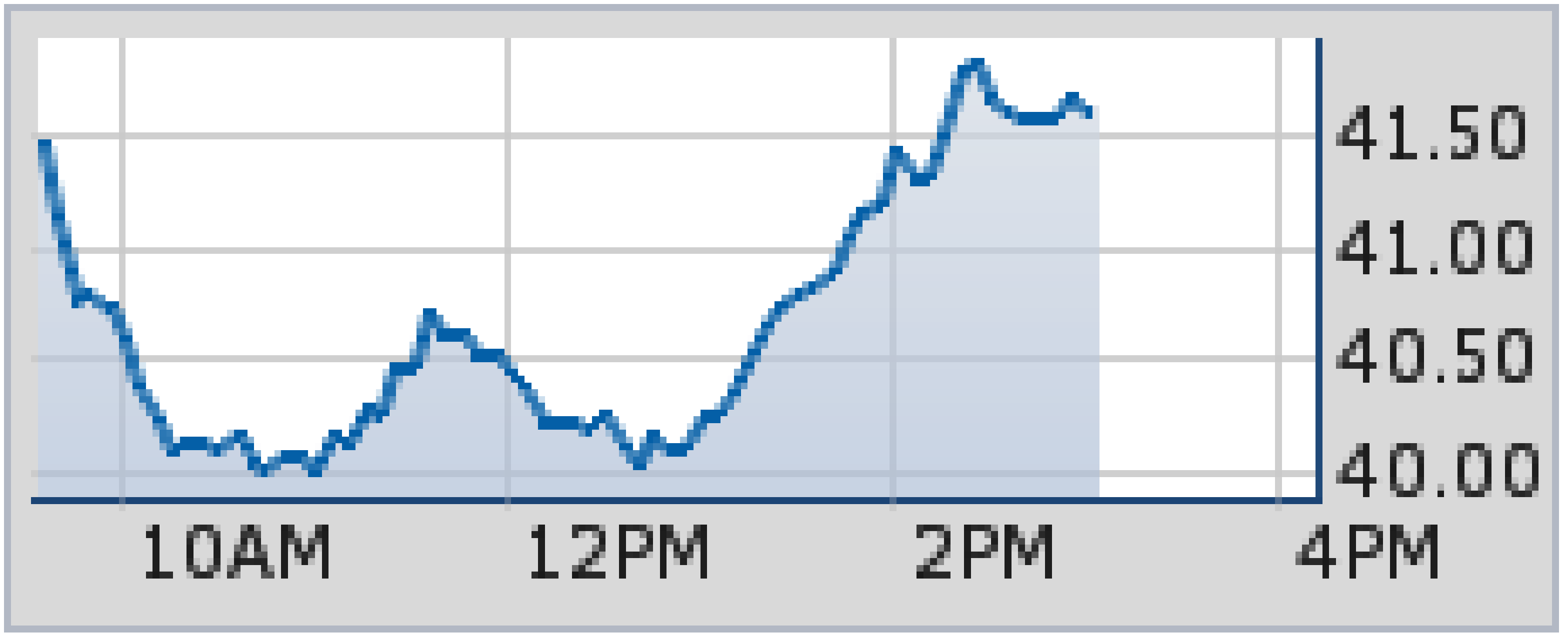
Intro: Energy Commodities-OVX Index

The Chicago Board Optiopns Exchange (CBOE) recently introduced a Crude Oil Volatility Index (ticker symbol OVX).

Intro: Energy Commodities-OVX Index

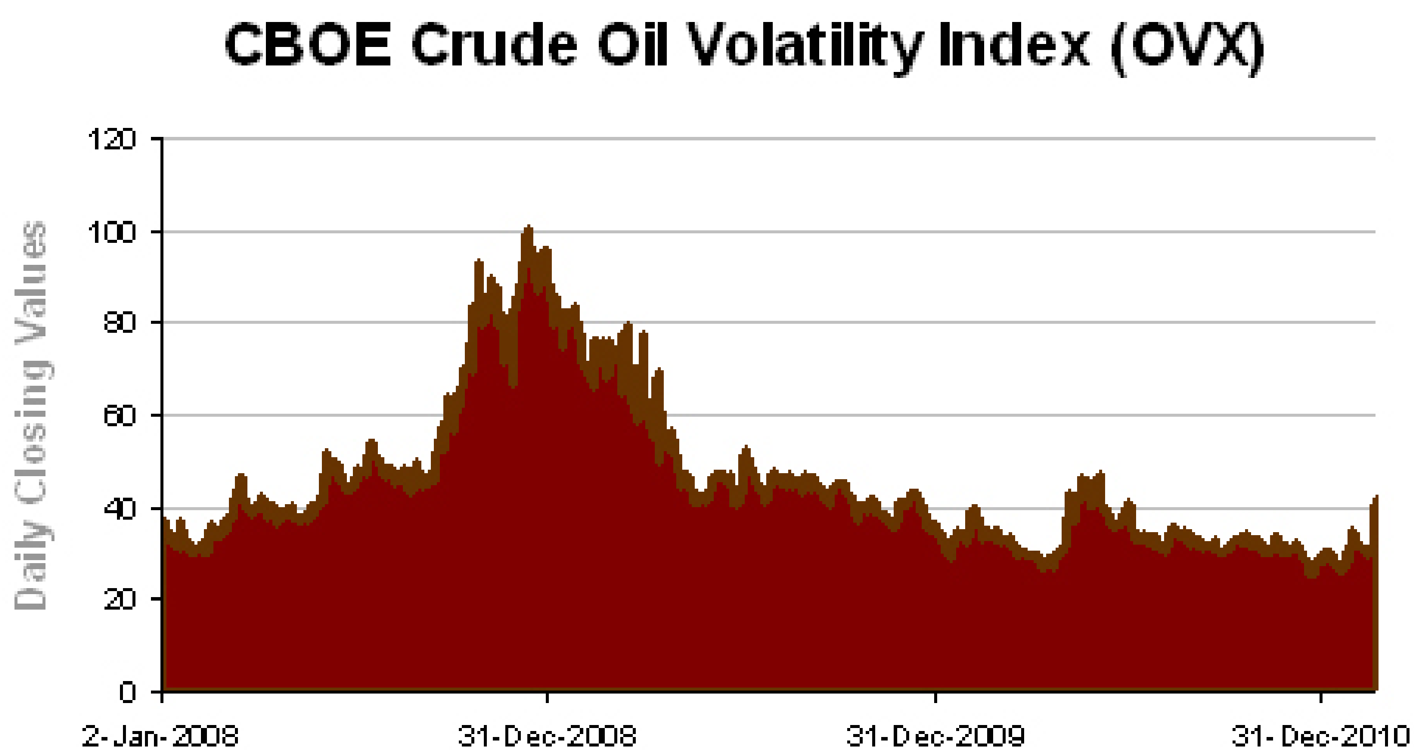
Courtesy-CBOE:

http://www.cboe.com/micro/oilvix/introduction.aspx.



Intro: Energy Commodities-OVX Index

This index also measures the conditional risk-neutral expectation of crude oil variance, but is computed from a cross-section of listed options on the United States Oil Fund (USO), which tracks the price of WTI as closely as possible. The CBOE Crude Oil ETF Volatility Index ('Oil VIX', Ticker - OVX) measures the market's expectation of 30-day volatility of crude oil prices by applying the VIX® methodology to United States Oil Fund, LP (Ticker - USO) options spanning a wide range of strike prices (see Figures below).



(Jan. 2, 2008 - Feb. 23, 2011)
Sources: CBOE and Bloomberg
www.cboe.com/OVX

Intro: Energy Commodities-OVX Index

We have to notice that crude oil and natural gas trade in units of 1,000 barrels and 10,000 British thermal units (mmBtu), respectivel.

Price are quoted as US dollars and cents per barrel or mmBtu.

Intro: Energy Commodities-Models for Stochastic Volatility

We consider Ornstein-Uhlenbeck process for commodity asset with stochastic volatility following continuous-time GARCH model or Pilipovic (1998) one-factor model.

The classical stochastic process for the spot dynamics of commodity prices is given by the Schwartz' model (1997). It is defined as the exponential of an Ornstein-Uhlenbeck (OU) process, and has become the standard model for energy prices possessing mean-reverting features.

Intro: Energy Commodities-Models for Stochastic Volatility

In this talk, we consider a risky asset in energy market with stochastic volatility following a mean-reverting stochastic process satisfying the following SDE (continuous-time GARCH(1,1) model):

$$d\sigma^2(t) = a(L - \sigma^2(t))dt + \gamma\sigma^2(t)dW_t,$$

where a is a speed of mean revertion, L is the mean reverting level (or equilibrium level), γ is the volatility of volatility $\sigma(t)$, W_t is a standard Wiener process.

Intro: Energy Commodities-Method

Using a change of time method we find an explicit solution of this equation and using this solution we are able to find the variance and volatility swaps pricing formula under the physical measure. Then, using the same argument, we find the option pricing formula under risk-neutral measure. We applied Brockhaus-Long (2000) approximation to find the value of volatility swap. A numerical example is presented for 'toy' data.

Intro: Energy Commodities-Motivation

The continuos-time GARCH model has also been explioted by Javaheri, Wilmott and Haug (2002) to calculate volatility swap for $S\&P500$ index. They used PDE approach and mentioned (page 8, sec. 3.3) that 'it would be interesting to use an alternative method to calculate $F(v,t)$ ($= E[\sigma^2(t)]$) and the other above quantities'.

Intro: Energy Commodities

This research exactly contains the alternative method, namely, 'change of time method', to get varinace and volatility swaps. The change of time method was also applied by Swishchuk (2004) for pricing variance, volatility, covariance and correlation swaps for Heston model. The first paper on pricing of commodity contracts was published by Black (1976).

Mean-Reverting Stochastic Volatility Model (MRSVM)

In this section we introduce MRSVM and study some properties of this model that we can use later for calculating variance and volatility swaps.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space with a sample space Ω, σ-algebra of Borel sets $\mathcal{F}$ and probability P. The filtration $\mathcal{F}_t, \quad t \in [0, T]$, is the natural filtration of a standard Brownian motion $W_t, \quad t \in [0, T]$, such that $\mathcal{F}_T = \mathcal{F}$.

Mean-Reverting Stochastic Volatility Model (MRSVM)

We consider a risky asset in energy market with stochastic volatility following a mean-reverting stochastic process the following stochastic differential equation:

$$d\sigma^2(t) = a(L - \sigma^2(t))dt + \gamma\sigma^2(t)dW_t, \tag{1}$$

where $a > 0$ is a speed (or 'strength') of mean revertion, $L > 0$ is the mean reverting level (or equilibrium level, or long-term mean), $\gamma > 0$ is the volatility of volatility $\sigma(t)$, W_t is a standard Wiener process.

Explicit Solution of MRSVM

Let

$$V_t := e^{at}(\sigma^2(t) - L). \tag{2}$$

Then, from (2) and (1) we obtain

$$dV_t = ae^{at}(\sigma^2(t) - L)dt + e^{at}d\sigma^2(t) = \gamma(V_t + e^{at}L)dW_t. \tag{3}$$

Explicit Solution of MRSVM

Using change of time approach to the equation (3) (see Ikeda and Watanabe (1981) or Elliott (1982)) we obtain the following solution of the equation (3)

$$V_t = \sigma^2(0) - L + \tilde{W}(\phi_t^{-1}),$$

or (see (2)),

$$\sigma^2(t) = e^{-at}[\sigma^2(0) - L + \tilde{W}(\phi_t^{-1})] + L, \tag{4}$$

where $\tilde{W}(t)$ is an $\mathcal{F}_t$-measurable standard one-dimensional Wiener process, $\phi_t^{-1}$is an inverse function to ϕ_t :

$$\phi_t = \gamma^{-2} \int_0^t (\sigma^2(0) - L + \tilde{W}(s) + e^{a\phi_s} L)^{-2} ds. \tag{5}$$

Explicit Solution of MRSVM

We note that

$$\phi_t^{-1} = \gamma^2 \int_0^t (\sigma^2(0) - L + \tilde{W}(\phi_t^{-1}) + e^{as} L)^2 ds, \tag{6}$$

which follows from (5).

Some Properties of the Process $\tilde{W}(\phi_t^{-1})$

We note that process $\tilde{W}(\phi_t^{-1})$ is $\tilde{\mathcal{F}}_t := \mathcal{F}_{\phi_t^{-1}}$-measurable and $\tilde{\mathcal{F}}_t$-martingale.

Then

$$E\tilde{W}(\phi_t^{-1}) = 0. \tag{7}$$

Let's calculate the second moment of $\tilde{W}(\phi_t^{-1})$ (see (6)):

$$\begin{aligned}
E\tilde{W}^2(\phi_t^{-1}) &= E < \tilde{W}(\phi_t^{-1}) >= E\phi_t^{-1} \\
&= \gamma^2 \int_0^t E(\sigma^2(0) - L + \tilde{W}(\phi_s^{-1}) + e^{as}L)^2 ds \\
&= \gamma^2[(\sigma^2(0) - L)^2 t + \frac{2L(\sigma^2(0)-L)(e^{at}-1)}{a} + \frac{L^2(e^{2at}-1)}{2a} \\
&+ \int_0^t E\tilde{W}^2(\phi_s^{-1}) ds].
\end{aligned} \tag{8}$$

Some Properties of the Process $\tilde{W}(\phi_t^{-1})$

From (8), solving this linear ordinary nonhomogeneous differential equation with respect to $E\tilde{W}^2(\phi_t^{-1})$,

$$\frac{dE\tilde{W}^2(\phi_t^{-1})}{dt} = \gamma^2[(\sigma^2(0)-L)^2+2L(\sigma^2(0)-L)e^{at}+L^2e^{2at}+E\tilde{W}^2(\phi_t^{-1})],$$

we obtain

$$\begin{aligned} E\tilde{W}^2(\phi_t^{-1}) &= \gamma^2[(\sigma^2(0)-L)^2\tfrac{e^{\gamma^2 t}-1}{\gamma^2}+\tfrac{2L(\sigma^2(0)-L)(e^{at}-e^{\gamma^2 t})}{a-\gamma^2} \\ &+ \tfrac{L^2(e^{2at}-e^{\gamma^2 t})}{2a-\gamma^2}]. \end{aligned}$$

Some Properties of the Process $\tilde{W}(\phi_t^{-1})$

We note, that

$$
\begin{aligned}
E\tilde{W}(\phi_s^{-1})\tilde{W}(\phi_t^{-1}) &= \gamma^2[(\sigma^2(0)-L)^2\tfrac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2} \\
&+ \tfrac{2L(\sigma^2(0)-L)(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a-\gamma^2} + \tfrac{L^2(e^{2a(t\wedge)}-e^{\gamma^2(t\wedge s)})}{2a-\gamma^2}],
\end{aligned} \tag{9}
$$

and the second moment for $\tilde{W}^2(\phi_t^{-1})$ above follows from (9).

Explicit Expression for the Process $\tilde{W}(\phi_t^{-1})$

It is turns out that we can find the explicit expression for the process $\tilde{W}(\phi_t^{-1})$.

From the expression (see Section 3.1)

$$V_t = \sigma^2(0) - L + \tilde{W}(\phi_t^{-1}),$$

we have the following relationship between $W(t)$ and $\tilde{W}(\phi_t^{-1})$:

$$\tilde{W}(\phi_t^{-1}) = \gamma \int_0^t [S(0) - L + Le^{at} + \tilde{W}(\phi_s^{-1})] dW(t).$$

Explicit Expression for the Process $\tilde{W}(\phi_t^{-1})$

It is a linear SDE with respect to $\tilde{W}(\phi_t^{-1})$ and we can solve it explicitly. The solution has the following look:

$$\begin{aligned} \tilde{W}(\phi_t^{-1}) &= \sigma^2(0)(e^{\gamma W(t)-\frac{\gamma^2 t}{2}} - 1) \\ &+ L(1-e^{at}) + aLe^{\gamma W(t)-\frac{\gamma^2 t}{2}} \int_0^t e^{as} e^{-\gamma W(s)+\frac{\gamma^2 s}{2}} ds. \end{aligned} \tag{10}$$

Explicit Expression for the Process $\tilde{W}(\phi_t^{-1})$

It is easy to see from (10) that $\tilde{W}(\phi_t^{-1})$ can be presented in the form of a linear combination of two zero-mean martingales $m_1(t)$ and $m_2(t)$:

$$\tilde{W}(\phi_t^{-1}) = m_1(t) + Lm_2(t),$$

where

$$m_1(t) := \sigma^2(0)(e^{\gamma W(t) - \frac{\gamma^2 t}{2}} - 1)$$

and

$$m_2(t) = (1 - e^{at}) + ae^{\gamma W(t) - \frac{\gamma^2 t}{2}} \int_0^t e^{as} e^{-\gamma W(s) + \frac{\gamma^2 s}{2}} ds.$$

Explicit Expression for the Process $\tilde{W}(\phi_t^{-1})$

Indeed, process $\tilde{W}(\phi_t^{-1})$ is a martingale (see Section 3.2), also it is well-known that process $e^{\gamma W(t)-\frac{\gamma^2 t}{2}}$ and, hence, process $m_1(t)$ is a martingale. Then the process $m_2(t)$, as the difference between two martingales, is also martingale.

Some Properties of the Mean-Reverting Stochastic Volatility $\sigma^2(t)$: First Two Moments, Variance and Covariation

From (4) we obtain the mean value of the first moment for mean-reverting stochastic volatility $\sigma^2(t)$:

$$E\sigma^2(t) = e^{-at}[\sigma^2(0) - L] + L. \tag{11}$$

It means that $E\sigma^2(t) \to L$ when $t \to +\infty$. We need this moment to value the variance swap.

Some Properties of the Mean-Reverting Stochastic Volatility $\sigma^2(t)$: First Two Moments, Variance and Covariation

Using formulae (4) and (9) we can calculate the second moment of $\sigma^2(t)$:

$$\begin{aligned} E(\sigma^2(t))^2 &= (e^{-at}(\sigma^2(0)-L)+L)^2 \\ &+ \gamma^2 e^{-2at}[(\sigma^2(0)-L)^2\tfrac{e^{\gamma^2 t}-1}{\gamma^2}+\tfrac{2L(\sigma^2(0)-L)(e^{at}-e^{\gamma^2 t})}{a-\gamma^2} \\ &+ \tfrac{L^2(e^{2at}-e^{\gamma^2 t})}{2a-\gamma^2}]. \end{aligned}$$

Some Properties of the Mean-Reverting Stochastic Volatility $\sigma^2(t)$: First Two Moments, Variance and Covariation

Combining the first and the second moments we have the variance of $\sigma^2(t)$:

$$
\begin{aligned}
Var(\sigma^2(t)) &= E\sigma^2(t)^2 - (E\sigma^2(t))^2 \\
&= \gamma^2 e^{-2at}[(\sigma^2(0) - L)^2 \tfrac{e^{\gamma^2 t}-1}{\gamma^2} + \tfrac{2L(\sigma^2(0)-L)(e^{at}-e^{\gamma^2 t})}{a-\gamma^2} \\
&+ \tfrac{L^2(e^{2at}-e^{\gamma^2 t})}{2a-\gamma^2}].
\end{aligned}
$$

Some Properties of the Mean-Reverting Stochastic Volatility $\sigma^2(t)$: First Two Moments, Variance and Covariation

From the expression for $\tilde{W}(\phi_t^{-1})$ (see (10)) and for $\sigma^2(t)$ in (4) we can find the explicit expression for $\sigma^2(t)$ through $W(t)$:

$$\begin{aligned} \sigma^2(t) &= e^{-at}[\sigma^2(0) - L + \tilde{W}(\phi_t^{-1})] + L \\ &= e^{-at}[\sigma^2(0) - L + m_1(t) + Lm_2(t)] + L \\ &= \sigma^2(0)e^{-at}e^{\gamma W(t)-\frac{\gamma^2 t}{2}} + aLe^{-at}e^{\gamma W(t)-\frac{\gamma^2 t}{2}} \int_0^t e^{as}e^{-\gamma W(s)+\frac{\gamma^2 s}{2}} ds, \end{aligned} \tag{12}$$

where $m_1(t)$ and $m_2(t)$ are defined above.

Some Properties of the Mean-Reverting Stochastic Volatility $\sigma^2(t)$: First Two Moments, Variance and Covariation

From (12) it follows that $\sigma^2(t) > 0$ as long as $\sigma^2(0) > 0$.

The covariation for $\sigma^2(t)$ may be obtained from (4), (7) and (9):

$$\begin{aligned} E\sigma^2(t)\sigma^2(s) &= e^{-a(t+s)}(\sigma^2(0)-L)^2 \\ &+ e^{-a(t+s)}\{\gamma^2[(\sigma^2(0)-L)^2\tfrac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2} \\ &+ \tfrac{2L(\sigma^2(0)-L)(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a-\gamma^2} \\ &+ \tfrac{L^2(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{2a-\gamma^2}]\} \\ &+ e^{-at}(\sigma^2(0)-L)L + e^{-as}(\sigma^2(0)-L)L + L^2. \end{aligned} \tag{13}$$

We need this covariance to value the volatility swap.

Variance Swap for MRSVM

To calculate the variance swap for $\sigma^2(t)$ we need $E\sigma^2(t)$. From (11) it follows that

$$E\sigma^2(t) = e^{-at}[\sigma^2(0) - L] + L.$$

Then $E\sigma_R^2 := EV$ takes the following form:

$$E\sigma_R^2 := EV := \frac{1}{T}\int_0^T E\sigma^2(t)dt = \frac{(\sigma^2(0) - L)}{aT}(1 - e^{-aT}) + L. \quad (14)$$

Recall, that $V := \frac{1}{T}\int_0^T \sigma^2(t)dt$.

Volatility Swap for MRSVM

To calculate the volatility swap for $\sigma^2(t)$ we need $E\sqrt{V} = E\sqrt{\sigma_R}$ and it means that we more than just $E\sigma^2(t)$, because the realized volatility $\sigma_R := \sqrt{V} = \sqrt{\sigma_R^2}$. Using Brockhaus-Long approximation we then get

$$E\sqrt{V} \approx \sqrt{EV} - \frac{Var(V)}{8(EV)^{3/2}}. \tag{15}$$

We have EV calculated in (14). We need

$$Var(V) = EV^2 - (EV)^2. \tag{16}$$

Variance Swap for MRSVM

From (14) it follows that $(EV)^2$ has the form:

$$(EV)^2 = \frac{(\sigma^2(0) - L)^2}{a^2T^2}(1-e^{-aT})^2+2\frac{(\sigma^2(0) - L)}{aT}(1-e^{-aT})L+L^2. \tag{17}$$

Variance Swap for MRSVM

Let us calculate EV^2 using (9) and (13):

$$
\begin{aligned}
EV^2 &= \frac{1}{T^2}\int_0^T\int_0^T E\sigma^2(t)\sigma^2(s)dtds \\
&= \frac{1}{T^2}\int_0^T\int_0^T [e^{-a(t+s)}(\sigma^2(0)-L)^2 \\
&+ e^{-a(t+s)}\{\gamma^2[(\sigma^2(0)-L)^2\frac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2} \\
&+ \frac{2L(\sigma^2(0)-L)(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a-\gamma^2} + \frac{L^2(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{2a-\gamma^2}]\} \\
&+ e^{-at}(\sigma^2(0)-L)L + e^{-as}(\sigma^2(0)-L)L + L^2]dtds
\end{aligned}
\tag{18}
$$

Variance Swap for MRSVM

After calculating the interals in the second, forth and fifth lines in (18) we have:

$$
\begin{aligned}
EV^2 &= \tfrac{1}{T^2}\int_0^T\int_0^T E\sigma^2(t)\sigma^2(s)dtds \\
&= \tfrac{(\sigma^2(0)-L)^2}{a^2T^2}(1-e^{-aT})^2 \\
&+ \tfrac{1}{T^2}\int_0^T\int_0^T e^{-a(t+s)}\{\gamma^2[(\sigma^2(0)-L)^2\tfrac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2} \\
&+ \tfrac{2L(\sigma^2(0)-L)(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a-\gamma^2} + \tfrac{L^2(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{2a-\gamma^2}]\}dtds \\
&+ \tfrac{(\sigma^2(0)-L)L}{aT}(1-e^{-aT}) + \tfrac{(\sigma^2(0)-L)L}{aT}(1-e^{-aT}) + L^2.
\end{aligned}
\tag{19}
$$

Variance Swap for MRSVM

Taking into account (16), (17) and (19) we arrive at the following expression for $Var(V)$:

$$\begin{aligned} Var(V) &= EV^2 - (EV)^2 \\ &= \tfrac{1}{T^2}\int_0^T\int_0^T e^{-a(t+s)}\{\gamma^2[(\sigma^2(0)-L)^2\tfrac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2} \\ &+ \tfrac{2L(\sigma^2(0)-L)(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a-\gamma^2} + \tfrac{L^2(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{2a-\gamma^2}]\}dtds \\ &= \tfrac{\sigma^2(0)-L}{T^2}\int_0^T\int_0^T e^{-a(t+s)}(e^{\gamma^2(t\wedge s)}-1)dtds \\ &+ \tfrac{2L\gamma^2(\sigma^2(0)-L)}{(a^2-\gamma^2)T^2}\int_0^T\int_0^T e^{-a(t+s)}(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})dtds \\ &+ \tfrac{\gamma^2L^2}{(2a-\gamma^2)T^2}\int_0^T\int_0^T e^{-a(t+s)}(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)})dtds. \end{aligned} \tag{20}$$

Variance Swap for MRSVM

After calculating the three integrals in (20) we obtain:

$$\begin{aligned} Var(V) &= EV^2 - (EV)^2 \\ &= \frac{1}{T^2}\int_0^T\int_0^T e^{-a(t+s)}\{\gamma^2[(\sigma^2(0) - L)^2\frac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2} \\ &+ \frac{2L(\sigma^2(0)-L)(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a-\gamma^2} + \frac{L^2(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{2a-\gamma^2}]\}dtds. \end{aligned} \tag{21}$$

Variance Swap for MRSVM

From (15) and (21) we get the volatility swap:

$$E\sqrt{V} \approx \sqrt{EV} - \frac{Var(V)}{8(EV)^{3/2}}. \tag{22}$$

Risk Neutral Stochastic Volatility Model (SVM)

Consider our model (1)

$$d\sigma^2(t) = a(L - \sigma^2(t))dt + \gamma\sigma^2(t)dW_t. \tag{23}$$

Risk Neutral Stochastic Volatility Model (SVM)

Let λ be 'market price of risk' and defind the following constants:

$$a^* := a + \lambda\sigma, \quad L^* := aL/a^*.$$

Then, in the risk-neutral world, the drift paramater in (23) has the following form:

$$a^*(L^* - \sigma^2(t)) = a(L - \sigma^2(t)) - \lambda\gamma\sigma^2(t). \tag{24}$$

Risk Neutral Stochastic Volatility Model (SVM)

Then the risk neutral stochastic volatility model has the following form

$$d\sigma^2(t) = (aL - (a + \lambda\gamma)\sigma^2(t))dt + \gamma\sigma^2(t)dW_t^*, \tag{25}$$

or, equivalently,

$$d\sigma^2(t) = a^*(L^* - \sigma^2(t))dt + \gamma\sigma^2(t)dW_t^*, \tag{26}$$

where

$$a^* := a + \lambda\gamma, \quad L^* := \frac{aL}{a + \lambda\gamma}. \tag{27}$$

Now, we have the same model in (26) as in (1), and we are going to apply our change of time method to this model (26) to obtain the values of variance and volatility swaps.

Variance and Volatility Swaps for Risk-Neutral SVM

Using the same arguments as in the previous section (where inplace of (4) we have to take (26)) we get the following expressions for variance and volatility swaps taking into account (27). For the variance swaps we have (see (14) and (27)):

$$E^*\sigma_R^2 := EV := \frac{1}{T}\int_0^T E\sigma^2(t)dt = \frac{(\sigma^2(0) - L^*)}{a^*T}(1 - e^{-a^*T}) + L^*. \tag{28}$$

Risk Neutral Stochastic Volatility Model (SVM)

For the volatility swap we obtain (see (22) and (27))

$$E^*\sqrt{V} \;\approx\; \sqrt{E^*V} - \frac{Var^*(V)}{8(E^*V)^{3/2}} \tag{29}$$

Numerical Example: Toy Data

To apply our formula for calculating these values we need to calibrate the parameters $a, \quad L, \quad \sigma_0^2$ and γ (T is monthly). These parameters may be obtained from futures prices (future work), e.g., using AECO Natural Gas Index, $S\&P$ GSCI or DJ-AIGCI.

Numerical Example: Toy Data

The parameters are the following:

Parameters			
a	γ	L	λ
4.6488	1.5116	2.7264	0.18

Numerical Example: Toy Data

For variance swap we use formula (14) and for volatility swap we use formula (22).

From this table we can calculate the values for risk adjusted parameters a^* and L^* :

$$a^* = a + \lambda\gamma = 4.9337,$$

and

$$L^* = \frac{aL}{a + \lambda\gamma} = 2.5690.$$

Numerical Example: Toy Data

For the value of $\sigma^2(0)$ we can take $\sigma^2(0) = 2.25$.

For variance swap and for volatility swap with risk adjusted papameters we use formula (28) and (29), respectively.

Figures

Figure 1 depicts variance swap (price vs. maturity) using formula (14).

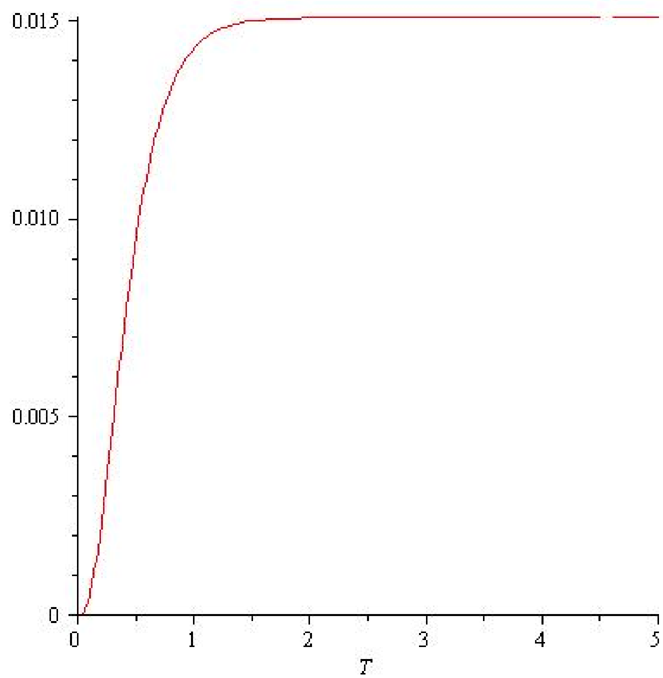


Fig. 1: Variance Swap

Figure 2 depicts volatility swap (price vs. maturity) using formula (22).

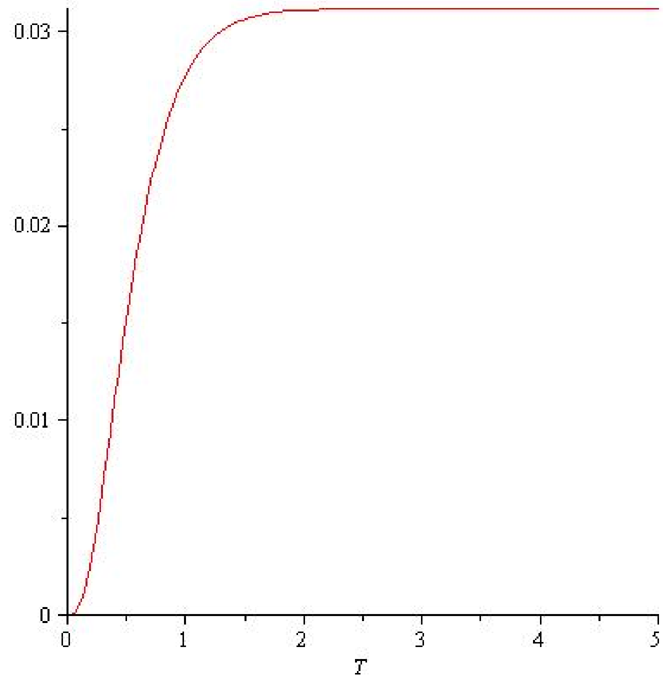


Fig. 2: Volatility Swap

Figure 3 depicts varinace swap with risk adjusted parameters (price vs. maturity) using formula (28).

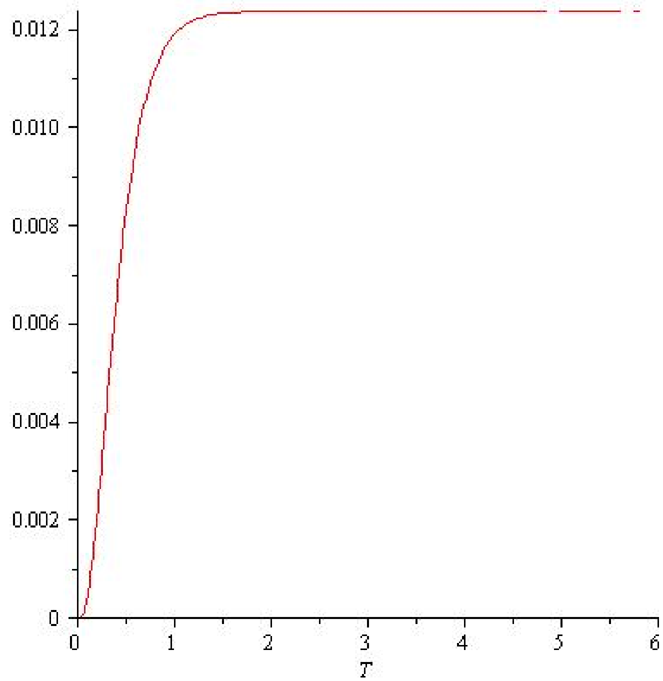


Fig. 3: Variance Swap (Risk Adjusted Parameters)

Figure 4 depicts volatility swap with risk adjusted parameters (price vs. maturity) using formula (29).

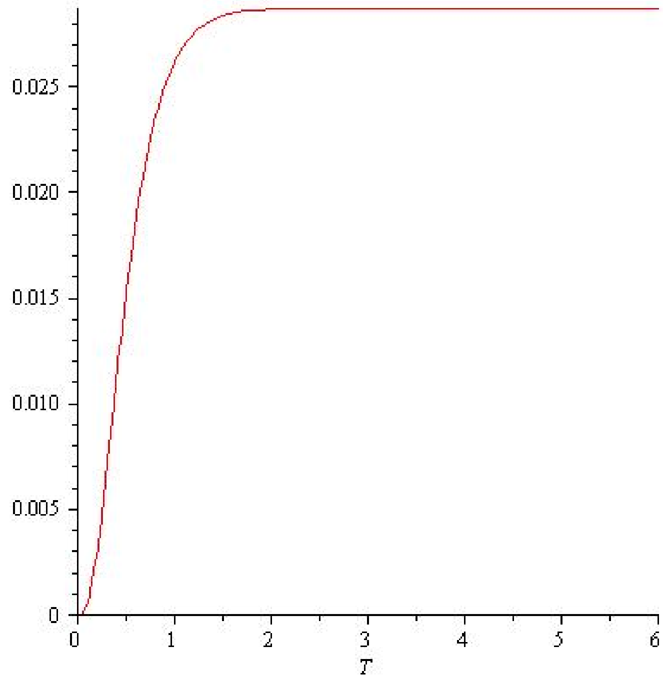


Fig. 4: Volatility Swap (Risk Adjusted Parameters)

Figure 5 depicts comparison of adjusted (green line) and non-adjusted price (red line) (naive strike vs. adjusted strike).

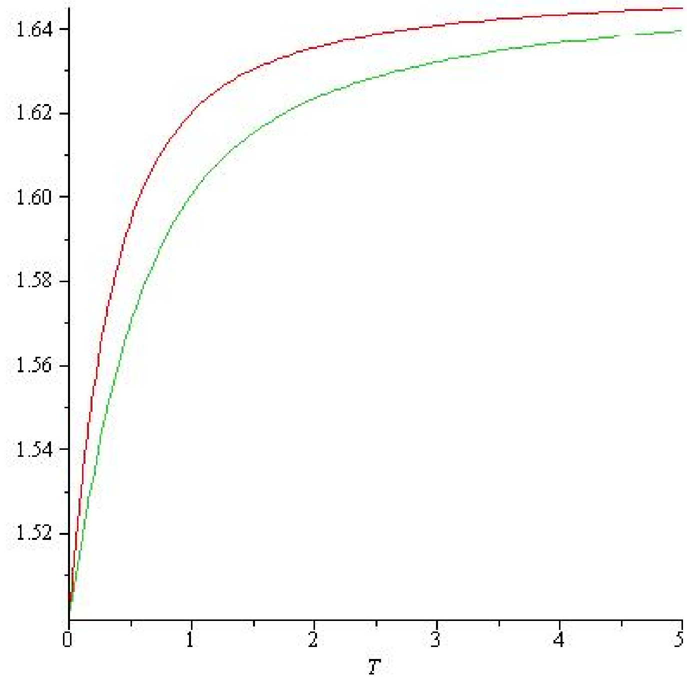


Fig. 5: Comparison: Adjusted and Non-Adjusted Price

Figure 6 depicts convexity adjustment. It's decreasing with swap maturity (the volatility of volatility over a long period of time is low).

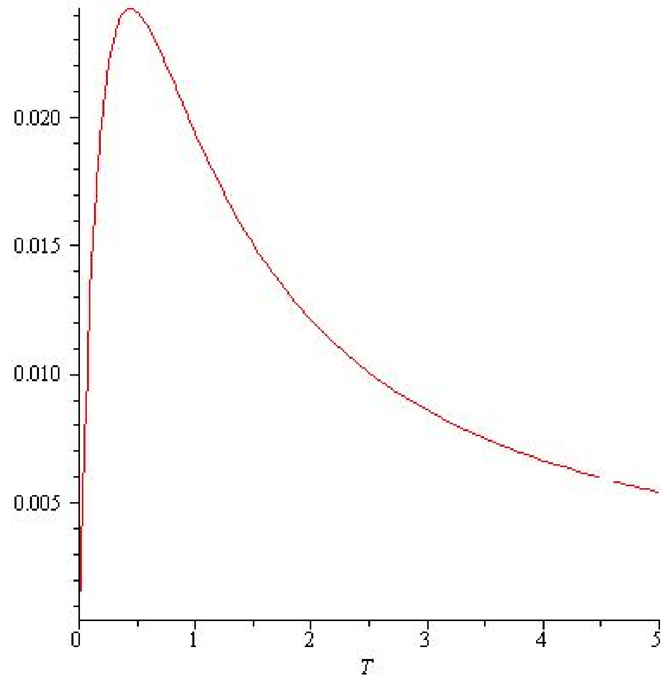


Fig. 6: Convexity Adjustment

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Future Work

1. Parameters Estimation ($S\&P$ GSCI, OVX or DJ-AIGCI).

2. Options on futures contracts.

3. Possible jumps in the model for variance.

The End

Thank You for Your Time and Attention!

Q&A Time!

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