Financial Mathematics: 
Historical Perspectives and Recent Developments

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Outline of Presentation

1. Introduction

2. History of Financial Mathematics (FM)

3. Some Basic Ideas, Methods and Results in FM

4. FM and Financial Industry

5. New Directions/Developments in FM
Introduction
What is it worth? ('Irises' (Les Iris)-Vincent van Gogh (1889))
Introduction

• *Finance* may be defined as the *study of how people allocate scarce resources over time*

• The outcomes of financial decisions (costs and benefits) are
  
  - *spread over time*

  - not known with certainty ahead of time, *i.e subject to an element of risk*

• Decision makers must therefore
  
  - be able to *compare the values of cash-flows at different dates*

  - take a *probabilistic view*
Introduction

Thus, financial mathematics is based on the idea in *making good decisions in the face of uncertainty.*

As long as uncertainty is involved, the *probability theory* is one of the the main instruments in financial mathematics.

One of the key objectives of financial mathematics is also to understand how to construct the best investment strategies that minimises risks in the real world.
Introduction

Cardano

Girolamo Cardano (1501-1576) was probably the first one who explored the ethics of *gambling* in his *'Liber de Ludo Aleae' ('Book on Games of Chance')* of 1564, which contains the first discussion of the idea of *mathematical probability* (fair dice, gambling).
Introduction

Pascal

Pascal's (1623-1662) Wager (you’ve got nothing to lose by betting that God exists) historically was groundbreaking because it charted new territory in probability theory, marked the first formal use of decision theory, and anticipated future philosophies such as existentialism, pragmatism and voluntarism.
The early development of probability, from Cardano, through Galileo and Fermat and Pascal up to Daniel Bernoulli, was driven by considering gambling problem.
Introduction

Jacob Bernoulli
These ideas about probability were collected by Jacob Bernoulli (1654-1705) (Daniel’s uncle), in his work ‘Ars Conjectandi’ (‘The Art of Conjecturing’). He introduced the law of large numbers, proving that if you repeat the same experiment (say rolling dice) a large number of times, then the observed mean (the average of the scores you have rolled) will converge to the expected mean.
Introduction

Laplace
(1749-1827)

Building on Jacob Bernoulli’s work, *probability theory* was developed by the likes of Laplace in the eighteenth century and the Fisher, Neyman and Pearson in the twentieth.
Introduction

For the first third of the twentieth century, probability was associated with inferring results, such as the life expectancy of a person, from observed data.

But as an inductive science (i.e., the results were inspired by experimental observations, rather than the deductive nature of mathematics built on axioms), probability was not fully integrated into maths until 1933.
Introduction

Kolmogorov

In 1933, Andrey Kolmogorov (1903-1987) identified probability with measure theory.

Kolmogorov defined probability to be any measure on a collection of events— not necessarily based on the frequency of events.
Introduction

Why is the measure theoretic approach so important in finance?

Financial mathematicians realised that an asset's price can be represented as an expectation under a special probability measure, called a risk-neutral measure, which bears no direct relation to the 'natural' probability of the asset price rising or falling based on past observations.
History of Financial Mathematics
History of Financial Mathematics

• The history of the modelling of risky asset (stock, foreign exchange rate, etc. e.g.) prices $S_t$ begins with Brownian motion (BM) $B_t$ ($\sigma$ is the volatility or standard deviation)

\[ S_t = \sigma B_t \]
Fig. 1. Path of Foreign Exchange Rate

Fig. 2. Path of Brownian Motion

The two pathes/trajectories look very similar!
History of Financial Mathematics

- The earliest attempts to model BM mathematically can be traced to *three sources*, each of which knew nothing about the others:
History of Financial Mathematics (cntd)

Thiele

- The first source was that of T. N. Thiele (1838-1910) of Copenhagen, who effectively created a model of BM while studying time series in 1880
The second was that of L. Bachelier of Paris (1870-1926), who created a model of BM while deriving the dynamic behavior of the Paris stock market, in 1900.
History of Financial Mathematics (cntd)

- The third was that of A. Einstein (1879-1955), who proposed a model of the motion of small particles suspended in a liquid, in an attempt to convince other physicists of the molecular nature of matter, in 1905.
History of Financial Mathematics (cntd)

• Of these three models, those of Thiele and Bachelier had little impact for a long time, while that of Einstein was immediately influential

• Peter Bernstein (1992): 'Despite its importance, Bachelier’s thesis was lost until it was rediscovered quite by accident in the 1950’s by Jimmie Savage, a mathematical statistician at Chicago’

• We know however that Kolmogorov and also Doob explicitly reference Bachelier, and Ito certainly knew of his work too

• Bernstein relates that Jimmie Savage alerted the economist Paul Samuelson to Bachelier’s work, who found Bachelier’s thesis in the MIT library
History of Financial Mathematics (cntd)

- Samuelson published in 1965 two papers of groundbreaking work

- In his papers he gives his economics arguments that \textit{prices must fluctuate randomly}, 65 years after Bachelier had assumed it!

- This paper along with Fama’s work (1965) on the behaviour of stock prices, form the basis of what has come to be known as \textit{‘the efficient market hypothesis’}
History of Financial Mathematics (cntd)

- Samuelson explains that *Bachelier’s model failed to ensure that stock prices always be positive, whereas geometric BM avoids these pitfalls*

- The derivation was almost identical to that used nearly a decade later to derive the *Black-Scholes formula*
History of Financial Mathematics (Brief Summary)

*Bachelier (1900)*: uses Brownian motion as underlying process to derive option price

*Black & Scholes (1973)*: publish their PDE-based option pricing formula

*Harrison & Pliska (1980)*: introduce the martingale approach into mathematical finance

*Financial Mathematics* has been established as a separate academic discipline only since the late eighties, with a number of dedicated journals
Some Basic Ideas, Methods, Results in FM
Some Contents of Financial Mathematics

Financial Mathematics is a collection of mathematical techniques that find applications in finance, e.g.:

Asset Pricing

Hedging and Risk Management

Portfolio Optimization
Approaches in Financial Mathematics

There are two main approaches in financial mathematics:

*Probability and Stochastic Processes*

*Partial Integro-Differential Equations*

Relationship: *Feynman-Kac Formula*
Preliminary Notions: Discounting

- **The Time Value of Money**: $1 in the hand today is worth more than the expectation of receiving $1 at some future date.

- **Premium**: Thus borrowing is not free: the borrower pays a premium to induce the lender to part with his/her money. This premium is the interest.
Preliminary Notions: Discounting (cntd.)

- Let $r$ denote the \textit{continuously compounded} interest rate, so that one unit of currency deposited in a riskless bank account grows to $e^{rT}$ units in time $T$.

- Thus an amount $X$ at time $T$ is the same as $X e^{-rT}$ now.

\textit{Discounting} allows us to compare amounts of money at different times.
Preliminary Notions: Discounting and Returns

- The *return* for $S_T$ on an investment $S_0$ is defined by

  \[ R = \ln \frac{S_T}{S_0}, \quad i.e. \quad S_T = S_0 e^{RT} \]

  The random variable $R$ is essentially the 'interest' obtained on the investment, and may be negative.

- Investors attempt to maximize their expected returns.

- *Fundamental Relationship in Finance:*

  \[ E[\text{Return}] = f(\text{Risk}) \]

  where $f$ is an increasing function.
Preliminary Notions: Financial Instruments (Securities)

- **Securities**: are contracts for future delivery of goods or money, e.g., shares, bonds and derivatives

- One distinguishes between underlying (primary) and derivative (secondary) instruments

- Examples of underlying instruments are shares, bonds, currencies, interest rates and indices

- A derivative (or contingent claim) is a financial instruments whose value is derived from an underlying asset

- Examples of derivatives are forward contracts, futures, options, swaps
Preliminary Notions: Why Using Derivatives?

- There are two main reasons for using derivatives: 1) **Hedging** and 2) **Speculation**

- Thus derivatives are essentially tools for *transferring risk*, and will allow one to diminish or increase one’s exposure to uncertain events

- An *option* gives the holder the *right*, but not the *obligation* to buy or sell an asset

- A *European Call Option* gives the holder the right to *buy* an asset $S$ (the *underlying*) for an agreed amount $K$ (the *strike price*) on a specified future date $T$ (*maturity*)
Modelling Stock Price

• Any model of stock price behaviour must be stochastic, i.e., incorporate the random nature of price behaviour. The simplest such models are random walks.

• Partition the interval $[0, T]$ into subintervals of length $\Delta t$

$$0 = t_0 \leq t_1 \leq ... \leq t_N = T, \quad N = \frac{T}{\Delta t}$$

• Let $X_{t_n}, n = 1, 2, ..., N$, be a family of random variables, and let $S_0$ be the stock price at $t = 0$. We might attempt to model the stock price process by

$$S_{t_{n+1}} = S_{t_n} + X_{t_{n+1}}$$
Modelling Stock Price (cntd)

• Thus

\[ S_t = S_0 + \sum_{n=1}^{t} X_n \]

• The intuition behind this is that the price at time \( t + \Delta t \) equals the price at time \( t \) plus a 'random shock,' modelled by \( X_t \).

• We also assume that these shocks are independent

• **Efficient Markets Hypothesis:** Stock price processes are Markov processes: future depends on the past only through present
Modelling Stock Price (cntd)

• To build a continuous version of our model, we use the Central Limit Theorem: If $X_n$ is largish family of i.i.d.r.v.s, then $\sum_n X_n$ is approximately normally distributed

• Thus: After a largish number of shocks, the stock price in our random walk model will be approximately normally distributed

• We seek a continuous-time version of the random walk—a stochastic process that is changing because of random shocks at every instant in time
Modelling Stock Price: Brownian Motion

• **Brownian Motion** is a *continuous-time stochastic process* $B_t, t \geq 0$, with the following properties:

1) Each change $B_t - B_s$ is *normally distributed* with mean 0 and variance $(t - s)$.

2) Each change $B_t - B_s$ is *independent* of all the previous values $B_u, u \leq s$.

3) Each sample path $B_t$ is *almost surely continuous*, and has $B_0 = 0$. 
Stochastic Calculus (cntd)

- It follows that

$$
\lim_{\Delta t \to 0} \sum E[\Delta B]^2 = \lim_{\Delta t \to 0} \Delta t = T
$$

where $T$ is the total elapsed time. Thus the quadratic variation of Brownian motion is non-zero.
Stochastic Calculus (cntd)

- We thus have the following rules for stochastic calculus:

\[
(dB_t)^2 = dt \\
 dB_t dt = (dt)^2 = 0.
\]
Stochastic Calculus (cntd)

- Suppose that \( f(t, x) \) is a \( C^{1,2} \)-function, and let \( X_t = f(t, B_t) \). Applying these rules to a second order Taylor series, we obtain the following result: **Ito's Lemma**

\[
dX_t = df(t, B_t) = \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} \right) dt + \frac{\partial f}{\partial B} dB_t
\]

- **Ordinary calculus** shows that for a function \( f(t, x) \) we have

\[
df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial B} dB
\]

- In **stochastic calculus**, we get another term \( \frac{1}{2} \frac{\partial^2 f}{\partial B^2} dt \), so-called **Itô's adjustment**, due to the non-zero quadratic variation of Brownian motion
Stochastic Calculus (cntd)

• Since Brownian motion has non-zero quadratic variation, Brownian sample paths are (a.s.) of \textit{unbounded variation}.

• This means that in general the \textit{Ito stochastic integral} \( \int_0^t f dB_t \) cannot be interpreted as a \textit{Riemann-Stieltjes integral}.

• Nevertheless, the \textit{stochastic integral can be defined using an approximation in a} \( L^2 \)-\textit{space}, rather than an (almost) pointwise limit.
Stock Price Process Parameters

- Let’s take another look at volatility. The geometric Brownian motion (GBM) model for stock price is

\[ dS_t = \mu S_t dt + \sigma S_t dB_t \]

Thus

\[ E\left[ \frac{dS}{S} \right]^2 = \sigma^2 dt \]

and thus \( \sigma^2 dt \) is the variance of the return of the stock price over a small period \( dt \).

- It follows that \( \sigma \) is the standard deviation of the annual return of the stock \( S \)

- This can be measured from market data
Modelling Stock Price: Geometric Brownian Motion

• For stock price, the Brownian motion model is inadequate, though, \((B_t\) may be negative). We expect the change in price to be proportional to the current price

• A better model for stock prices is given by the stochastic differential equation

\[
dS_t = \mu S_t dt + \sigma S_t dB_t
\]
Modelling Stock Price: Geometric Brownian Motion

- This share price process is called a *geometric Brownian motion*

\[ S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)} \]

- Here \( \mu \) is the *drift*, i.e. the rate at which the share price increases in the absence of risk. The differential \( dB_t \) models the randomness (risk), and the parameter \( \sigma \), known as the *volatility*, models how sensitive the share price is to these random events.
Black-Scholes Model: PDE Approach

• Consider \((B, S)\) market with a share \(S_t\) whose price process satisfies the stochastic differential equation (SDE)

\[
dS_t = \mu S_t dt + \sigma S_t dB_t
\]

• Let the risk-free interest rate be \(r\), and let \(B_t\) be the riskless bank account, with dynamics

\[
 dB_t = r B_t dt.
\]

• Let \(V(t, S_t)\) be derivative security whose value depends on both the share price and time (forward, futures, option, etc.).
Black-Scholes Model: PDE Approach (cntd)

• Then $V(t, S_t)$ satisfies the following partial differential equation (PDE):

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$ 

• This is the famous Black-Scholes PDE. It is second-order parabolic PDE, i.e., essentially a heat equation.

• We do not care about the drift $\mu$ of the underlying asset $S$! It does not appear in the BS PDE!
Black-Scholes Model: Risk-Neutral Approach

• Since we do not care about the drift rate $\mu$ of an underlying asset, we may assume that *all assets have the same drift*

• The riskless asset (bank account) has drift $r$, which we can actually see. We thus assume that *all assets have the same return*, namely the risk-free rate $r$. 
Black-Scholes Model: Risk-Neutral Approach

- Mathematically, this corresponds to a change of measure from real world, unknowable probability measure $P$ to a knowable, risk-neutral measure $Q$.

- In the risk-neutral world, the dynamics of $S_t$ are

$$dS_t = rS_t dt + \sigma S_t dB_t$$

- Mathematically, this is accomplished using the Cameron-Girsanov Theorem.

- Amazingly, a change of measure from $P$ to $Q$ changes only the drift and not the volatility.
Black-Scholes Model: Risk-Neutral Approach

- We can calculate option prices in the risk-neutral world, because the asset price dynamics/distributions are known.

- But: *Prices in the real-and risk-neutral world are the same!* *It is just probabilities that are changed.*

- *Fundamental Theorem of Asset Pricing:* There are no arbitrage opportunities (free lunches) if and only if there exists a risk-neutral measure.
PDE=Risk-Neutral

• Consider a derivative security (so-called European call option) $V(t, S_t)$ on a share $S_t$ with strike price $K$ and maturity $T$. The volatility of the underlying share $S$ is $\sigma$ and the risk-free rate is $r$.

• To find the price of $V$, we must solve the following **Boundary Value Problem**:

$$\begin{align}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV &= 0 \\
V(T, S_T) &= \Phi(S_T) = \max(S_T - K, 0)
\end{align}$$
Black-Scholes Model: Risk-Neutral Approach

- Risk-neutral valuation/pricing associated with risk-neutral measure $Q$ (different from initial/physical/natural measure $P$)

- Or, It’s associated with *martingale measure* $Q$ (the same meaning)
What is Martingale? (PDE=Risk-Neutral)

**Martingale (real life)**

- In Real Life: *martingale* is a strap, or set of straps, attached at one end to the noseband (standing martingale) or reins (running martingale) of a horse and at the other end to the girth. It is used to prevent the horse from raising its head too high.

**Martingale (trading risk)**

- In Probability Theory: *martingale* is a model of a fair game where knowledge of past events never helps predict the mean of the future winnings.
Theorem: In the risk-neutral world, the discounted value process
\[ e^{-rt} V(t, S_t) \]
is a martingale.

It follows that the expected value of \( e^{-rt} V(t, S_t) \) at any time is its current value, and thus the value of the call option with strike \( K \) and maturity \( T \) is given by
\[ V_0 = E_Q[e^{-rT} V(T, S_T)] = e^{-rT} E_Q[\max(S_T - K, 0)]. \]
In the same way, the value of any derivative security (European-style) $V$ with maturity $T$ and payoff (boundary condition) $V(T, S_T) = \Phi(S_T)$ is simply

$$V = e^{-rT} E_Q[\Phi(S_T)]$$

This is a version of the **Feynman-Kac Theorem**, which gives the solution to a large class of parabolic PDE’s as an expectation of a diffusion
Black-Scholes Pricing Formula

The price $C$ of a European Call Option in a risk-neutral world is the well-known **Black-Scholes formula**, where

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

with

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and $N(x)$ is the cumulative probability distribution function of a standard normal variable.
Black-Scholes Pricing Formula (Interpretation)

\[ C = e^{-rT}[Se^{rT}N(d_1) - KN(d_2)]. \]

\(Se^{rT}N(d_1)\) represents the future value of the underlying asset conditional on the end stock value \(S_T\) being greater than the strike price \(K\). The second term in the brackets \(KN(d_2)\) is the value of the known payment \(K\) times the probability \(N(d_2)\) that the strike price will be paid. The terms inside the brackets are discounted by the risk-free rate \(r\) to bring the payments into present value terms. Thus the evaluation inside the brackets is made using the risk-neutral or martingale probabilities.
Financial Mathematics and Financial Industry
Financial Mathematics and Financial Industry

- Physicists began to follow the jobs from academia to Wall Street in the late of 1970s. They arrived on Wall Street in the midst of a financial revolution and they are known as 'quants' because they do quantitative finance. Quants occupy a revealing niche in modern capitalism.

- In the mid-80s, Wall Street turned to the quants-brainy financial engineers-to invent new ways to boost profits. Their methods for minting money worked brilliantly...until one of them devastated the global economy.
Quants Generation
Financial Mathematics and Financial Industry

- Financial innovation currently has a poor reputation and some might feel that mathematicians should think twice before becoming involved with "filthy lucre".

- However, Aristotle tells us that Thales, the father of western science, became rich by applying his scientific knowledge to speculation, Galileo left the University of Padua to work for Cosimo II de Medici, and wrote 'On the Discoveries of Dice' becoming the first quant.

- Around a hundred years after Galileo left Padua, Sir Isaac Newton, left Cambridge to become warden of the Royal Mint, and lost the modern equivalent of £3,000,000 in the South Sea Bubble.
Financial Mathematics and Financial Industry (cntd)

• In the 1970s the late *Fisher Black of Goldman Sacks, Myron Scholes of Stanford and Robert Merton of Harvard* had figured out how to price and hedge these options in a way that seemed to guarantee profits.

• The *Black-Scholes model* has been the quants’ gold standard ever since.
Financial Mathematics and Financial Industry (cntd)

- But it *gets more complicated than that.*

- For example:
  - *markets are not perfectly efficient-prices do not always adjust to right level,*
  - *people are not perfectly rational* 
  - *distribution of market data do not follow bell-shaped curve.*
Fig. 1. Standard Normal and $\alpha$-Stable Densities
Fig. 2. Tails for Normal and $\alpha$-Stable Curves
Financial Mathematics and Financial Industry: Coffee & Volatility Smile

Coffee's Options

- One consequence of this is sometimes called the *volatility smile*, in which options that benefit from market drops cost more than options that benefit from market rises.
Fig. 3. Volatility 'smile'
Fig. 4. Volatility 'smile' and 'skew'
Fig. 6. Volatility surface

Figure 1.3. Implied volatilities of vanilla options on the EUR/USD exchange rate on November 5, 2001.
Financial Mathematics and Financial Industry (cntd)

- Another consequence is that when you need financial models the most-on days like Black Monday in 1987 when the Dow dropped 20 percent-they might break down.

- The risks of relying on simple models are heightened by investors’ desire to increase their leverage by playing with borrowed money. In that case one bad bet can doom a hedge fund.
Dr. Merton and Dr. Scholes won the *Nobel in economic* science in 1997 for the stock option model. Only a year later *Long Term Capital Management (LTCM)*, a highly leverage hedge fund whose directors included the two Novelists, collapsed and had to be *bailed out to the tune of $3.65 billion by a group of banks*.

Afterward, a Merrill Lynch memorandum noted that the financial models 'may provide a greater sense of security than warranted; therefore *reliance on these models should be limited*.'
In 2008, it was hardly unthinkable that a math wizard like David X. Li might someday earn a Nobel Prize for determining correlation.

In 2000, while working at JPMorgan Chase, Li published a paper in *The Journal of Fixed Income* titled 'On Default Correlation: A Copula Function Approach'. (In statistics, a *copula is used to couple the behavior of two or more variables*).
Financial Mathematics and Financial Industry (cntd)

• For five years, Li’s formula, known as a Gaussian copula function- a piece of financial technology that allowed hugely complex risks to be modelled with more ease and accuracy than ever before.

• Then the model fell apart in 2008-users of Li’s formula had not expected: The cracks became full-fledged canyons in 2008-when ruptures in the financial system’s foundation swallowed up trillions of dollars and put the survival of the global banking system in serious peril
How could one formula pack such a devastating punch? *The answer lies in the bond market*, the multimillion-dollar system that allows pension funds, insurance companies, and hedge fund to lend trillions of dollars to companies, countries, and home buyers (mortgages)

*Another answer is correlation*-the degree to which one variable moves in line with another—and measuring it is important part of determining how risky mortgage bonds are
Financial Mathematics and Financial Industry (cntd)

- The damage was foreseeable and, in fact, foreseen. In 1998, before Li had even invented his copula function, Paul Wilmott wrote that 'the correlation between financial quantities are notoriously unstable.', and argued that no theory should be built on such unpredictable parameters.

- 'The relationship between two assets can never be captured by a single scalar quantity', Wilmott said.

- 'Li can not be blamed', says Gilkes of CreditSights. In financial markets, everybody doing the same thing is the classic recipe for a bubble and inevitable bust.
Financial Mathematics and Financial Industry (cntd)

- One of the most outspoken critics is Nassim Nickolas Taleb, a former trader and now a professor at New York University. He got a rock-star reception at the World Economic Forum in Davos in 2009.

- In his best-selling book 'The Black Swan' (Random House, 2007), Dr. Taleb, who made a fortune trading currency on Black Monday, argues that finance and history are dominated by rare and unpredictable events.
Financial Mathematics and Financial Industry (cntd)

- Steven Shreve, the Orion Hoch Professor of mathematical sciences at Carnegie Mellon University and one of the founders of Carnegie Mellon’s Bachelor’s, Master’s and Ph.D. programs in quantitative finance, wrote in his article ‘Don't Blame the Quants’ (Forbes, 2008):

’The quants know better than anyone how their models can fail. For banks, the only way to avoid a repetition of the current crisis is to measure and control all their risks, including the risk that their models give incorrect results’.
New Directions/Developments in FM
Some Prospectives in Financial Mathematics

- Alternatives to Black-Scholes
  - Stochastic volatility models
  - Jump-diffusion models
  - Fractal statistics (applied to many systems in nature and finance, and popularized by Benoit Mandelbrot of IBM; look the same at every scale)
  - Lévy processes
Figure 2.4. Examples of Lévy processes: linear drift (left) and Brownian motion

Figure 2.5. Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion
Fig. 8. Paths of Lévy Processes

**Figure 7.10.** Simulated path of a normal inverse Gaussian (left) and an inverse Gaussian process
Main Original Contributors to the Theory of Lévy Processes (1930-1940)

Paul Lévy (1886-1971)
Alexander Khinchine (1934-1959)
Kiyoshi Itô (1915-2008)
Some Prospectives in Financial Mathematics

- Stochastic Interest-rate modelling
- Pricing in incomplete markets
- Pricing/measuring/hedging credit risk
- Stochastic correlation models
Some Prospectives in Financial Mathematics (cnt’d)

- Real options
- Entropy-based option pricing
- Non-standard finance (based on non-standard analysis)
- Environmental and Energy Finance
Some Prospectives in Financial Mathematics:
Energy Finance

- Energy Finance: use financial instruments to manage storage impact, seasonality, mean-reversion, illiquidity, decentralized energy markets
Some Prospectives in Financial Mathematics: Environmental Finance

- Environmental Finance - use of financial instruments to protect the ecological environment (climate exchanges for trading greenhouse gases (GHG) in Chicago, Europe, China, Canada, Australia)
Some Prospectives in Financial Mathematics: Energy Finance - Carbon Finance

- **Carbon Finance**-investments in GHG emission reduction projects and use financial instruments that are tradable on the carbon markets
Some Prospectives in Financial Mathematics: Energy Finance - Weather Derivatives

- **Weather Derivatives** - use financial instruments to reduce risk associated with adverse or unexpected weather conditions (derivatives are non-tradable)

- **Renewable Energy Finance** - use financial instruments to manage wind, solar, hydro & marine, water, etc., energy
Some Prospectives in Financial Mathematics: Systemic Risk, Big Data, Limit Order Books/Markets

• Systemic Risk (very recent)

• Big Data Science—*Bid Data in Finance*

• *Limit Order Books/Markets*
Some Prospectives in Financial Mathematics: Systemic Risk

- **Systemic risk, or instabilities**, occur in many complex systems: In ecology (diversity of species), in climate change, in material behavior (phase transitions), etc. Mathematical methodologies do overlap.

- Two types of trading in equities are widely practiced today: High-frequency (limit-order and market) trading and statistical arbitrage or market neutral (generalized) pairs trading.

- These types of trading account for well over *two thirds the volume traded today*.

- It is not yet clear *how to quantify the systemic risk*, or the market instabilities generated by these types of trading.
Some Prospectives in Financial Mathematics: Big Data in Finance

**Big data** has now become a driver of model building and analysis in a number of areas, including **finance**.

Main problem: how to deal with big data arising in electronic markets for algorithmic and high-frequency (milliseconds) trading that contain two types of orders, **limit orders** and **market orders**.

More than half of the markets in today’s highly competitive and relentlessly fast-paced financial world now use a **limit order book (LOB)** mechanism to facilitate trade.
Some Prospectives in Financial Mathematics: Limit Order Books/Markets

Orders to buy and sell an asset arrive at an exchange:

1. **Market buy/sell order** - specifies number of shares to be bought/sold at the best available price, right away.

2. **Limit buy/sell order** - specifies a price and a number of shares to be bought/sold at that price, when possible.

3. **Order cancellation** - agents who have submitted a limit order may cancel the order before it is executed.
Some Prospectives in Financial Mathematics: Limit Order Books/Markets II

- *Market orders* are executed immediately

- *Limit orders* are queued for later execution, but may cancel

- The *Limit-Order Book* is the collection of queued limit orders awaiting execution or cancellation
Some Prospectives in Financial Mathematics: Big Data in Finance-Lobster Data

Description of a Big Data: LOBster Data

https://lobsterdata.com/info/DataSamples.php
sample files.

The sample files contain an 'orderbook' file, a 'message' file and a readme summarizing the data's properties. All sample files are based on the official NASDAQ Historical TotalView-ITCH sample.

demo code.

We have prepared small demo codes for Matlab and R to help you get started with LOBSTER's data. The demo files for Matlab and R contain a small code sample, a sample file and a readme. The download links and further code is available in the code help.

download samples.

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<td>Intel</td>
<td>INTC</td>
<td>1</td>
<td>3.3</td>
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</tbody>
</table>
More experienced researchers might be interested in higher level order books. The files provided below contain the limit order book evolution between 09:30:00 and 10:30:00 on the same day as the files above.

- Apple: AAPL Levels: [30] [50]
- Microsoft: MSFT Levels: [30] [50]
- SPDR Trust Series I: SPY Levels: [30] [50]

Please note that if there are unoccupied price levels in the requested price range, LOBSTER’s output contains dummy variables to guarantee a symmetric output. Dummy variables are easily identified by a volume of 0.

more information.

A detailed description of LOBSTER’s output structure can be found here. Details on the access options are available here. The process of joining LOBSTER is outlined here.
Some Prospectives in Financial Mathematics: Big Data in Finance-Lobster Data II

Description of the LOBster Data-Actual Files: http://LOBSTER.wiwi.hu-berlin.de

LOBster generates a 'message' and an 'orderbook' file for each active trading day of a selected ticker. The 'orderbook' file contains the evolution of the LOB up to the requested number of levels. The 'message' file contains indicators for the type of event causing an update of the LOB in the requested price range. All events are timestamped to seconds after midnight, with decimal precision of at least milliseconds and up to nanoseconds depending on the requested period. 'Message' file-3.3 MB, 'Orderbook' file 4.9 MB, if you print it out (do not do that!)-1,370 pages!!!
Some Prospectives in Financial Mathematics: Big Data in Finance-Lobster Data III

Description of the LOBster Data:

http://LOBSTER.wiwi.hu-berlin.de
output.

LOBSTER generates a 'message' and an 'orderbook' file for each active trading day of a selected ticker. The 'orderbook' file contains the evolution of the limit order book up to the requested number of levels. The 'message' file contains indicators for the type of event causing an update of the limit order book in the requested price range. All events are timestamped to seconds after midnight, with decimal precision of at least milliseconds and up to nanoseconds depending on the requested period.

Both the 'message' and 'orderbook' files are provided in the .CSV format and can easily be read with any statistical software package.

Below the structure of the message and orderbook file are described in detail.

message file.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Event Type</th>
<th>Order ID</th>
<th>Size</th>
<th>Price</th>
<th>Direction</th>
</tr>
</thead>
<tbody>
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<td>3</td>
<td>206833312</td>
<td>100</td>
<td>118600</td>
<td>-1</td>
</tr>
</tbody>
</table>

variable explanation
nanoseconds depending on the period requested

- **Event Type:**
  - 1: Submission of a new limit order
  - 2: Cancellation (partial deletion of a limit order)
  - 3: Deletion (total deletion of a limit order)
  - 4: Execution of a visible limit order
  - 5: Execution of a hidden limit order
  - 7: Trading halt indicator (detailed information below)

- **Order ID:** Unique order reference number
- **Size:** Number of shares
- **Price:** Dollar price times 10000 (i.e. a stock price of $91.14 is given by 911400)
- **Direction:**
  - -1: Sell limit order
  - 1: Buy limit order
  - Note: Execution of a sell (buy) limit order corresponds to a buyer (seller) initiated trade, i.e. buy (sell) trade.

**order book file.**

<table>
<thead>
<tr>
<th>Ask Price 1</th>
<th>Ask Size 1</th>
<th>Bid Price 1</th>
<th>Bid Size 1</th>
<th>Ask Price 2</th>
<th>Ask Size 2</th>
<th>Bid Price 2</th>
<th>Bid Size 2</th>
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</thead>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>

**variable explanation.**

- **Ask Price 1:** Level 1 ask price (best ask price)
- **Ask Size 1:** Level 1 ask volume (best ask volume)
- **Bid Price 1:** Level 1 bid price (best bid price)
- **Bid Size 1:** Level 1 bid volume (best bid volume)
- **Ask Price 2:** Level 2 ask price (second best ask price)
- **Ask Size 2:** Level 2 ask volume (second best ask volume)
Some Prospectives in Financial Mathematics: Big Data in Finance-Lobster Data IV

Description of the LOBster Data-Actual Files
http://LOBSTER.wiwi.hu-berlin.de

Snapshot of the 'Orderbook' file
<table>
<thead>
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<th>Value</th>
<th>Type</th>
<th>Value</th>
<th>Type</th>
<th>Value</th>
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Some Prospectives in Financial Mathematics: Big Data in Finance-Lobster Data V

Description of the LOBster Data-Actual Files
http://LOBSTER.wiwi.hu-berlin.de

Snapshot of the 'Message' file
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Some Recent Discoveries in Financial Mathematics: Semi-Markov Evolution of Limit Order Books/Markets

R. Cont and A. de Larrard (SIAM J. Finan. Math, 2013) introduced a tractable stochastic model for the dynamics of a limit order book, computing various quantities of interest such as the probability of a price increase or the diffusion limit of the price process.
Some Recent Discoveries in Financial Mathematics: Semi-Markov Evolution of Limit Order Books/Markets II

Among the various assumptions made in this article, we seek to challenge two of them while preserving analytical tractability:

- the inter-arrival times between book events (limit orders, market orders, order cancellations) are assumed to be independent and exponentially distributed
- the arrival of a new book event at the bid or the ask is independent from the previous events
As suggested by empirical observations, we extend R. Cont and A. de Larrard (SIAM J. Finan. Math, 2013) framework to:

1) arbitrary distributions for book events inter-arrival times (possibly non-exponential) and

2) both the nature of a new book event and its corresponding inter-arrival time depend on the nature of the previous book event.

We do so by resorting to Markov renewal processes to model the dynamics of the bid and ask queues.
Some Recent Discoveries in Financial Mathematics: Semi-Markov Evolution of Limit Order Books/Markets IV

We justify and illustrate our approach by calibrating our model to the five stocks Amazon, Apple, Google, Intel and Microsoft on June 21st 2012 (Courtesy: https://lobster.wiwi.hu-berlin.de/info/DataSamples.php).

When calibrating the empirical distributions of the inter-arrival times to the Weibull and Gamma distributions (Amazon, Apple, Google, Intel and Microsoft on June 21st 2012), we find that the shape parameter is in all cases significantly different than 1 (∼ 0.1 to 0.3), which suggests that the exponential distribution is typically not rich enough to capture the behaviour of these inter-arrival times.
# Numerical Results: Apple Bid

<table>
<thead>
<tr>
<th>Apple Bid</th>
<th>$H(1, 1)$</th>
<th>$H(1, -1)$</th>
<th>$H(-1, -1)$</th>
<th>$H(-1, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Weibull $\theta$</strong></td>
<td>75.9 (71.6-80.5)</td>
<td>180.9 (172.6-189.7)</td>
<td>31.5 (29.5-33.6)</td>
<td>78.2 (73.4-83.3)</td>
</tr>
<tr>
<td><strong>Weibull $k$</strong></td>
<td>0.317 (0.313-0.321)</td>
<td>0.400 (0.394-0.405)</td>
<td>0.271 (0.267-0.274)</td>
<td>0.300 (0.296-0.304)</td>
</tr>
<tr>
<td><strong>Gamma $\theta$</strong></td>
<td>2187 (2094-2284)</td>
<td>1860 (1787-1935)</td>
<td>2254 (2157-2355)</td>
<td>2711 (2592-2835)</td>
</tr>
<tr>
<td><strong>Gamma $k$</strong></td>
<td>0.206 (0.202-0.210)</td>
<td>0.276 (0.271-0.282)</td>
<td>0.168 (0.165-0.171)</td>
<td>0.196 (0.192-0.199)</td>
</tr>
</tbody>
</table>

*Apple Bid: Fitted Weibull and Gamma parameters. 95 % confidence intervals in brackets. June 21\textsuperscript{st} 2012.*
Some Recent Discoveries in Financial Mathematics: Semi-Markov Evolution of Limit Order Books/Markets V

Comparison of CDFs for Empirical and Theoretical Weibull, Gamma and Exponential distributions (stock-Google-June 21st 2012-Bid side)
We now specify formally the "state process", which is semi-Markov process and which will keep track of the state of the limit order book at time $t$ (stock price and sizes of the bid and ask queues),

$$
\tilde{L}_t := (s_t, q^b_t, q^a_t),
$$

where $s_t := \frac{s^a_t + s^b_t}{2}$ is a mid-price,

$$
s_t := s_0 + \sum_{i=1}^{N(t)} X_k,
$$

$X_k = \{-\delta, +\delta\}$, $\delta$-tick size, $q^a_t, q^b_t$ are sizes of bid and ask queues, $N(t)$-number of price changes (renewal process).
Some Recent Discoveries in Financial Mathematics: Semi-Markov Evolution of Limit Order Books/Markets VI

In the context of Cont and Larrard (SIAM J. Finan. Math., 2013), this process $\tilde{L}_t$ was proved to be Markovian. Here, we will need to "add" to this process the process $(V^b_t, V^a_t)$ keeping track of the nature of the last book event at the bid and the ask to make it Markovian: in this sense we can view it as being semi-Markovian. The process:

$$L_t := (s_t, q^b_t, q^a_t, V^b_t, V^a_t)$$

is Markovian, where $V^b_t, V^a_t$ are processes for events of increase or decrease the bid or ask queue by 1, respectively.
Why Study Financial Mathematics?

- **Financial mathematics** is interesting because it synthesizes a highly technical and abstract branch of maths, measure theoretic probability, with practical applications that affect peoples' everyday lives.

- **Financial mathematics** is exciting because, by employing advanced mathematics, we are developing the theoretical foundations of finance and economics.
Conclusion

KEEP CALM AND STUDY FINANCIAL MATHEMATICS

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Some Resources I


Some Resources II

LOBSTER Data: https://lobsterdata.com/info/DataSamples.php

LOBSTER Files: http://LOBSTER.wiwi.hu-berlin.de

Wikipedia
The End

Thank You for Your Time and Attention!

e-mail: aswish@ucalgary.ca

Q&A time!

UNIVERSITY OF CALGARY