Change of Time Methods
in
Quantitative Finance

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Preface

"Time changes everything except something within us, which is always surprised, by change”, -Thomas Hardy.

The present book is devoted to the history of change of time methods (CTM), connection of CTM with stochastic volatilities and finance, and many applications of CTM. A reader may consider this book as a brief introduction to the theory of CTM and as a handbook that can be used to apply to many real life problems. As Winston Churchill once said, ’...I only read for pleasure or for profit,’ similarly, someone may read the present book for pleasure, someone for profit (but see disclaimer below!) and many for both. The author’s intension was to satisfy all the readers who like change of time methods, stochastic volatilities and finance. There is one book that the most close to the present book-’Change of Time and Change of Measure’ by O. Barndorff-Nielsen and A. Shiryaev (Springer, 2010). The difference between the present book and the latter book is that the present book focuses more on applications and presents some novel models, e.g., the delayed version of Heston model, that not covered by the monograph of Barndorff-Nielsen and Shiryaev. For some extent, someone may consider the present book as a useful complement to the latter monograph. I hope that the present book will attract a wide audience from graduate students and quants to researchers in mathematical finance, and also to practitioners in finance and energy areas.

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Disclaimer

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Dedication

To my late parents
Achowledgements

I would like to thank very much to many of my colleagues and graduate students, with whom I discussed or obtained some results presented in the book, and also to all the participants of the 'Lunch at the Lab' finance seminar at our Department of Mathematics and Statistics, University of Calgary, where all the results were first presented and tested.

Many thanks go to my family, wife Mariya, son Victor and daughter Julia (who found and recommended the picture in the Preface), whose continuous support encouraged me on writing and creating.
"Both Aristotle and Newton believed in absolute time...Time was completely separate from and independent of space...However, we have had to change our ideas about space and time..." - Stephen Hawking 'A Brief History of Time'

1 Introduction to the Change of Time Methods

In this Chapter, we consider a history for the change of time methods (CTM), connection of CTM with finance and stochastic volatilities.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, $t \geq 0$. We shall frequently use in this book the notion of: 1) Brownian motion $B_t$ (or Wiener process $W_t$): a process with independent Gaussian (normal) increments and continuous trajectories (A. Einstein (1905) used it when analysing the chaotic motion of particles in a liquid); 2) stochastic differential equation $dX_t = a(X_t, t)dt + \sigma(X_t, t)dB_t$ (describing diffusion process $X_t$ with drift $a$ and diffusion $\sigma$) with local Lipschitz and linear growth conditions for the coefficients $a$ and $\sigma$; 3) martingale $M_t$ (stochastic model for fair game), meaning $E|M_t| < +\infty$ and $E(M_t|\mathcal{F}_s) = M_s$, $s \leq t$, where $M_t$ is a stochastic process on the above mentioned filtered probability space; 4) Lévy process $L_t$ (process that contains deterministic drift, diffusion and jumps) (i.e., stochastically continuous process with stationary and independent increments) (see, e.g., Jacod & Shiryaev (1987), Applebaum (2003)).

The main idea of change of time method is to find a simple representation for a stochastic process with a complicated structure, using some simple process and change of time process. For example, if we consider a Brownian motion $B_t$ (or Lévy process $L_t$) as a simple process and $X_t$ as a complicated process, that satisfies the following stochastic differential equation $dX_t = a(X_t, t)dt + \sigma(X_t, t)dB_t$ (or $dX_t = a(X_t, t)dt + \sigma(X_t, t)dL_t$) on the latter filtered probability space, then the question is: can we represent $X_t$ in the following form

$$X_t = B_{T_t}, \quad (or \quad X_t = L_{T_t})$$

where $T_t$ is a change of time process? In many cases, the answer is 'yes'. In this book, we shall show you those cases and many applications of them.

In general, the procedure of change of time means that we proceed from old (or physical, calendar) time $t$ to a new (operational, business) time $t'$ with $t' = T_t$ in a such way to be able to construct our initial complicated process $X_t$ (old one) through a simple process $\hat{X}_t$ (new one) that satisfies the relation $X_t = \hat{X}_{T_t}$. If we define $t = T'_t$, then $\hat{X}_{t'} = X_{T'_t}$, $t' > 0$. 

1
1.1 A Brief History of Change of Time Method

To the best of the author’s knowledge, Wolfgang Doeblin (see Doeblin W. (1940), Lévy (1955), Lindvall (1991) and also Bru and Yor (2002) for details) was the first one who introduced change of time method into the theory of stochastic processes. He took the martingale point of view in his analysis of the paths of an inhomogeneous real-valued diffusion $X_t, t \geq 0$, starting from $x$, with drift coefficient $a(x)_{t}$ and diffusion coefficient $\sigma(x)_{t}$. If we define $Y_t := X_t - x - \int_0^t a(X_s, s)ds$ and $h_t := \int_0^t \sigma^2(X_s, s)ds$, then he proved first that $Y_t$ and $Y_t^2 - h_t$ are martingales without mentioning the notion of martingale (see Doeblin (1940)). Secondly, Doeblin introduced the time change $\theta(\tau) := \inf\{t : m_t > \tau, \tau > 0\}$ and showed that $B(\tau) := Y_{\theta(\tau)}$ is a Brownian motion. In fact, he proved that there exists a Brownian motion $B(s), s \geq 0$, such that $Y_t := B(h_t)$. All in all, Doeblin has obtained the representation of $X_t$ as

$$X_t = x + B(h_t) + \int_0^t a(X_s, s)ds. \quad (1)$$

Of course, at that time the general notion of martingale did not exist (see for more details Bru and Yor (2002)). The notion of positive martingale and its denomination (which Doeblin does not use) are due to Ville, in his 1939 thesis (see Ville (1939)) (martingale property was also used by Levy under another name, 'condition C', since 1934). A several years later, K. Ito (1942a, 1942b (see also Ito (1951a, 1951b)) presented $X_t$ in the form of stochastic differential equation

$$X_t = x + \int_0^t a(X_s, s)ds + \int_0^t \sigma(X_s, s)dB(s), \quad (2)$$

where $B(s)$ is a Brownian motion. If we compare Doeblin’s and Ito’s representations (1) and (2), then we can obtain

$$\int_0^t \sigma(X_s, s)ds = B(h_t). \quad (3)$$

The latter result (3) would be understood, of course, in a general setting many years later with the Dambis (1965) and Dubins-Schwartz (1965) representation of a continuous martingale $M_t$ as

$$M_t = B(< M >_t), \quad (4)$$

where $B(u)$ is a Brownian motion and $< . >$ is the quadratic variation. The idea of associating with a diffusion $X_t$ a compensated process $Y_t$ which follows the trajectories of a standard Brownian motion is presented in the works of Levy on additive processes (see Levy (1934, 1937, 1948)) and in the seminal paper of Kolmogorov (1931). However, as Bru and Yor (2002) mentioned, "Doeblin’s method goes much further and the change of time which he adopts..."
It seems to be original. It is usually attributed to Volkonskii (1958); see e.g. Dynkin (1965); Williams (1979). In any case, there does not seem to be much use for random time-changes in the study of diffusion before the end of the fifties”. Though, we would like to mention that Bochner (1949) also used the notion of ‘change of time’, namely, time-changed Brownian motion, before the beginning of fifties.

Girsanov (1960) used change of time method to find a nontrivial weak solution to the following stochastic differential equation \( dX_t = |X_t|^\alpha dB_t \), where \( X_0 = 0 \), \( B_t \) is a Brownian motion and \( 0 < \alpha < 1/2 \).

It worth to mention here that the change of time method is closely associated with the embedding problem: to embed a process \( X(t) \) in Brownian motion is to find a Brownian motion (or Wiener process) \( B(t) \) and an increasing family of stopping times \( T_t \) such that \( B(T_t) \) has the same joint distribution as \( X(t) \). Skorokhod (1965) first treated the embedding problem, showing that the sum of any sequence of independent random variables with mean zero and finite variation could be embedded in Brownian motion using stopping times. See also Monroe (1972).

Dambis (1965), Dubins & Schwartz (1965) independently showed that every continuous martingale could be embedded in Brownian motion (in the sense of (4) above). Feller (1936, 1966) introduced subordinated process \( X(T_t) \) for a Markov process \( X(t) \) with \( T_t \) a process having independent increments. \( T_t \) was called ‘randomized operational time’. Huff (1969) showed that every process of pathwise bounded variation could be embedded in Brownian motion. Knight (1971) discovered a multivariate extension of the Dambis (1965) and Dubins & Schwartz (1965) result. Monroe (1972) proved that every right continuous martingale could be embedded in a Brownian motion. Meyer (1971) and Papangelou (1972) independently discovered Knight’s (1971) result for point processes.

Clark (1973) introduced change of time method into financial economics. Monroe (1978) proved that a process can be embedded in Brownian motion if and only if this process is a local semimartingale. Johnson (1979) introduced a time-changed stochastic volatility model in continuous time. Ikeda & Watanabe (1981) introduced and studied the change of time to find the solutions of stochastic differential equations. Rosiński & Woyczyński (1986) considered time changes for integrals over a stable Lévy processes. Lévy processes can also be used as a time change for other Lévy processes (subordinators). Johnson & Shanno (1987) studied pricing of options using a time-changed stochastic volatility model.

Madan & Seneta (1990) introduced variance gamma process (i.e., Brownian motion with drift time changed by a gamma process). Kallsen & Shiryaev (2001) showed that the Rosiński-Woyczyński-Kallenberg result cannot be extended to any other Lévy processes other than the symmetric \( \alpha \)-stable processes. Kallenberg (1992) considered time change representations for stable integrals. Geman, Madan & Yor (2001) considered time changes (‘business times’) for Lévy processes.
Barndorff-Nielsen, Nicolato & Shephard (2002) studied the relationship between subordination and stochastic volatility models using a change of time (they called it the $T_t$-‘chronometer’). Carr, Geman, Madan, Yor (2003) used subordinated processes to construct stochastic volatility for Lévy processes, ($T_t$ being ‘business time’).

Carr, Geman, Madan & Yor (2003) also used a change of time to introduce stochastic volatility into a Lévy model to achieve leverage effect and a long-term skew.


The book ‘Option Prices as Probabilities’ by C. Profeta, B. Roynette and M. Yor (Springer, 2010) is also relies on time change methods.

The recent book 'Change of Time and Change of Measure' by Barndorff-Nielsen and Shiryaev (2010) states the main ideas and results of the stochastic theory of 'change of time and change of measure'.

1.2 Change of Time Method and Stochastic Volatility

The volatility is a measure for variation of price $S_t$ of a financial instrument over time $t \geq 0$. We use the symbol $\sigma$ for volatility and it corresponds to standard deviation which quantifies the amount of variation or dispersion of a set of data values $S_t$. Of course, $\sigma$ can be positive constant, positive deterministic function of time $\sigma(t)$ or a stochastic process $\sigma(t, \omega)$, $\omega \in \Omega$, e.g., that satisfies some stochastic differential equation. The model for volatility that initiated the stochastic volatility model was implied volatility model: this volatility $\sigma \equiv \sigma(K, T)$ can be derived from the Black-Scholes formula for the European call option price and demonstrates the smile effect, e.g., dependency of volatility from strike price $K$ and maturity $T$ (see Fouque, Papanicolaou and Sircar (2000)). This smile effect tells us that the Balck-Scholes model with a constant volatility is not adequate to statistical and probabilistical structures of observable prices $S_t$. Merton (1973) was the first one who replaced the constant volatility $\sigma$ by a deterministic function $\sigma = \sigma(t), \ t \geq 0$. In such models there is no smile effect across strike, however smile effect appears for different maturities. Another way to obtain smile effect with nonstochastic volatility is to add to the deterministic volatility $\sigma(t)$ one more variable, namely, phase one, $S : \sigma \equiv \sigma(t, S)$ (see Dupire (1994)). We can go further and assume that volatility depends not only from $t$ and $S$, but also on all preceding values $S_u, u \leq t$, i.e., the volatility
\( \sigma(t, S_u; u \leq t) \) depends on all past observed prices, or volatility depends on its own past values. The latter case will be considered in Chapter 8. Besides smile effect, mean reversion (i.e., returning of the volatility to the mean) is another important property of stochastic volatility. That’s why most of modern models of stochastic volatility are assumed that the volatility is generated by another source of randomness than initial Brownian motion \( B_t \), say by process \( Y_t \), which correlates with \( B_t \), and this process follows some mean-reversion process, say, Ornstein-Uhlenbeck or CIR process (see Cont and Tankov (2004)).

The connection of change of time with stochastic volatility can be described by the following representation: 

\[
X_t = \hat{X}_{T(t)} = \int_0^t \sigma(s, \omega) d\tilde{X}_s,
\]

where \( X_t \) is a given process, \( \sigma(t, \omega) \) is a stochastic volatility, \( \hat{X}_t \) is a simple initial process and \( T(t) \) is a change of time process. In many cases in finance, the process \( \tilde{X}_t \) is a Brownian motion or Lévy process. In general case, the process \( \tilde{X}_t \) is a semimartingale, meaning \( \tilde{X}_t = \tilde{X}_0 + A_t + M_t \), where \( A_t \) is a process of bounded variation and \( M_t \) is a local martingale (see, e.g., Barndorff-Nielsen and Shiryaev (2010)).

The most typical example of this connection between change of time and stochastic volatility is the following one. Let 

\[
M_t = \int_0^t \sigma(s, \omega) dB_s, t \geq 0,
\]

where \( B_s \) is a Brownian motion, \( \sigma(s, \omega) \) is a positive process such that \( \int_0^t \sigma^2_s ds < +\infty \). Then \( M_t \) can be presented in the following way:

\[
M_t = \hat{B}_{\hat{T}_t},
\]

where \( T_t := \int \sigma^2_s ds, \hat{B}_t := M_{\hat{T}_t} \), and \( \hat{T}_t = \inf\{s : \int_0^s \sigma^2_u du \geq t\} \). We note that \( \hat{B}_t := M_{\hat{T}_t} \) is a Brownian motion with respect to the filtration \( \hat{\mathcal{F}}_t := \mathcal{F}_{\hat{T}_t} \).

Another interesting example associated with the \( \alpha \)-stable processes \( L^\alpha_s \), \( 0 < \alpha \leq 2 \) (see Applebaum (2003)). Let 

\[
X_t = \int_0^t \sigma(s, \omega) dL^\alpha_s, T(t) := \int_0^t |\sigma(s, \omega)|^\alpha ds < +\infty, \text{ and } \hat{T}_t = \inf\{s \geq 0 : T_s > t\}.
\]

Then

\[
\hat{L}^\alpha_t := X_{\hat{T}_t}, \quad t \geq 0,
\]

is an \( \alpha \)-stable process. The proof follows from the Doob optional sampling theorem and the characteristic property of semimartingales (see Jacod & Shiryaev (1987)). We note that process \( X_t \) can be represented through change of time in the following way:

\[
X_t = \hat{L}^\alpha_{\hat{T}_t}.
\]

We shall use this approach in Chapters 6 and 7 for Lévy-based and multi-factor financial models.

It is worth to mention that the only change of time process \( T_t \) that retains the Gaussian property of the time changed Brownian motion \( B_{T_t} \) is deterministic one.

The probability literature has demonstrated that stochastic volatility models and their time-changed Brownian motion relatives are fundamental (see Shephard (2005a, 2005b)).
1.3 Change of Time Method and Finance

The change of time method in finance is related directly to the notion of volatility, the measure for variation of price of a financial instrument (stock, etc.) over time $t \geq 0$. Often the change of time is called an 'operational time' (the term first coined by Fellner (1966)) or 'business time' (Carr, Madan & Yor (2001)). This time measures the intensity of the variations/fluctuations of the prices in the financial markets. The notion of change of time is very important in finance because many prices in the financial markets can be expressed in the form of Brownian motion with the changed time, called the operational or business time.

The role of Brownian motion in finance is also hard to underestimate: besides its important role in probability and stochastic processes (central limit theorem, functional limit theorem and so-called time-changed Brownian motion processes), it was the main component in modeling of the dynamics of financial asset prices $S_t, t \geq 0$. We would like to mention Bachelier model (1900) $S_t = S_0 + \mu t + \sigma B_t$ and Samuelson model (1965) $\log \frac{S_t}{S_0} = \mu t + \sigma B_t$ for asset prices $S_t$, where $B_t$ is a Brownian motion. Another important process in finance is Poisson process, antipode to Brownian motion, which was first used by Lundberg (1903) to model the dynamics of the capital of insurance companies.

Both Brownian motion and Poisson process are main components in constructing more general class of processes in finance and insurance, Lévy processes (see Lévy (1934, 1935, 1937, 1954), Applebaum (2003)). To go further, we mention that even more general processes recently have been used to construct many financial and insurance model, namely, processes with independent increments and semimartingales (see Shiryaev (2008)). The latter processes are not necessarily homogeneous as in the case of Lévy processes. As we can see later, we can obtain these kind of processes in finance if we consider Bachelier (1903) and Samuelson (1965) models, mentioned above, in change of time mode, namely, $S_t = S_0 + \mu T_t + \sigma B_T$, and $\log \frac{S_t}{S_0} = \mu T_t + \sigma B_T$, respectively, where $T_t$ is a change time process. Another important example in finance can be constructed if we take gamma process $T_t$ and a Brownian motion $B(t)$, independent of each other, and then form a new process $G(t)$ such that $G(t) = \mu t + \beta T_t + B_T$. This process is called the variance gamma process (or VG process) (see Madan & Seneta (1990)). To catch the leverage and clustering effects, other models of stock prices in finance can be obtained using, for example, exponential Lévy models that include both stochastic volatility and change of time, and also the models based on fractional Brownian motion (see Barndorff-Nielsen and Shiryaev (2010)).

Many stochastic differential equations, that practitioners use in finance, can be solved using change of time method as well. One of such equations is Ornstein-Uhlenbeck equation (see Ornstein & Uhlenbeck (1930)) (we present this equation in general form):

$$dX_t = (a(t) - b(t)X_t)dt + c(t)dW(t),$$
where $W(t)$ is a Wiener process (or Brownian motion) and $a, b, c$ are deterministic functions of time $t \geq 0$. The solution of this equation is:

$$X_t = \exp(-\int_0^t b(s)ds)[X_0 + \int_0^t a(s) \exp(\int_0^s b(u)du)ds + \int_0^t c(s) \exp(\int_0^s b(u)du)dW_s].$$

The solution of this equation can also be presented in the following form, using change of time method (see Swishchuk (2007) and Ikeda & Watanabe (1981)):

$$X_t = \exp(-\int_0^t b(s)ds)[X_0 + \int_0^t a(s) \exp(\int_0^s b(u)du)ds + B_{T_t}],$$

where $T_t = \int_0^t (c(s)) \exp(\int_0^s b(u)du)^2ds$-change of time process, and $B_t$ is a new Brownian motion. The latter Brownian motion can be obtained from the previous one, $W(t)$, by the following formulae:

$$B_t := \int_0^{\hat{T}_t} c(s) \exp(\int_0^s b(u)du)dW(s),$$

where $\hat{T}_t = \inf\{s : T_s > t\}$, where $T_t$ is defined above.

We shall use this approach in the next Chapter 2 to find the solutions of many stochastic differential equations.

2 Structure of the Book

Chapter 2 is devoted to the general theory of change of time method and many approaches in this field.

Chapter 3 gives yet one more derivation (among many others) of Black-Scholes option pricing formula using change of time method. In this chapter we also present a brief introduction to the option pricing theory.

Chapter 4 models and prices variance, volatility, covariance and correlation swaps for classical Heston model of a stock price. We also give a numerical example based on S&P60 Canada Index.

Chapter 5 introduces a new delayed Heston model for pricing of variance and volatility swaps, and also for hedging volatility swaps using variance swaps. We use here change of time method as well, and also calibrate all the parameters based on real data. This model improves the market volatility surface fitting by 44% compare with classical Heston model.

Chapters 6 introduces multi-factor Lévy-based financial and energy models. Change of time method is used to price many financial and energy derivatives.

Chapter 7 deals with explicit option pricing formula for a mean-reverting asset in energy markets using change of time method. We also present here a numerical example for AECO natural gas index.
Finally, Chapter 8 deals with variance and volatility swaps in energy markets using CTM as well.

All chapters contain their own list of References.

**Warning!**: We note that we use throughout the book two notations for the change of time process, namely, $T_t$ and $\phi_t$, with respect to Barndorff-Nielsen and Shiryaev’s and Ikeda and Watanabe’s books. For example, in Chapters 1, 2 and 7 we use $T_t$ and in Chapters 3-5 and 8 we use $\phi_t$.

# 3 References


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Chapter 2: Change of Time Methods: Definitions, Theory and Applications

'It is always more easy to discover and proclaim general principles than to apply them',-Winston Churchill.

1 Change of Time Methods: Definitions, Properties and Theory

In this Chapter, we consider a general theory of change of time method. One of probabilistic methods which is useful in solving stochastic differential equations (SDEs) arising in finance is the change of time method. In this Introduction we give definition of CT, describe CTM in martingale, semimartingale and the SDEs settings. We also point out the association of CTM with subordinators and stochastic volatilities. Applications of CTM such as Black-Scholes formula, option pricing formula for a mean-reverting asset, variance and volatility swap prices for Heston and delayed Heston models, and variance and volatility swap prices in energy markets are described as well.

1.1 Change of Time Process: Definition and Properties

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space with a sample space $\Omega$, $\sigma$-algebra $\mathcal{F}$ of subsets of $\Omega$ and probability measure $P$. The filtration $\mathcal{F}_t$, $t \geq 0$, is a nondecreasing right-continuous family of sub-$\sigma$-algebras of $\mathcal{F}$.

Definition of Change of Time Process. A time change process is a right-continuous increasing $[0, +\infty]$-valued and $\mathcal{F}_t$-adapted process $(T_t)_{t \in \mathbb{R}_+}$ such that $\lim_{t \to +\infty} = +\infty$. $T_t$ is also a stopping time for any $t \in \mathbb{R}_+$.

By $\hat{\mathcal{F}}_t := \mathcal{F}_{T_t}$ we define the time-changed filtration $(\hat{\mathcal{F}}_t)_{t \in \mathbb{R}_+}$. The inverse time change $(\hat{T}_t)_{t \in \mathbb{R}_+}$ is defined as $\hat{T}_t := \inf\{s \in \mathbb{R}_+ : T_s > t\}$. We note that $\hat{T}_t$ is increasing process and $\lim_{t \to +\infty} \hat{T}_t = +\infty$. Furthermore, $\hat{T}_t$ is an $\mathcal{F}_t$-stopping time. Let $X_t$ be an $\mathcal{F}_t$-adapted process and define $X_{\hat{T}_t}$. Then $X_{\hat{T}_t}$ is $\hat{\mathcal{F}}_t$-adapted process and this process is called the time change of $X_t$ by $T_t$.

One of the examples of change of time is the following. Let $A_t$ be an $\mathcal{F}_t$-adapted, increasing, right-continuous random process with $A_0 = 0$. Define

$$\hat{T}_t = \inf\{s : A_s > t\}, \quad t \geq 0.$$ 

Then the process $\hat{T}_t$ is a change of time process. We call $A_t$ as the process generating the change of time $\hat{T}_t$. We note that the process $T_t$ (see above definition) coincides with $A_t$ in this case. It means that changes of time
processes $T_t$ and $\hat{T}_t$ are mutually inverse processes: someone may construct $\hat{T}_t$ using $T_t$,

$$\hat{T}_t = \inf\{s : T_s > t\},$$

or, may construct $T_t$ using $\hat{T}_t$:

$$T_t = \inf\{s : \hat{T}_s > t\}.$$  

We also note that

$$T_{\hat{T}_t} = t \quad \text{and} \quad \hat{T}_{T_t} = t.$$  

We would like to also mention the change of time in Lebesque-Stiltjes integrals, which is well-known from calculus. If we take $A_t, A_0 = 0$, as a deterministic increasing continuous function, $f(t)$ as a nonnegative Borel function on $[0, +\infty)$, and put

$$\hat{A}_t = \inf\{s : A_s > t\},$$

then we have

$$\int_0^{\hat{A}_a} f(t) dA_t = \int_0^a f(\hat{A}_t) dt, \quad a > 0.$$  

We note that $A_t = \int \{s : \hat{A}_s > t\}$ and $A_{\hat{A}_t} = t$. The latter expression can be written in the symmetric form as well:

$$\int_0^{A_a} f(t) d\hat{A}_t = \int_0^a f(A_t) dt, \quad a > 0.$$  

There are many stochastic generalizations of the last two relationships for the case of $A$ and $f$ being some stochastic processes. One of them is the following one: let $f(t, \omega)$ be a progressively measurable nonegative stochastic process, and let $B_t(\omega)$ be an $\mathcal{F}_t$-measurable right-continuous process with bounded variation. Then

$$\int_0^{\hat{T}_a} f(t, \omega) dB_t(\omega) = \int_0^a f(\hat{T}_t, \omega) dB_{\hat{T}_t}(\omega),$$

where $\hat{T}_t$ is the inverse change of time process. For example, if $f(t, \omega) = F(T(t))$, where $T_t = A_t$, and $A_t$ is a continuous and strictly increasing process generating the change of time $\hat{T}_t$ (see above), then

$$\int_0^{\hat{T}_a} F(T_t) dB_t(\omega) = \int_0^a F(t) dB_{\hat{T}_t}(\omega).$$

Of course, if $B_t = t$, then

$$\int_0^{\hat{T}_a} F(T_t) dt = \int_0^a F(t) d\hat{T}_t,$$

and, if $B_t = T_t$, then

$$\int_0^{\hat{T}_a} F(T_t) dT_t = \int_0^a F(t) dt.$$  

1.2 CTM: Martingale and Semimartingale Settings

The general theory of time changes for martingale and semimartingale theories is well known (see Ikeda and Watanabe (1981)). We will give a brief overview of those results.

The following result on martingales and change of time process belongs to Dambis (1965) and Dubins and Schwartz (1965): Suppose $M_t$ is a continuous local martingale such that $\lim_{t \to +\infty} <M>_t = +\infty$ a.s., and define $\hat{T}_t := \inf\{u : <M>_u > t\}$ and $\hat{F}_t = \mathcal{F}_{\hat{T}_t}$. Then the time changed process $B(t) := M_{\hat{T}_t}$ is an $\hat{F}_t$-Brownian motion. Also, $M_t = B(<M>_t)$. Thus, $M_t$ can be presented by a $\hat{F}_t$-Brownian motion $B(t)$ and an $\hat{F}_t$-stopping time $<M>_t$. Here, $<\cdot>$ defines predictable quadratic variation. One of the examples of this result was considered in section 1.2. for a continuous local martingale $M_t = \int_0^t \sigma_s(\omega)dB(s)$, where $B_t$ is a Brownian motion and $\sigma_t(\omega)$ is a positive process such that $\int_0^t \sigma_s^2(\omega)ds < +\infty$. In this case, $\hat{T}_t = \inf\{s : \int_0^s \sigma_u^2(\omega)du \geq t\}$ and $\hat{F}_t = \int_0^t \sigma_u^2(\omega)du ds$.

This result was generalized by Knight (1971) for $d$-dimensional case: Let $M^i_t$ be square integrable local continuous martingales, $i = 1, 2, \ldots, d$, such that $<M^i, M^j>_t = 0$ if $i \neq j$ and $\lim_{t \to +\infty} <M^i>_t = +\infty$ a.s. If $\hat{T}^i_t = \inf\{u : <M^i>_u > t\}$, then $\hat{B}(t) = (B^1(t), B^2(t), \ldots, B^d(t))$ is a $d$-dimensional Brownian motion, where $B^i(t) = M^i_{\hat{T}^i_t}$, $i = 1, 2, \ldots, d$.

One of the main properties of semimartingale $X_t$ with respect to the CTM is the following (see Liptser and Shiryaev (1989)): If $X_t$ is a semimartingale with respect to a filtration $\mathcal{F}_t$, then the changed time process $X_{\hat{T}_t}$ is also a semimartingale with respect to the filtration $\hat{\mathcal{F}}_t$ (see sec. 1.1).

If we have the triplet of predictable characteristics $(B_t, C_t, \nu)$ for a semimartingale $X_t$, then the triplet of the time changed semimartingale $X_{\hat{T}}$ is determined as $(\hat{B}_t, \hat{C}_t, I_{\hat{G}_{\hat{T}_t}})$ (see Kallsen and Shiryaev (2001)).

The connection of semimartingales, Brownian motions and CTM is described by the Monroe result (see Monroe (1978)): if $X_t$ is a semimartingale, then there exists a filtered probability space with Brownian motion $\hat{B}_t$ and a change of time $T_t$ on it such that distribution of $X_t$ coincides with the distribution of $\hat{B}_{T_t}$, namely,

\[ X_t \overset{\text{law}}{=} \hat{B}_{T_t}. \]  

Let us consider now a counting process $N_t$ with respect to the filtration $\mathcal{F}_t$ and with the continuous compensator $A_t$ such that $N_t = A_t + M_t$, where $M_t$ is a local martingale. Here, $<M>_t = A_t$. Let us then define time change as $\hat{T}_t = \inf\{s : <M>_s > t\}$. If we suppose that $<M>_{+\infty} = +\infty$, then the following process

\[ \hat{N}_t := N_{\hat{T}_t} \]

is a standard Poisson process with the intensity parameter $\lambda = 1$. We note that the initial counting process $N_t$ can be expressed in the following way:

\[ N_t = \hat{N}_{T_t}, \]
where $T_t = < M >_t$. We note that here $M_t = \check{M}_t$, where $\check{M}_t = \check{N}_t - t$ is a Poisson martingale (see Liptser and Shiryaev (2001) for more details).

Suppose that we have a nondecreasing Lévy process $X_t$ and a Brownian motion $\check{B}_t$ independent of $X_t$. Then we can find a change of time $T_t$ such that

$$X_t = \check{B}_{T_t}$$

holds with probability one. This change of time $T_t$ can be found as

$$T_t = \inf\{s : \check{B}_s = X_t\}.$$

We mention that a semimartingale $X_t$ can be presented in the form (1) with continuous change of time $T_t$ if and only if the process $X_t$ is a continuous local martingale (see Huff (1969) and Cherny and Shiryaev (2002) for more details).

### 1.3 CTM: Subordinators and Stochastic Volatility

We note, that if the process $\check{T}_t$ (see sec. 1.1) is a Lévy process, then $\check{T}_t$ is called subordinator. Feller (1966) introduced a subordinated process $X_{\tau_t}$ for a Markov process $X_t$ and $\tau_t$ a process with independent increments. $\tau_t$ was called a ‘randomized operational time’. Increasing Lévy processes can also be used as a time change for other Lévy processes (see Applebaum (2004), Barndorf-Nielsen et al. (2001), Barndorf-Nielsen et al. (2003), Bertoin (1996), Cont et al. (2004) and Schoutens (2003)). Lévy processes of this kind are called subordinators. They are very important ingredients for building Lévy based models in finance (see Cont et al. (2004) and Schoutens (2003)). If $S_t$ is a subordinator, then its trajectories are almost surely increasing, and $S_t$ can be interpreted as a ‘time deformation’ and used to ‘time change’ other Lévy processes. Roughly, if $(X_t)_{t \geq 0}$ is a Lévy process and $(S_t)_{t \geq 0}$ is a subordinator independent of $X_t$, then the process $(Y_t)_{t \geq 0}$ defined by $Y_t := X_{S_t}$ is a Lévy process (see Cont et al. (2004)). This time scale has the financial interpretation of business time (see Geman et al. (2001)), that is, the integrated rate of information arrival. Using subordinator $S_t$ and a Brownian motion $\check{B}_t$ that is independent of $S_t$we can construct many stochastic processes $X_t = \check{B}_{S_t}$. For example: for Cauchy process $S_t = \inf\{s : B_s > t\}$, where $B_s$ is a standard Brownian motion independent of $\check{B}_t$; for generalized hyperbolic Lévy processes $S_t$ is generated by the nonnegative infinitely divisible random variable having generalized inverse Gaussian distribution (the normal inverse Gaussian and hyperbolic Lévy processes are particular cases of the generalized hyperbolic Lévy processes).

The time change method was used to introduce stochastic volatility into a Lévy model to achieve the leverage effect and a long-term skew (see Carr et al. (2003)). In the Bates (1996) model the leverage effect and long-term skew were achieved using correlated sources of randomness in the price process and
the instantaneous volatility. The sources of randomness are thus required to be Brownian motions. In the Barndorff-Nielsen et al. (2001, 2002) model the leverage effect and long-term skew are generated using the same jumps in the price and volatility without a requirement for the sources of randomness to be Brownian motions. Another way to achieve the leverage effect and long-term skew is to make the volatility govern the time scale of the Lévy process driving jumps in the price. Carr et al. (2003) suggested the introduction of stochastic volatility into an exponential-Lévy model via a time change. The generic model here is

$$S_t = \exp(\int_0^t \sigma_s^2 ds)$$

where $$\sigma_t$$ := \int_0^t \sigma_s^2 ds. The volatility process should be positive and mean-reverting (i.e., an Ornstein-Uhlenbeck or Cox-Ingersoll-Ross processes). Barndorff-Nielsen et al. (2003) reviewed and placed in the context some of their recent work on stochastic volatility models including the relationship between subordination and stochastic volatility.

In general setting, the connection between stochastic volatility and change of time can be described in the following way. Let

$$X_t = \int_0^t H_s dB_s,$$

where $$H_s$$ is the adapted process such that \(\int_0^t H_s^2 ds < \infty\), \(\int_0^\infty H_s^2 ds = \infty\), and $$B_t$$ is a Brownian motion. Then the process $$\hat{B}_t := X_{\hat{T}_t}$$, where $$\hat{T}_t = \inf\{s : <X>_s > t\}$$, is a Brownian motion. Moreover, the process $$X_t$$ has the following representation $$X_t = \hat{B}_{\hat{T}_t}$$, where $$\hat{T}_t = <X>_t = \int_0^t H_s^2 ds$$.

In the case of \(\alpha\)-stable processes $$Y_t^\alpha$$ instead of Brownian motion $$B_t$$ in the integral $$X_t = \int_0^t H_s dY_s^\alpha$$, we have similar result for $$\hat{T}_t = \int_0^\infty |H_s|^\alpha ds < \infty$$ and $$\int_0^\infty |H_s|^\alpha ds = \infty$$. If we set $$\hat{T}_t = \inf\{s : T_s >> t\}$$, then $$\hat{Y}_t^\alpha = X_{\hat{T}_t}$$ is an \(\alpha\)-stable process and $$X_t = \hat{Y}_{\hat{T}_t}$$.

The main difference between the change of time method and the subordinator method is that in the former case the change of time process $$T_t$$ depends on the process $$X_t$$, but in the latter case, the subordinator $$S_t$$ and Lévy process $$X_t$$ are independent.

### 1.4 CTM: Stochastic Differential Equations (SDEs) Setting

#### 1.4.1 General Result

We consider the following generalization of the previous results to the SDE of the following form (without a drift)

$$dX(t) = \alpha(t, X(t))dW(t), \quad (2.1)$$

where $$W(t)$$ is a Brownian motion and $$\alpha(t, X)$$ is a continuous and measurable by $$t$$ and $$X$$ function on $$[0, +\infty) \times R$$.

The reason to consider this equation is the following one: if we solve the equation, then we can solve and more general equation with a drift $$\beta(t, X)$$ by drift transformation method or Girsanov transformation (see Ikeda and Watanabe (1981), Chapter 4, Section 4).
Theorem 2.1. (Ikeda and Watanabe (1981), Chapter IV, Theorem 4.3)
Let \( \tilde{W}(t) \) be an one-dimensional \( \mathcal{F}_t \)-Wiener process with \( \tilde{W}(0) = 0 \), given on a probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) and let \( X(0) \) be an \( \mathcal{F}_0 \)-adopted random variable. Define a continuous process \( V = V(t) \) by the equality

\[
V(t) = X(0) + \tilde{W}(t).
\]

Let \( \phi_t \) be the change of time process (see Section 2.3):

\[
T_t = \int_0^t \alpha^{-2}(T_s, X(0) + \tilde{W}(s))ds.
\]

If

\[
X(t) := V(\hat{T}_t) = X(0) + \tilde{W}(\hat{T}_t)
\]

and \( \hat{\mathcal{F}}_t := \mathcal{F}_{\hat{T}_t} \), then there exists \( \hat{\mathcal{F}}_t \)-adopted Wiener process \( W = W(t) \) such that \( (X(t), W(t)) \) is a solution of (1) on probability space \( (\Omega, \mathcal{F}, \hat{\mathcal{F}}_t, P) \). Here, \( \hat{T}_t \) is the inverse process to \( T_t \) in (2.3).

Proof of this Theorem may be found in Ikeda and Watanabe (1981), Chapter IV, Theorem 4.3.

We note that in this case,

\[
M(t) := \tilde{W}(\hat{T}_t)
\]

is a martingale with quadratic variation

\[
< M > (t) = \hat{T}_t = \int_0^{\hat{T}_t} \alpha^2(T_s, X)dT_s = \int_0^t \alpha(s, X)^2ds,
\]

and \( \hat{T}_t \) satisfies the equation

\[
\hat{T}_t = \int_0^t \alpha^2(s, X(0) + \tilde{W}(\hat{T}_s))ds.
\]

We also remark, that

\[
W(t) = \int_0^t \alpha^{-1}(s, X(s))d\tilde{W}(\hat{T}_s) = \int_0^t \alpha^{-1}(s, X(s))dM(s)
\]

and

\[
X(t) = X(0) + \int_0^t \alpha(s, X)dW(s).
\]
1.4.2 Corollary.

The solution of the following SDE

\[
dX(t) = a(X(t))dW(t)
\]

may be presented in the following form

\[
X(t) = X(0) + \tilde{W}(\hat{T}_t),
\]

where \(a(X)\) is a continuous measurable function, \(\tilde{W}(t)\) is an one-dimensional \(\mathcal{F}_t\)-Wiener process with \(\tilde{W}(0) = 0\), given on a probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) and let \(X(0)\) is an \(\mathcal{F}_0\)-adopted random variable. In this case,

\[
T_t = \int_0^t a^{-2}(X(0) + \tilde{W}(s))ds,
\]

and

\[
\hat{T}_t = \int_0^t a^2(X(0) + \tilde{W}(\hat{T}_s))ds.
\]

(See Ikeda and Watanabe (1981), Chapter IV, Example 4.2).

We note that

\[
M(t) := \tilde{W}(\hat{T}_t)
\]

is a martingale with quadratic variation

\[
<M>(t) = \hat{T}_t = \int_0^{\hat{T}_t} a^2(X) dT_s = \int_0^t a(X)^2 ds.
\]

We also remark, that

\[
W(t) = \int_0^t a^{-1}(X(s))d\tilde{W}(\hat{T}_s) = \int_0^t a^{-1}(X(0) + \tilde{W}(\hat{T}_s))d\tilde{W}(\hat{T}_s)
\]

and

\[
X(t) = X(0) + \int_0^t a(X(s))dW(s).
\]

1.4.3 One-factor Diffusion Models and their Solutions Using CTM

In this section, we introduce well-known one-factor diffusion models (used in finance) described by SDEs and driven by a Brownian motion (so-called Gaussian models).

For one-factor Gaussian models we define the following well-known processes:

1. The Geometric Brownian Motion: \(dS(t) = \mu S(t)dt + \sigma S(t)dW(t)\);
2. The Continuous-Time GARCH Process: \( dS(t) = \mu(b - S(t))dt + \sigma S(t)dW(t) \);

3. The Ornstein-Uhlenbeck (1930) Process: \( dS(t) = -\mu S(t)dt + \sigma dW(t) \);

4. The Vasićek (1977) Process: \( dS(t) = \mu(b - S(t))dt + \sigma dW(t) \);

5. The Cox-Ingersoll-Ross (1985) Process: \( dS(t) = k(\theta - S(t))dt + \gamma \sqrt{S(t)}dW(t) \);

6. The Ho and Lee (1986) Process: \( dS(t) = \theta(t)dt + \sigma dW(t) \);

7. The Hull and White (1990) Process: \( dS(t) = (a(t) - b(t))S(t)dt + \sigma(t)dW(t) \);

8. The Heath, Jarrow and Morton (1987) Process: Define the forward interest rate \( f(t, s) \), for \( t \leq s \), characterized by the following equality \( P(t, u) = \exp[-\int_t^u f(t, s)ds] \) for any maturity \( u \). \( f(t, s) \) represents the instantaneous interest rate at time \( s \) as ‘anticipated’ by the market at time \( t \). It is natural to set \( f(t, t) = r(t) \). The process \( f(t, u)_{0 \leq t \leq u} \) satisfies an equation

\[
\int_0^t a(v, u)dv + \int_0^t b(f(v, u))dW(v),
\]

where the processes \( a \) and \( b \) are continuous. We note that the last SDE may be written in the following form: \( df(t, u) = b(f(t, u))(\int_t^u b(f(t, s))ds + b(f(t, u))d\dot{W}(t)) \), where \( \dot{W}(t) = W(t) - \int_0^t q(s)ds \) and

\[
q(t) = \int_0^t b(f(t, s))ds - \frac{a(t, u)}{b(f(t, u))}.
\]

We use the change of time method to get the solutions of the above-mentioned SDEs.

\( W(t) \) below is a standard Brownian motion, and \( \dot{W}(t) \) is a \((\hat{T}_t)_{t \in \mathbb{R}_+} \)-adapted standard Brownian motion on \((\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t \in \mathbb{R}_+}, P)\). We use \( T_t \) and \( \hat{T}_t \) notations instead of \( \phi_t \) and \( \phi_t^{-1} \), for time change and inverse time change, respectively.

1. The Geometric Brownian Motion: \( dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \). Solution \( S(t) = e^{\mu t}[S(0) + \hat{W}(\hat{T}_t)] \), where \( \hat{T}_t = \sigma^2 \int_0^t [S(0) + \hat{W}(\hat{T}_s)]^2ds \).

2. The Continuous-Time GARCH Process: \( dS(t) = \mu(b - S(t))dt + \sigma S(t)dW(t) \). Solution \( S(t) = e^{-\mu t}[S(0) + \hat{W}(\hat{T}_t)] \), where \( \hat{T}_t = \sigma^2 \int_0^t [S(0) + \hat{W}(\hat{T}_s)]^2ds \).

3. The Ornstein-Uhlenbeck Process: \( dS(t) = -\mu S(t)dt + \sigma dW(t) \). Solution \( S(t) = e^{-\mu t}[S(0) + \hat{W}(\hat{T}_t)] \), where \( \hat{T}_t = \sigma^2 \int_0^t [S(0) + \hat{W}(\hat{T}_s)]^2ds \).

4. The Vasićek Process: \( dS(t) = \mu(b - S(t))dt + \sigma dW(t) \), solution \( S(t) = e^{-\mu t}[S(0) - b + \hat{W}(\hat{T}_t)] \), where \( \hat{T}_t = \sigma^2 \int_0^t [S(0) - b + \hat{W}(\hat{T}_s)] + b^2ds \).

5. The Cox-Ingersoll-Ross Process: \( dS(t) = k(\theta - S(t))dt + \gamma S(t)dW(t) \), solution \( S(t) = e^{-kt}[S_0^2 - \theta^2 + \hat{W}(\hat{T}_t)] + \theta^2, \) where \( \hat{T}_t = \gamma^{-2} \int_0^t [e^{\gamma T_s}(S_0^2 - \theta^2 + \hat{W}(s)) + \theta^2 e^{2\gamma T_s}]^{-1}ds \).

6. The Ho and Lee Process: \( dS(t) = \theta(t)dt + \sigma dW(t) \). Solution \( S(t) = S(0) + \hat{W}(\sigma^2 t) + \int_0^t \theta(s)ds \).

7. The Hull and White Process: \( dS(t) = (a(t) - b(t))S(t)dt + \sigma(t)dW(t) \). Solution \( S(t) = \exp[-\int_0^t b(s)ds][S(0) - \frac{a(s)}{b(s)} + \hat{W}(\hat{T}_t)] \), where \( \hat{T}_t = \int_0^t \sigma^2(s)[S(0) - \frac{a(s)}{b(s)}] + \hat{W}(\hat{T}_s) + \exp[\int_0^t b(u)du]_{b(s)}^2ds \).
8. The Heath, Jarrow and Morton Process: \( f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dW(v) \). Solution \( f(t, u) = f(0, u) + \hat{W}(\hat{T}_t) + \int_0^t a(v, u)dv \), where \( \hat{T}_t = \int_0^t b^2(f(0, u) + \hat{W}(\hat{T}_s) + \int_0^s a(v, u)dv)ds \).

2 Applications of CTM: Overview

In this section we give an overview on applications of change of time methods in quantitative finance. These applications include: yet one more derivation of Black-Scholes formula; derivation of option pricing formula for a mean-reverting asset in energy finance; pricing of variance, volatility, covariance and correlation swaps for Heston model; pricing of variance and volatility swaps in energy markets; pricing of variance and volatility swaps and hedging of volatility swaps for delayed Heston model.

2.1 Black-Scholes Formula by Change of Time Method

In the early 1970’s, Black and Scholes (1973) made a major breakthrough by deriving pricing formula for vanilla option written on the stock. Their model and its extensions assume that the probability distribution of the underlying cash flow at any given future time is lognormal.

There are at least three proofs of their result, including PDE and martingale approaches (see Wilmott et al (1995), Elliott and Kopp (1999)).

One of the aims of this application is to give yet one more derivation of Black-Scholes result by change of time method.

2.2 Variance and Volatility Swaps by Change of Time Method: Heston Model

Volatility swaps are forward contracts on future realized stock volatility, variance swaps are similar contract on variance, the square of the future volatility, both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

A stock’s volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility \( \sigma_R \) is the annualized standard deviation of the stock’s returns during the period of interest, where the subscript ’R’ denotes the observed or ‘realized’ volatility.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility (and only to volatility).

Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999) explained the properties and the theory of both variance and volatility swaps. They derived
an analytical formula for theoretical fair value in the presence of realistic volatility skews, and pointed out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap.

Javaheri A, Wilmott, P. and Haug, E. G. (2002) discussed the valuation and hedging of a GARCH(1,1) stochastic volatility model. They used a general and exible PDE approach to determine the first two moments of the realized variance in a continuous or discrete context. Then they approximate the expected realized volatility via a convexity adjustment.

Brockhaus and Long (2000) provided an analytical approximation for the valuation of volatility swaps and analyzed other options with volatility exposure.


Although options market participants talk of volatility, it is variance, or volatility squared, that has more fundamental significance (see Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999)).

Modeling and pricing of variance, volatility, covariance and correlation swaps for Heston model have been considered in Swishchuk (2004). In this paper, a new probabilistic approach, change of time method, was proposed to study variance and volatility swaps for financial markets with underlying asset and variance that follow the Heston (1993) model. We also studied covariance and correlation swaps for the financial markets. As an application, we provided a numerical example using $S&P60$ Canada Index to price swap on the volatility.

Variance swaps for financial markets with underlying asset and stochastic volatilities with delay were modelled and priced in the paper Swishchuk (2005a). We found some analytical close forms for expectation and variance of the realized continuously sampled variance for stochastic volatility with delay both in stationary regime and in general case. The key features of the stochastic volatility model with delay are the following: i) continuous-time analogue of discrete-time GARCH model; ii) mean-reversion; iii) contains the same source of randomness as stock price; iv) market is complete; v) incorporates the expectation of log-return. As applications, we provided two numerical examples using $S&P60$ Canada Index (1998-2002) and $S&P500$ Index (1990-1993) to price variance swaps with delay. Variance swaps for stochastic volatility with delay is very similar to variance swaps for stochastic volatility in Heston model (see Swishchuk (2004)), but simpler to model and to price it.

Variance swaps for multi-factor stochastic volatility models with delay have been studied in Swishchuk (2006).
One of the aim of this application is to value variance and volatility swaps for Heston (1993) model using change of time method.

**Remark 1.1.** A extensive reviews of the literature on stochastic volatility is given in Shephard (2005a, 2005b). A detailed introduction to variance and volatility swaps, including a history and market products, may be found in Carr and Madan (1998) and Demeterfi et al (1999). The pricing of a range of volatility derivatives, including volatility and variance swaps and swaptions, is studied in Howison et al (2004). This paper also contains a lot of volatility models, including those with jumps. Volatility model with jumps was first considered in Naik (1993). Parameter estimation in a stochastic drift hidden Markov model with a cap and with applications to the theory of energy finance and interest rate modeling is studied in Hernandez et al (2005).

**Remark 1.2.** The fact that stochastic volatility models, such the Heston model and others, are able to fit skews and smiles, while simultaneously providing sensible Greeks, have made these models a popular choice in the pricing of options and swaps. Some ideas of how to calculate the Greeks for volatility contracts may be found in Howison et al (2004).

**Remark 1.3.** We note, that the change of time method was used in Swishchuk and Kalemanova (2000) to study stochastic stability of interest rates with and without jumps, in Swishchuk (2004) to model and to price variance and volatility swaps for Heston model and in Swishchuk (2005) to price European call option for commodity prices that follow mean-reverting model.

### 2.3 Change of Time Method: Delayed Heston Model

Heston model (Heston (1993)) is one of the most popular stochastic volatility models in the industry as semi-closed formulas for vanilla option prices are available, few (five) parameters need to be calibrated, and it accounts for the mean-reverting feature of the volatility.

One might be willing, in the variance diffusion, to take into account not only its current state but also its past history over some interval \([t − \tau, t]\), where \(\tau > 0\) is a constant and is called the delay. Starting from the discrete-time \(\text{GARCH}(1,1)\) model (Bollerslev (1986)), a first attempt was made in this direction in Kazmerchuk et al. (2005), where a non-Markov delayed continuous-time \(\text{GARCH}\) model was proposed. We present a variance drift adjusted version of the Heston model which leads to a significant improvement of the market volatility surface fitting by 44% (compared to Heston). The numerical example we performed with recent market data shows a significant reduction of the average absolute calibration error \(^1\) (calibration on

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\(^1\)The average absolute calibration error is defined to be the average of the absolute values of the differences between market and model implied Black & Scholes volatilities.
12 dates ranging from Sep. 19th to Oct. 17th 2011 for the FOREX underlying EURUSD). Our model has two additional parameters compared to the Heston model, can be implemented very easily and was initially introduced for volatility derivatives pricing purpose. The main idea behind our model is to take into account some past history of the variance process in its (risk-neutral) diffusion. Using a change of time method for continuous local martingales, we derive a closed formula for the Brockhaus&Long approximation of the volatility swap price in this model. We also consider dynamic hedging of volatility swaps using a portfolio of variance swaps.

One of the aim of this application is to get variance and volatility swap prices using change of time method and to get hedge ratio for volatility swaps in the delayed Heston model model.

2.4 Change of Time Method: Multi-factor Lévy-based Models for Pricing of Financial and Energy Derivatives

We also introduce one-factor and multi-factor \( \alpha \)-stable Lévy-based models to price financial and energy derivatives. These models include, in particular, as one-factor models, the Lévy-based geometric motion model, the Ornstein-Uhlenbeck (1930), the Vasicek (1977), the Cox-Ingersoll-Ross (1985), the continuous-time GARCH, the Ho-Lee (1986), the Hull-White (1990) and the Heath-Jarrow-Morton (1992) models, and, as multi-factor models, various combinations of the previous models. For example, we introduce new multi-factor models such as the Lévy-based Heston model, the Lévy-based SABR/LIBOR market models, and Lévy-based Schwartz-Smith and Schwartz models. Using the change of time method for SDEs driven by \( \alpha \)-stable Lévy processes we present the solutions of these equations in simple and compact forms. We then apply this method to price many financial and energy derivatives such as variance swaps, options, forward and futures contracts.

One of the aim of this application is to consider various applications of the change of time method for Lévy-based SDEs arising in financial and energy markets: swap and option pricing, interest derivatives pricing and forward and futures contracts pricing.

2.5 Mean-Reverting Asset Model by Change of Time Method: Option Pricing Formula

Some commodity prices, like oil and gas, exhibit the mean reversion, unlike stock price. It means that they tend over time to return to some long-term mean. This mean-reverting model is a one-factor version of the two-factor model made popular in the context of energy modeling by Pilipovic (1997).
Black’s model (1976) and Schwartz’s model (1997) have become a standard approach to the problem of pricing options on commodities. These models have the advantage of mathematical convenience, in that they give rise to closed-form solutions for some types of option (see Wilmott (2000)). Bos, Ware and Pavlov (2002) presented a method for evaluation of the price of a European option based on $S_t$ using a semi-spectral method. They did not have the convenience of a closed-form solution, however, they shown that values for certain types of option may nevertheless be found extremely efficiently. They used the following partial differential equation (see, for example, Wilmott, Howison and Dewynne (1995))

$$C'_t + R(S,t)C'_S + \sigma^2 S^2 C''_{SS}/2 = rC$$

for option prices $C(S,t)$, where $R(S,t)$ depends only on $S$ and $t$, and corresponds to the drift induced by the risk-neutral measure, and $r$ is the risk-free interest rate. Simplifying this equation to the singular diffusion equation they were able to calculate numerically the solution.

The working paper Swishchuk (2005b) presents explicit expression for a European option price, $C(S,t)$, for mean-reverting asset $S_t$, using change of time method under both physical and risk-neutral measures.

We note, that recent book by Geman (2005) covers hard and soft commodities (energy, agriculture and metals) and analysis economic and geopolitical issues in commodities markets, commodity price and volume risk, stochastic modeling of commodity spot prices and forward curves, real options valuation and hedging of physical assets in the energy industry.

One of the aim of this application is to obtain explicit expression for a European call option price on mean-reverting model of commodity asset using change of time method. As we can see, if mean-reverting level equals to zero then the option pricing formula coincides with Black-Scholes result.

### 2.6 Variance and Volatility Swaps by Change of Time Method: Energy Markets

One of applications of CTM is devoted to the pricing of variance and volatility swaps in energy market. We found explicit variance swap formula and closed form volatility swap formula (using change of time) for energy asset with stochastic volatility that follows continuous-time mean-reverting GARCH (1,1) model. Numerical example is presented for AECO Natural Gas Index (1 May 1998-30 April 1999). Variance swaps are quite common in commodity, e.g., in energy market, and they are commonly traded. We consider Ornstein-Uhlenbeck process for commodity asset with stochastic volatility following continuous-time GARCH model or Pilipovic (1998) one-factor model. The classical stochastic process for the spot dynamics of commodity prices is given by the Schwartz’ model (1997). It is defined as the exponential
of an Ornstein-Uhlenbeck (OU) process, and has become the standard model for energy prices possessing mean-reverting features.

In this book, we consider a risky asset in energy market with stochastic volatility following a mean-reverting stochastic process satisfying the following SDE (continuous-time GARCH(1,1) model):

$$d\sigma^2(t) = a(L - \sigma^2(t))dt + \gamma\sigma^2(t)dW_t,$$

where \(a\) is a speed of mean reversion, \(L\) is the mean reverting level (or equilibrium level), \(\gamma\) is the volatility of volatility \(\sigma(t)\), \(W_t\) is a standard Wiener process. Using a change of time method we find an explicit solution of this equation and using this solution we are able to find the variance and volatility swaps pricing formula under the physical measure. Then, using the same argument, we find the option pricing formula under risk-neutral measure. We applied Brockhaus-Long (2000) approximation to find the value of volatility swap. A numerical example for the AECO Natural Gas Index for the period 1 May 1998 to 30 April 1999 is presented.

Commodities are emerging as an asset class in their own. The range of products offered to investors range from exchange traded funds (ETFs) to sophisticated products including principal protected structured notes on individual commodities or baskets of commodities and commodity range-accrual or variance swap. More and more institutional investors are including commodities in their asset allocation mix and hedge funds are also increasingly active players in commodities. Example: Amaranth Advisors lost USD 6 billion during September 2006 from trading natural gas futures contracts, leading to the fund’s demise. Concurrent with these developments, a number of recent papers have examined the risk and return characteristics of investments in individual commodity futures or commodity indices composed of baskets of commodity futures. However, since all but the most plain-vanilla investments contain an exposure to volatility, it is equally important for investors to understand the risk and return characteristics of commodity volatilities.

The focusing on energy commodities derives from two reasons: 1) energy is the most important commodity sector, and crude oil and natural gas constitute the largest components of the two most widely tracked commodity indices: the Standard & Poors Goldman Sachs Commodity Index (S&P GSCI) and the Dow Jones-AIG Commodity Index (DJ-AIGCI); 2) existence of a liquid options market: crude oil and natural gas indeed have the deepest and most liquid options marketss among all commodities. The idea is to use variance (or volatility) swaps on futures contracts. At maturity, a variance swap pays off the difference between the realized variance of the futures contract over the life of the swap and the fixed variance swap rate. And since a variance swap has zero net market value at initiation, absence of arbitrage implies that the fixed variance swap rate equals to conditional risk-neutral expectation of the realized variance over the life of swap. Therefore, e.g., the
time-series average of the payoff and/or excess return on a variance swap is a measure of the variance risk premium.

Variance risk premia in energy commodities, crude oil and natural gas, has been considered by A. Trolle and E. Schwartz (2009). The same methodology as in Trolle & Schwartz (2009) was used by Carr & Wu (2009) in their study of equity variance risk premia. The idea was to use variance swaps on futures contracts. The study in Trolle & Schwartz (2009) is based on daily data from January 2, 1996 until November 30, 2006—a total of 2750 business days. The source of the data is NYMEX. Trolle & Schwartz (2009) found that: 1) the average variance risk premia are negative for both energy commodities but more strongly statistically significant for crude oil than for natural gas; 2) the natural gas variance risk premium (defined in dollars terms or in return terms) is higher during the cold months of the year (seasonality and peaks for natural gas variance during the cold months of the year); 3) energy risk premia in dollar terms are time-varying and correlated with the level of the variance swap rate. In contrast, energy variance risk premia in return terms, particularly in the case of natural gas, are much less correlated with the variance swap rate.

The S&P GSCI is comprised of 24 commodities with the weight of each commodity determined by their relative levels of world production over the past five years. The DJ-AIGCI is comprised of 19 commodities with the weight of each component determined by liquidity and world production values, with liquidity being the dominant factor. Crude oil and natural gas are the largest components in both indices. In 2007, their weight were 51.30% and 6.71%, respectively, in the S&P GSCI and 13.88% and 11.03%, respectively, in the DJ-AIGCI. The Chicago Board Options Exchange (CBOE) recently introduced a Crude Oil Volatility Index (ticker symbol OVX). This index also measures the conditional risk-neutral expectation of crude oil variance, but is computed from a cross-section of listed options on the United States Oil Fund (USO), which tracks the price of WTI as closely as possible. The CBOE Crude Oil ETF Volatility Index ('Oil VIX', Ticker - OVX) measures the market’s expectation of 30-day volatility of crude oil prices by applying the VIX methodology to United States Oil Fund, LP (Ticker - USO) options spanning a wide range of strike prices. We have to notice that crude oil and natural gas trade in units of 1,000 barrels and 10,000 British thermal units (mmBtu), respectively. Price are quoted as US dollars and cents per barrel or mmBtu. The continuous-time GARCH model has also been exploited by Javaheri, Wilmott and Haug (2002) to calculate volatility swap for S&P500 index. They used PDE approach and mentioned (page 8, sec. 3.3) that ‘it would be interesting to use an alternative method to calculate $F(v, t)$ and the other above quantities’. This paper exactly contains the alternative method, namely, ‘change of time method’, to get variance and volatility swaps. The change of time method was also applied by Swishchuk (2004) for pricing variance, volatility, covariance and correlation swaps for Heston model. The first paper on pricing of commodity contracts was pub-
lished by Black (1976).

One of the aim of this application is to get variance and volatility swap prices for Heston model using change of time method.

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Chapter 3: Change of Time Method (CTM) and Black-Scholes Formula

'It is said that there is no such thing as a free lunch. But the universe is the ultimate free lunch’,-Alan Guth (MIT).

1 Introduction to Option Pricing and Black-Scholes Formula

In this Chapter, we consider applications of the change of time method to yet one more time derive the well-known Black-Scholes formula for European call options. In the early 1970’s, Black and Scholes, (1973), made a major breakthrough by deriving a pricing formula for a vanilla option written on the stock. Their model and its extensions assume that the probability distribution of the underlying cash flow at any given future time is lognormal. We mention that there are many proofs of this result, including PDE and martingale approaches, (see Wilmott et. al. (1995), Elliott and Kopp (1999)). The present approach, using change of time of getting the Black-Scholes formula was first shown in Swishchuk (2007).

1.1 A Brief and Quick Introduction to Option Pricing Theory

We use the term asset to describe any financial object whose value is known at present but is liable to change in the future. Some examples include shares of a company, commodities (oil, electricity, gas, gold, etc.), currencies, etc. Now we give the definitions of options.

Definition 1 (European Call Option). A European call option gives its holder the right (but not the obligation) to purchase from the writer a prescribed asset for a prescribed price at a prescribed time in the future.

Definition 2 (European Put Option). A European call option gives its holder the right (but not the obligation) to sell to the writer a prescribed asset for a prescribed price at a prescribed time in the future.

The prescribed purchase price is known as the exercise price or strike price, and the prescribed time in the future is known as the expiry date or maturity. The key question in option pricing theory is: how much should the holder pay for the privilege of holding an option? Or, how do we compute a fair option price or value?

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, $t \in [0, T]$, and $\mathcal{F}_T = \mathcal{F}$. We denote by $T$ the expiry date, by $K$ the strike price and by $S(t)$ the asset price at time $t \geq 0$. Then, e.g., $S(T)$ is the asset price at the expiry date, which is not known (uncertain or random) at the time when the option is taken out. If $S(T) > K$ at expiry $T$, then the holder of a European call
option may buy the asset for \( K \) and sell it in the market for \( S(T) \), gaining
an amount \( S(T) - K \). If \( K \geq S(T) \), on the other hand, then the holder gains
nothing or zero. Therefore, the value or price of the European call option at
the expiry date, denoted by \( C(T) \), is

\[
f_C(T) = \max\{S(T) - K, 0\}.
\]

As for a European put option, the situation is opposite. If \( S(T) < K \) at
expiry \( T \), then the holder of a European put option may buy the asset for
\( S(T) \) in the market and sell it in the market for \( K \), gaining an amount
\( K - S(T) \). If \( K \leq S(T) \), on the other hand, then the holder gains nothing or
zero. Therefore, the value or price of the European put option at the expiry
date, denoted by \( P(T) \), is

\[
f_P(T) = \max\{K - S(T), 0\}.
\]

We call \( f_C(T) \) and \( f_P(T) \) the payoff functions. The shapes of the corresponding
payoff diagrams look like (ice) hockey sticks with the kinks at \( K \).

European call and put options are the simplest and classical examples of
so-called financial derivatives, meaning those derivatives indicate that their
values is derived from the underlying asset (do not mix up this term with
the mathematical meaning of a derivative!). There are many other financial
derivatives, such as forwards, futures, swaps, etc.

Next question in option pricing theory is: how to determine a fair value or
price of the option at time \( t = 0 \)? We denote this value or price (for European
call option) by \( C(0) \). To answer this question we introduce two key concepts:
discounting for interest and the no arbitrage principle (sometimes referred to
as no free lunch opportunity).

**Discounting interest:** If we have some money in a risk-free savings ac-
count (bank deposit) and this investment grows accordingly to a continuously
compounded interest rate \( r > 0 \), then its value increases by a factor \( e^{rt} \) over
a time length \( t \). We will use \( r \) to denote the annual rate (so that time is mea-
sured in years). The simplest example is risk-free bank account with amount
of money \( B(t) \) at time \( t > 0 \). If the initial deposit is \( B(0) \), then at time \( t > 0 \)
it will be \( B(t) = e^{rt}B(0) \). Hence, \( B(t) \) satisfies the equation \( dB(t) = rB(t)dt \),
\( B(0) > 0, r > 0, t \geq 0 \). Suppose also that we have an amount \( C(0) \) at time
time zero, then it is worth \( C(t) = e^{rt}C(0) \) at time \( t \) or \( C(T) = e^{rT}C(0) \) at
expiry \( T \). It means that to have \( C(T) \) amount of money on saving account at
time \( T \) we have to have \( C(0) = e^{-rT}C(T) \) amount of money at time \( t = 0 \).

**No Arbitrage Principle:** This principle means that there is never an op-
portunity to make a risk-free profit that gives a greater return than that
provided by the interest from bank deposit. Arguments based on the No
Arbitrage principle are the main tools of financial mathematics.

The key role for criteria of No Arbitrage plays the problem of change
of measures which is crucial in mathematical finance. We call this measure
"martingale measure" or "risk-neutral measure", and denote it by \( Q \), to
make it different from the initial or physical probability measure $P$. The technic of change of measure is based on the construction of a new probability measure $Q$ equivalent to the given measure $P$ and such that a process $\tilde{S}(t)$, built on initial process $S(t)$ satisfies some ‘fairness’ condition. In the case of mathematical finance, this process $\tilde{S}(t) = e^{-rt}S(t)$ is a martingale with respect to the new measure $Q$. As long as asset value $S(t)$ at time $t$ is random or unknown, we have to calculate the expected value of this asset. It means that expectation should be taking with respect to this martingale measure $Q$. Returning to our payoff functions, it means that we have to calculate this expected value (we denote it by $E_Q$, compare with $E_P$-expectation with respect to the initial measure $P$) with respect to the risk-neutral measure $Q$:

$$E_Q[\max\{S(T) - K, 0\}]$$

Therefore, if the initial (at time $t = 0$) fair price of option is $C(0)$, then the value $E_Q[\max\{S(T) - K\}]$ is equal to $e^{rT}C(0)$ or

$$C(0) = e^{-rT}E_Q[\max\{S(T) - K, 0\}].$$

This formulae gives the answer to our question: the fair price of the option at time $t = 0$ is defined by the last formulae.

The situation for the European put option is similar, taking into account the payoff function $P(T) = \max\{K - S(T), 0\}$ in this case. Therefore, the fair price $P(0)$ of the European put option at time $t = 0$ is

$$P(0) = e^{-rT}E_Q[\max\{K - S(T), 0\}].$$

We note, that fair prices of European call and put options satisfy the following so-called call-put parity:

$$C(0) + Ke^{-rT} = P(0) + S(0),$$

where $S(0)$ is the initial (at time $t = 0$) asset price. See Elliott et al. (1999) or Wilmott et al. (1995) for more details on option pricing.

1.2 Black-Scholes Formula

The well-known BLACK-SCHOLES (1973) formula states that if we have a $(B, S)$-security market consisting of a riskless asset $B(t)$ with a constant continuously compounded interest rate $r$:

$$dB(t) = rB(t)dt, \quad B(0) > 0, \quad r > 0. \quad (1)$$

The risky asset, (stock), $S(t)$ is assumed to have dynamics

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad S(0) > 0. \quad (2)$$

Here: $\mu \in R$ is the appreciation rate and $\sigma > 0$ is the volatility. Then price for a European call option with pay-off function $f(T) = \max(S(T) - K, 0)$, ($K > 0$ is the strike price), has the following form:
\[ C(T) = S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-), \quad (3) \]

where
\[ y_\pm := \ln\left(\frac{S(0)}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)T \quad (4) \]

and
\[ \Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx. \quad (5) \]

### 1.3 Solution of SDE for Geometric Brownian Motion using Change of Time Method

**Lemma 1.** The solution of the equation (3.2) has the following form:
\[ S(t) = e^{\mu t}(S(0) + \tilde{W}(\phi^{-1})), \quad (6) \]

where \(\tilde{W}(t)\) is a one-dimensional Wiener process,
\[ \phi^{-1} = \sigma^2 \int_0^t [S(0) + \tilde{W}(\phi^{-1})]^2 ds \]

and
\[ \phi = \sigma^{-2} \int_0^t [S(0) + \tilde{W}(s)]^{-2} ds. \]

**Proof.**
Set
\[ V(t) = e^{-\mu t}S(t), \quad (7) \]

where \(S(t)\) is defined in (2).

Applying Itô’s formula to \(V(t)\) we obtain
\[ dV(t) = \sigma V(t)dW(t). \quad (8) \]

Equation (8) is similar to equation (9) of Chapter 2, with
\[ a(X) = \sigma X. \]

Therefore, the solution of equation (3.8) using the change of time method (see Corollary 2.1, Section 2.4, Chapter 2) is (see (10) and (11))
\[ V(t) = S(0) + \tilde{W}(\phi^{-1}), \quad (9) \]

where \(\tilde{W}(t)\) is an one-dimensional Wiener process,
\[ \phi^{-1} = \sigma^2 \int_0^t [S(0) + \tilde{W}(\phi^{-1})]^2 ds \]

and
\[ \phi = \sigma^{-2} \int_0^t [S(0) + \tilde{W}(s)]^{-2} ds. \]

From (7) and (9) it follows that the solution of equation (2) has the representation (6).
1.4 Properties of the Process $\tilde{W}(\phi^{-1}_t)$

**Lemma 2.** Process $\tilde{W}(\phi^{-1}_t)$ is a mean-zero martingale with quadratic variation

$$<\tilde{W}(\phi^{-1}_t)> = \phi^{-1}_t = \sigma^2 \int_0^t [S(0) + \tilde{W}(\phi^{-1}_s)]^2 ds$$

and has the following representation

$$\tilde{W}(\phi^{-1}_t) = S(0)(e^{\sigma W(t) - \frac{\sigma^2}{2} t} - 1). \quad (10)$$

**Proof.**

From Corollary 2.1, Section 2.4, Chapter 2, it follows that $\tilde{W}(\phi^{-1}_t)$ is a martingale with quadratic variation

$$<\tilde{W}(\phi^{-1}_t)> = \phi^{-1}_t = \sigma^2 \int_0^t [S(0) + \tilde{W}(\phi^{-1}_s)]^2 ds.$$ 

and the process $W(t)$ has the following look

$$W(t) = \sigma^{-1} \int_0^t [S(0) + \tilde{W}(\phi^{-1}_s)]^{-1} d\tilde{W}(\phi^{-1}_s) \quad (11)$$

From (11) we obtain the following SDE for $\tilde{W}(\phi^{-1}_s)$

$$d\tilde{W}(\phi^{-1}_s) = \sigma [S(0) + \tilde{W}(\phi^{-1}_s)] dW(t).$$

Solving this equation we have the explicit expression (3.10) for $\tilde{W}(\phi^{-1}_s)$

$$\tilde{W}(\phi^{-1}_s) = S(0)(e^{\sigma W(t) - \frac{\sigma^2}{2} t} - 1).$$

Q.E.D

We note that $E[\tilde{W}(\phi^{-1}_s)] = 0$ and $E[\tilde{W}(\phi^{-1}_s)]^2 = S^2(0)(e^{\sigma^2 t} - 1)$, where $E := E_P$ is an expectation under physical measure $P$.

Since

$$E[e^{\sigma W(t) - \frac{\sigma^2}{2} t}]^n = e^{\frac{\sigma^2}{2} n(n-1)t}, \quad (12)$$

we can obtain all the moments for the process $\tilde{W}(\phi^{-1}_s)$:

$$E[\tilde{W}(\phi^{-1}_s)]^n = S^n(0) \sum_{k=0}^n C^n_k e^{\frac{\sigma^2}{2} k(k-1)} (-1)^{n-k}, \quad (13)$$

where $C^n_k := \frac{n!}{k!(n-k)!}$, $n! := 1 \times 2 \times 3... \times n$.

**Corollary 1.** From Lemma 2 (see (6), (10) and (12)) it follows that we can also obtain all the moment for the asset price $S(t)$ in (9), sin

$$E[S(t)]^n = e^{n\mu t} E[S(0) + \tilde{W}(\phi^{-1}_s)]^n = e^{n\mu t} S^n(0) E[e^{\sigma W(t) - \frac{\sigma^2}{2} t}]^n = e^{n\mu t} S^n(0) e^{\frac{\sigma^2}{2} n(n-1)t}. \quad (14)$$
For example, variance of \( S(t) \) is going to be

\[
\text{Var} S(t) = E S^2(t) - (E S(t))^2 = S^2(0) e^{2\mu t} (e^{\sigma^2 t} - 1),
\]

where \( E S(t) = S(0) e^{\mu t} \) (see (9)).

## 2 Black-Scholes Formula by Change of Time Method

In risk-neutral world the dynamic of stock price \( S(t) \) has the following look:

\[
dS(t) = rS(t)dt + \sigma S(t)dW^*(t),
\]

(15)

where

\[
W^*(t) := W(t) + \frac{\mu - r}{\sigma}. \tag{16}
\]

Following Section 3.2, from (6) we have the solution of the equation (15)

\[
S(t) = e^{rt}[S(0) + \tilde{W}^*(\phi_t^{-1})],
\]

(17)

where

\[
\tilde{W}^*(\phi_t^{-1}) = S(0)(e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} - 1)
\]

and \( W^*(t) \) is defined in (16).

Let \( E_Q \) be an expectation under risk-neutral measure (or martingale measure) \( Q \) (i.e., process \( e^{-rT}S(t) \) is a martingale under the measure \( Q \)).

Then the option pricing formula for European call option with pay-off function

\[
f_C(T) = \max[S(T) - K, 0]
\]

has the following look

\[
C(T) = e^{-rT} E_Q[f(T)] = e^{-rT} E_{P^*}[\max(S(T) - K, 0)]. \tag{19}
\]

**Proposition 3.1.**

\[
C(T) = S(0)\Phi(y_+) - Ke^{-rT}\Phi(y_-), \tag{20}
\]

where \( y_+ \) and \( \Phi(y) \) are defined in (4) and (5).

**Proof.** Using change of time method we have the following representation for the process \( S(t) \) (see (17))

\[
S(t) = e^{rt}[S(0) + \tilde{W}^*(\phi_t^{-1})],
\]
where $\tilde{W}^*(\phi_t^{-1})$ is defined in (18). From (17)-(19), after substitution $\tilde{W}^*(\phi_t^{-1})$ into (17) and $S(T)$ into (19), it follows from (*) (see sec. 1.1) that

$$C(T) = e^{-rT}E_Q[\max(S(T) - K, 0)]$$
$$= e^{-rT}E_Q[\max(e^{rt}[S(0) + \tilde{W}^*(\phi_t^{-1})] - K, 0)]$$
$$= e^{-rT}E_Q[\max(e^{rt}S(0)e^{\sigma W^*(T) - \frac{\sigma^2 T}{2}} - K, 0)]$$
$$= e^{-rT}E_Q[\max(S(0)e^{\sigma Y^*(T)} + (r - \frac{\sigma^2}{2})T - K, 0)]$$
$$= e^{-rT}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \max[S(0)e^{\sigma u\sqrt{T} + (r - \frac{\sigma^2}{2})T} - K, 0]e^{-u^2/2}du. \tag{21}$$

Let $y_0$ be a solution of the following equation

$$S(0)e^{\sigma y\sqrt{T} + (r - \frac{\sigma^2}{2})T} = K,$$

namely,

$$y_0 = \frac{\ln\left(\frac{K}{S(0)}\right) - (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}.$$

Then (21) may be presented in the following form

$$C(T) = e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{y_0}^{+\infty} (S(0)e^{\sigma u\sqrt{T} + (r - \frac{\sigma^2}{2})T} - K)e^{-u^2/2}du. \tag{22}$$

Finally, straightforward calculation of the integral in the right-hand side of (22) gives us the Black-Scholes result

$$C(T) = \frac{1}{\sqrt{2\pi}} \int_{y_0}^{+\infty} S(0)e^{\sigma u\sqrt{T} - \frac{\sigma^2 u^2}{2}}e^{-u^2/2}du - Ke^{-rT}[1 - \Phi(y_0)]$$
$$= \frac{S(0)}{\sqrt{2\pi}} \int_{y_0 - \sigma \sqrt{T}}^{+\infty} e^{-u^2/2}du - Ke^{-rT}[1 - \Phi(y_0)]$$
$$= S(0)[1 - \Phi(y_0 - \sigma \sqrt{T})] - Ke^{-rT}[1 - \Phi(y_0)]$$
$$= S(0)\Phi(y_+) - Ke^{-rT}\Phi(y_-),$$

where $y_\pm$ and $\Phi(y)$ are defined in (4) and (5). Q.E.D.

**References**


Chapter 4: CTM and Variance, Volatility, Covariance and Correlation Swaps for Classical Heston Model

'Criticism is easy; achievement is difficult,' - Winston Churchill.

1 Introduction.

In this Chapter, we use CTM to price variance and volatility swaps for financial markets with underlying asset and variance that follow the classical Heston (1993) model. We also find covariance and correlation swaps for the model. As an application, we provide a numerical example using S&P 60 Canada Index to price swap on the volatility.

In the early 1970’s, Black and Scholes (1973) made a major breakthrough by deriving pricing formulas for vanilla options written on the stock. The Black-Scholes model assumes that the volatility term is a constant. This assumption is not always satisfied by real-life options as the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see Hull (2000)), and the assumption of constant volatility $\sigma$ in financial model (such as the original Black-Scholes model) is incompatible with derivatives prices observed in the market.

The above issues have been addressed and studied in several ways, such as:

(i) Volatility is assumed to be a deterministic function of the time: $\sigma \equiv \sigma(t)$ (see Wilmott et al. (1995)); Merton (1973) extended the term structure of volatility to $\sigma := \sigma_t$ (deterministic function of time), with the implied volatility for an option of maturity $T$ given by $\hat{\sigma}^2_T = T \int_0^T \sigma^2_u du$;

(ii) Volatility is assumed to be a function of the time and the current level of the stock price $S(t)$: $\sigma \equiv \sigma(t, S(t))$ (see Hull (2000)); the dynamics of the stock price satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma(t, S(t)) S(t) dW_1(t),$$

where $W_1(t)$ is a standard Wiener process;

(iii) The time variation of the volatility involves an additional source of randomness, besides $W_1(t)$, represented by $W_2(t)$, and is given by

$$d\sigma(t) = a(t, \sigma(t)) dt + b(t, \sigma(t)) dW_2(t),$$

where $W_2(t)$ and $W_1(t)$ (the initial Wiener process that governs the price process) may be correlated (see Buff (2002), Hull and White (1987), Heston (1993)).
The volatility depends on a random parameter $x$ such as $\sigma(t) \equiv \sigma(x(t))$, where $x(t)$ is some random process (see Elliott and Swishchuk (2002), Griego and Swishchuk (2000), Swishchuk (1995), Swishchuk (2000), Swishchuk et al. (2000));

Another approach is connected with stochastic volatility, namely, uncertain volatility scenario (see Buff (2002)). This approach is based on the uncertain volatility model developed in Avellaneda et al. (1995), where a concrete volatility surface is selected among a candidate set of volatility surfaces. This approach addresses the sensitivity question by computing an upper bound for the value of the portfolio under arbitrary candidate volatility, and this is achieved by choosing the local volatility $\sigma(t, S(t))$ among two extreme values $\sigma_{\min}$ and $\sigma_{\max}$ such that the value of the portfolio is maximized locally;

The volatility $\sigma(t, S_t)$ depends on $S_t := S(t + \theta)$ for $\theta \in [-\tau, 0]$, namely, stochastic volatility with delay (see Kazmerchuk, Swishchuk and Wu (2002));

In the approach (i), the volatility coefficient is independent of the current level of the underlying stochastic process $S(t)$. This is a deterministic volatility model, and the special case where $\sigma$ is a constant reduces to the well-known Black-Scholes model that suggests changes in stock prices are lognormal distributed. But the empirical test by Bollerslev (1986) seems to indicate otherwise. One explanation for this problem of a lognormal model is the possibility that the variance of $\log(S(t)/S(t-1))$ changes randomly. This motivated the work of Chesney and Scott (1989), where the prices are analyzed for European options using the modified Black-Scholes model of foreign currency options and a random variance model. In their works the results of Hull and White (1987), Scott (1987) and Wiggins (1987) were used in order to incorporate randomly changing variance rates.

In the approach (ii), several ways have been developed to derive the corresponding Black-Scholes formula: one can obtain the formula by using stochastic calculus and, in particular, the Ito’s formula (see Øksendal (1998), for example).

A generalized volatility coefficient of the form $\sigma(t, S(t))$ is said to be *level-dependent*. Because volatility and asset price are perfectly correlated, we have only one source of randomness given by $W_1(t)$. A time and level-dependent volatility coefficient makes the arithmetic more challenging and usually precludes the existence of a closed-form solution. However, the *arbitrage argument* based on portfolio replication and a completeness of the market remain unchanged.

The situation becomes different if the volatility is influenced by a second “non-tradable” source of randomness. This is addressed in the approach (iii), (iv) and (v) we usually obtains a *stochastic volatility model*, which is general enough to include the deterministic model as a special case. The concept of stochastic volatility was introduced by Hull and White (1987), and subsequent development includes the work of Wiggins (1987), Johnson
and Shanno (1987), Scott (1987), Stein and Stein (1991) and Heston (1993). We also refer to Frey (1997) for an excellent survey on level-dependent and stochastic volatility models. We should mention that the approach (iv) is taken by, for example, Griego and Swishchuk (2000).

Hobson and Rogers (1998) suggested a new class of non-constant volatility models, which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model. The volatility is non-constant and can be regarded as an endogenous factor in the sense that it is defined in terms of the past behaviour of the stock price. This is done in such a way that the price and volatility form a multidimensional Markov process. Volatility swaps are forward contracts on future realized stock volatility, variance swaps are similar contract on variance, the square of the future volatility, both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

A stock’s volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility $\sigma_R$ is the annualized standard deviation of the stock’s returns during the period of interest, where the subscript R denotes the observed or ”realized” volatility.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility(and only to volatility).

Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999) explained the properties and the theory of both variance and volatility swaps. They derived an analytical formula for theoretical fair value in the presence of realistic volatility skews, and pointed out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap.

Javaheri A, Wilmott, P. and Haug, E. G. (2002) discussed the valuation and hedging of a GARCH(1,1) stochastic volatility model. They used a general and exible PDE approach to determine the first two moments of the realized variance in a continuous or discrete context. Then they approximate the expected realized volatility via a convexity adjustment.

Brockhaus and Long (2000) provided an analytical approximation for the valuation of volatility swaps and analyzed other options with volatility exposure.


In the paper we propose a new probabilistic approach to the study of stochastic volatility model (Section 3), Heston (1993) model, to model variance and volatility swaps (Section 2). The Heston asset process has a variance $\sigma_t^2$ that follows a Cox, Ingersoll & Ross (1985) process. We find some analytical close forms for expectation and variance of the realized both con-
tinuously (Section 3.4) and discrete sampled variance (Section 3.5), which are needed for study of variance and volatility swaps, and price of pseudo-variance, pseudo-volatility, the problems proposed by He & Wang (2002) for financial markets with deterministic volatility as a function of time. This approach was first applied to the study of stochastic stability of Cox-Ingersoll-Ross process in Swishchuk and Kalemanova (2000).

The same expressions for $E[V]$ and for $Var[V]$ (like in present paper) were obtained by Brockhaus & Long (2000) using another analytical approach. Most articles on volatility products focus on the relatively straightforward variance swaps. They take the subject further with a simple model of volatility swaps.

We also study covariance and correlation swaps for the securities markets with two underlying assets with stochastic volatilities (Section 4).

As an application of our analytical solutions, we provide a numerical example using S&P60 Canada Index to price swap on the volatility (Section 5).

2 Variance and Volatility Swaps.

Volatility swaps are forward contracts on future realized stock volatility, variance swaps are similar contract on variance, the square of the future volatility, both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

A stock’s volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility $\sigma_R(S)$ is the annualized standard deviation of the stock’s returns during the period of interest, where the subscript $R$ denotes the observed or ”realized” volatility for the stock $S$.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility (and only to volatility) (see Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999)).

A stock volatility swap is a forward contract on the annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R(S) - K_{vol}),$$

where $\sigma_R(S)$ is the realized stock volatility (quoted in annual terms) over the life of contract,

$$\sigma_R(S) := \sqrt{\frac{1}{T} \int_0^T \sigma_s^2 ds},$$

$\sigma_t$ is a stochastic stock volatility, $K_{vol}$ is the annualized volatility delivery price, and $N$ is the notional amount of the swap in dollar per annualized volatility point. The holder of a volatility swap at expiration receives $N$ dollars for every point by which the stock’s realized volatility $\sigma_R$ has exceeded
the volatility delivery price \( K_{\text{vol}} \). The holder is swapping a fixed volatility \( K_{\text{vol}} \) for the actual (floating) future volatility \( \sigma_R \). We note that usually \( N = \alpha I \), where \( \alpha \) is a converting parameter such as 1 per volatility-square, and I is a long-short index (+1 for long and -1 for short).

Although options market participants talk of volatility, it is variance, or volatility squared, that has more fundamental significance (see Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999)).

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

\[
N(\sigma_R^2(S) - K_{\text{var}}),
\]

where \( \sigma_R^2(S) \) is the realized stock variance (quoted in annual terms) over the life of the contract,

\[
\sigma_R^2(S) := \frac{1}{T} \int_0^T \sigma_s^2 ds,
\]

\( K_{\text{var}} \) is the delivery price for variance, and \( N \) is the notional amount of the swap in dollars per annualized volatility point squared. The holder of variance swap at expiration receives \( N \) dollars for every point by which the stock’s realized variance \( \sigma_R^2(S) \) has exceeded the variance delivery price \( K_{\text{var}} \).

Therefore, pricing the variance swap reduces to calculating the realized volatility square.

Valuing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract \( P \) on future realized variance with strike price \( K_{\text{var}} \) is the expected present value of the future payoff in the risk-neutral world:

\[
P = E\{e^{-rT}(\sigma_R^2(S) - K_{\text{var}})\},
\]

where \( r \) is the risk-free discount rate corresponding to the expiration date \( T \), and \( E \) denotes the expectation.

Thus, for calculating variance swaps we need to know only \( E\{\sigma_R^2(S)\} \), namely, mean value of the underlying variance.

To calculate volatility swaps we need more. From Brockhaus-Long (2000) approximation (which is used the second order Taylor expansion for function \( \sqrt{x} \)) we have (see also Javaheri et al (2002), p.16):

\[
E\{\sqrt{\sigma_R^2(S)}\} \approx \sqrt{E\{V\}} - \frac{Var\{V\}}{8E\{V\}^{3/2}},
\]

where \( V := \sigma_R^2(S) \) and \( \frac{Var\{V\}}{8E\{V\}^{3/2}} \) is the convexity adjustment.

Thus, to calculate volatility swaps we need both \( E\{V\} \) and \( Var\{V\} \).

The realized continuously sampled variance is defined in the following way:

\[
V := \text{Var}(S) := \frac{1}{T} \int_0^T \sigma_t^2 dt.
\]
The realised discrete sampled variance is defined as follows:

\[
Var_n(S) := \frac{n}{(n-1)T} \sum_{i=1}^{n} \log^2 \frac{S_i}{S_{t_{i-1}}},
\]

where we neglected by \( \frac{1}{n} \sum_{i=1}^{n} \log \frac{S_i}{S_{t_{i-1}}} \) since we assume that the mean of the returns is of the order \( \frac{1}{n} \) and can be neglected. The scaling by \( \frac{n}{T} \) ensures that these quantities annualized (daily) if the maturity \( T \) is expressed in years (days).

\( Var_n(S) \) is unbiased variance estimation for \( \sigma_t \). It can be shown that (see Brockhaus & Long (2000))

\[
V := Var(S) = \lim_{n \to +\infty} Var_n(S).
\]

Realised discrete sampled volatility is given by:

\[
Vol_n(S) := \sqrt{Var_n(S)}.
\]

Realised continuously sampled volatility is defined as follows:

\[
Vol(S) := \sqrt{Var(S)} = \sqrt{V}.
\]

The expressions for \( V, Var_n(S) \) and \( Vol(S) \) are used for calculation of variance and volatility swaps.

3 Variance and Volatility Swaps for Heston Model of Securities Markets

3.1 Stochastic Volatility Model.

Let \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \) be probability space with filtration \( \mathcal{F}_t \), \( t \in [0,T] \).

Assume that underlying asset \( S_t \) in the risk-neutral world and variance follow the following model, Heston (1993) model:

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sigma_t dw^1_t, \\
\frac{d\sigma_t^2}{\sigma_t^2} &= k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dw^2_t,
\end{align*}
\]

(1)

where \( r_t \) is deterministic interest rate, \( \sigma_0 \) and \( \theta \) are short and long volatility, \( k > 0 \) is a reversion speed, \( \gamma > 0 \) is a volatility (of volatility) parameter, \( w^1_t \) and \( w^2_t \) are independent standard Wiener processes.

The Heston asset process has a variance \( \sigma_t^2 \) that follows Cox-Ingersoll-Ross (1985) process, described by the second equation in (1).

If the volatility \( \sigma_t \) follows Ornstein-Uhlenbeck process (see, for example, Øksendal (1998)), then Ito’s lemma shows that the variance \( \sigma_t^2 \) follows the process described exactly by the second equation in (1).
3.2 Explicit expression for $\sigma_t^2$.

In this section we propose a new probabilistic approach to solve the equation for variance $\sigma_t^2$ in (1) explicitly, using change of time method (see Ikeda and Watanabe (1981)).

Define the following process:

$$v_t := e^{kt} (\sigma_0^2 - \theta^2). \tag{2}$$

Then, using Ito formula (see Øksendal (1995)) we obtain the equation for $v_t$:

$$dv_t = \gamma e^{kt} \sqrt{e^{-kt} v_t + \theta^2} dw_t^2. \tag{3}$$

Using change of time approach to the general equation (see Ikeda and Watanabe (1981))

$$dX_t = \alpha(t, X_t) dw_t^2,$$

we obtain the following solution of the equation (3):

$$v_t = \sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1}),$$

or (see (2)),

$$\sigma_t^2 = e^{-kt} (\sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1})) + \theta^2, \tag{4}$$

where $\tilde{w}^2(t)$ is an $\mathcal{F}_t$-measurable one-dimensional Wiener process, $\phi_t^{-1}$ is an inverse function to $\phi_t$:

$$\phi_t = \gamma^{-2} \int_0^t \{e^{k\phi_s}(\sigma_0^2 - \theta^2 + \tilde{w}^2(t)) + \theta^2 e^{2k\phi_s}\}^{-1} ds.$$

3.3 Properties of processes $\tilde{w}^2(\phi_t^{-1})$ and $\sigma_t^2$.

The properties of $\tilde{w}^2(\phi_t^{-1}) := b(t)$ are the following:

$$Eb(t) = 0; \tag{5}$$

$$E(b(t))^2 = \gamma^2 \left\{ \frac{e^{kt} - 1}{k} (\sigma_0^2 - \theta^2) + \frac{e^{2kt} - 1}{2k} \theta^2 \right\} \tag{6};$$

$$Eb(t)b(s) = \gamma^2 \left\{ \frac{e^{k(t\wedge s)} - 1}{k} (\sigma_0^2 - \theta^2) + \frac{e^{2k(t\wedge s)} - 1}{2k} \theta^2 \right\}; \tag{7}$$

where $t \wedge s := \min(t, s)$.

Using representation (4) and properties (5)-(7) of $b(t)$ we obtain the properties of $\sigma_t^2$. Straightforward calculations give us the following results:

$$E\sigma_t^2 = e^{-kt} (\sigma_0^2 - \theta^2) + \theta^2;$$

$$E\sigma_t^2 \sigma_s^2 = \gamma^2 e^{-k(t+s)} \left\{ \frac{e^{k(t\wedge s)} - 1}{k} (\sigma_0^2 - \theta^2) \right. \right.$$ 

$$+ \frac{e^{2k(t\wedge s)} - 1}{2k} \theta^2 \left. \right\} + e^{-k(t+s)} (\sigma_0^2 - \theta^2)^2$$

$$+ e^{-kt} (\sigma_0^2 - \theta^2) \theta^2 + e^{-ks} (\sigma_0^2 - \theta^2) \theta^2 + \theta^4. \tag{8}$$
3.4 Valuing Variance and Volatility Swaps

From formula (8) we obtain mean value for $V$:

$$E\{V\} = \frac{1}{T} \int_{0}^{T} E\sigma_{t}^{2} dt$$

$$= \frac{1}{T} \int_{0}^{T} \{e^{-kt} (\sigma_{0}^{2} - \theta^{2}) + \theta^{2}\} dt$$

$$= \frac{1-e^{-kT}}{kT} (\sigma_{0}^{2} - \theta^{2}) + \theta^{2}. \quad (9)$$

The same expression for $E[V]$ may be found in Brockhaus and Long (2000).

Substituting $E[V]$ from (9) into formula

$$P = e^{-rT} (E\{\sigma_{H}^{2}(S)\} - K_{var}) \quad (10)$$

we obtain the value of the variance swap.

Variance for $V$ equals to:

$$Var(V) = EV^{2} - (EV)^{2}. \quad \text{(11)}$$

From (9) we have:

$$(EV)^{2} = \frac{1 - 2e^{-kT} + e^{-2kT}}{k^{2}T^{2}} (\sigma_{0}^{2} - \theta^{2})^{2} + \frac{2(1 - e^{-kT})}{kT} (\sigma_{0}^{2} - \theta^{2})\theta^{2} + \theta^{4}. \quad \text{(11)}$$

Second moment may found as follows using formula (8):

$$EV^{2} = \frac{1}{T^{2}} \int_{0}^{T} \int_{0}^{T} E\sigma_{t}^{2}\sigma_{s}^{2} dt ds$$

$$= \frac{\gamma^{2}}{T^{2}} \int_{0}^{T} \int_{0}^{T} e^{-k(t+s)} \{\frac{e^{k(t+s)}-1}{k} (\sigma_{0}^{2} - \theta^{2}) + \frac{e^{2k(t+s)}-1}{2k} \theta^{2}\} dt ds$$

$$+ \frac{1-2e^{-kT} + e^{-2kT}}{k^{2}T^{2}} (\sigma_{0}^{2} - \theta^{2})^{2} + \frac{2(1-e^{-kT})}{kT} (\sigma_{0}^{2} - \theta^{2})\theta^{2} + \theta^{4}. \quad (12)$$

Taking into account (11) and (12) we obtain:

$$Var(V) = EV^{2} - (EV)^{2}$$

$$= \frac{\gamma^{2}}{T^{2}} \int_{0}^{T} \int_{0}^{T} e^{-k(t+s)} \{\frac{e^{k(t+s)}-1}{k} (\sigma_{0}^{2} - \theta^{2}) + \frac{e^{2k(t+s)}-1}{2k} \theta^{2}\} dt ds. \quad \text{(12)}$$

After calculations the last expression we obtain the following expression for variance of $V$:

$$Var(V) = \frac{\gamma^{2}e^{-2kT}}{2k^{2}T^{4}} [(2e^{2kT} - 4e^{kT}kT - 2)(\sigma_{0}^{2} - \theta^{2})$$

$$+ (2e^{2kT}kT - 3e^{2kT} + 4e^{kT} - 1)\theta^{2}]. \quad (13)$$
Similar expression for $\text{Var}[V]$ may be found in Brockhaus and Long (2000).

Substituting $EV$ from (9) and $\text{Var}(V)$ from (13) into formula

$$P = \{e^{-rT}(E\{\sigma_R(S)\} - \text{Var})\}$$

with

$$E\{\sigma_R(S)\} = E\{\sqrt{\sigma^2_R(S)}\} \approx \sqrt{E\{V\}} - \frac{\text{Var}\{V\}}{8E\{V\}^{3/2}},$$

we obtain the value of volatility swap.

### 3.5 Calculation of $E\{V\}$ in discrete case.

The realised discrete sampled variance:

$$\text{Var}_n(S) := \frac{n}{(n-1)T} \sum_{i=1}^{n} \log^2 \frac{S_{ti}}{S_{t_{i-1}}},$$

where we neglected by $\frac{1}{n} \sum_{i=1}^{n} \log \frac{S_{ti}}{S_{t_{i-1}}}$ for simplicity reason only. We note that

$$\log \frac{S_{ti}}{S_{t_{i-1}}} = \int_{t_{i-1}}^{t_i} (r_t - \sigma_t^2/2)dt + \int_{t_{i-1}}^{t_i} \sigma_t dw^1_t.$$

$$E\{\text{Var}_n(S)\} = \frac{n}{(n-1)T} \sum_{i=1}^{n} E\{\log^2 \frac{S_{ti}}{S_{t_{i-1}}}\}.$$

$$E\{\log^2 \frac{S_{ti}}{S_{t_{i-1}}}\} = (\int_{t_{i-1}}^{t_i} r_t dt)^2 - \int_{t_{i-1}}^{t_i} r_t dt \int_{t_{i-1}}^{t_i} E\sigma_t^2 dt$$

$$+ \frac{1}{4} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} E\sigma_t^2 \sigma_s^2 ds dt$$

$$- E(\int_{t_{i-1}}^{t_i} \sigma_t^2 dt \int_{t_{i-1}}^{t_i} \sigma_s dw^1_t) + \int_{t_{i-1}}^{t_i} E\sigma_t^2 dt.$$

We know the expressions for $E\sigma_t^2$ and for $E\sigma_t^2 \sigma_s^2$, and the fourth expression is equal to zero. Hence, we can easily calculate all the above expressions and, hence, $E\{\text{Var}_n(S)\}$ and variance swap in this case.

**Remark 1.** Some expressions for price of the realised discrete sampled variance $\text{Var}_n(S) := \frac{n}{(n-1)T} \sum_{i=1}^{n} \log^2 \frac{S_{ti}}{S_{t_{i-1}}}$, (or pseudo-variance) were obtained in the Proceedings of the 6th PIMS Industrial Problems Solving Workshop, PIMS IPSW 6, UBC, Vancouver, Canada, May 27-31, 2002. Editor: J. Macki, University of Alberta, Canada, June, 2002, pp.45-55.
4 Covariance and Correlation Swaps for Two Assets with Stochastic Volatilities.

4.1 Definitions of Covariance and Correlation Swaps

Option dependent on exchange rate movements, such as those paying in a currency different from the underlying currency, have an exposure to movements of the correlation between the asset and the exchange rate, this risk may be eliminated by using covariance swap.

A covariance swap is a covariance forward contact of the underlying rates $S_1$ and $S_2$ which payoff at expiration is equal to

$$N(Cov_R(S_1, S_2) - K_{cov}),$$

where $K_{cov}$ is a strike price, $N$ is the notional amount, $Cov_R(S_1, S_2)$ is a covariance between two assets $S_1$ and $S_2$.

Logically, a correlation swap is a correlation forward contract of two underlying rates $S_1$ and $S_2$ which payoff at expiration is equal to:

$$N(Corr_R(S_1, S_2) - K_{corr}),$$

where $Corr(S_1, S_2)$ is a realized correlation of two underlying assets $S_1$ and $S_2$, $K_{corr}$ is a strike price, $N$ is the notional amount.

Pricing covariance swap, from a theoretical point of view, is similar to pricing variance swaps, since

$$Cov_R(S_1, S_2) = 1/4\{\sigma^2_R(S_1S_2) - \sigma^2_R(S_1/S_2)\}$$

where $S_1$ and $S_2$ are given two assets, $\sigma^2_R(S)$ is a variance swap for underlying assets, $Cov_R(S_1, S_2)$ is a realized covariance of the two underlying assets $S_1$ and $S_2$.

Thus, we need to know variances for $S_1S_2$ and for $S_1/S_2$ (see Section 4.2 for details). Correlation $Corr_R(S_1, S_2)$ is defined as follows:

$$Corr_R(S_1, S_2) = \frac{Cov_R(S_1, S_2)}{\sqrt{\sigma^2_R(S_1)}\sqrt{\sigma^2_R(S_2)}},$$

where $Cov_R(S_1, S_2)$ is defined above and $\sigma^2_R(S_1)$ in section 3.4.

Given two assets $S_1^t$ and $S_2^t$ with $t \in [0, T]$, sampled on days $t_0 = 0 < t_1 < t_2 < ... < t_n = T$ between today and maturity $T$, the log-return each asset is:

$$R_i^j := \log\left(\frac{S_i^j}{S_{i-1}^j}\right), \quad i = 1, 2, ..., n, \quad j = 1, 2.$$

Covariance and correlation can be approximated by

$$Cov_n(S_1, S_2) = \frac{n}{(n-1)T} \sum_{i=1}^{n} R_i^1 R_i^2$$
and
\[ \text{Corr}_n(S^1, S^2) = \frac{\text{Cov}_n(S^1, S^2)}{\sqrt{\text{Var}_n(S^1)} \sqrt{\text{Var}_n(S^2)}}, \]
respectively.

### 4.2 Valuing of Covariance and Correlation Swaps

To value covariance swap we need to calculate the following
\[ P = e^{-rT} (E\text{Cov}(S^1, S^2) - K_{cov}). \]  
(15)

To calculate \( E\text{Cov}(S^1, S^2) \) we need to calculate \( E\{\sigma_R^2(S^1 S^2) - \sigma_R^2(S^1/S^2)\} \) for a given two assets \( S^1 \) and \( S^2 \).

Let \( S^i_t \), \( i = 1, 2 \), be two strictly positive Ito’s processes given by the following model
\[
\begin{align*}
\frac{dS^i_t}{S^i_t} &= \mu^i_t dt + \sigma^i_t dw^i_t, \\
\sigma^i_t^2 &= k_i (\theta^2_i - (\sigma^i_t)^2) dt + \gamma^i \sigma^i_t dw^j_t, \quad i = 1, 2, \quad j = 3, 4,
\end{align*}
\]  
(16)

where \( \mu^i_t, \quad i = 1, 2 \), are deterministic functions, \( k^i, \quad \theta^i, \quad \gamma^i, \quad i = 1, 2 \), are defined in similar way as in (1), standard Wiener processes \( w^j_t, \quad j = 3, 4 \), are independent, \([w^i_t, w^j_t] = \rho_t dt, \rho_t \) is deterministic function of time, \([,] \) means the quadratic covariance, and standard Wiener processes \( w^i_t, \quad i = 1, 2 \), and \( w^j_t, \quad j = 3, 4 \), are independent.

We note that
\[ d\ln S^i_t = m^i_t dt + \sigma^i_t dw^i_t, \]
(17)
where
\[ m^i_t := (\mu^i_t - (\sigma^i_t)^2), \]
(18)
and
\[ \text{Cov}_R(S^1_T, S^2_T) = \frac{1}{T} [\ln S^1_T, \ln S^2_T] = \frac{1}{T} \int_0^T \sigma^1_t dw^1_t \int_0^T \sigma^2_t dw^2_t = \frac{1}{T} \int_0^T \rho_t \sigma^1_t \sigma^2_t dt. \]  
(19)

Let us show that
\[ [\ln S^1_T, \ln S^2_T] = \frac{1}{4} ([\ln(S^1_T S^2_T)] - [\ln(S^1_T/S^2_T)]). \]
(20)

Remark first that
\[ d\ln(S^1_T S^2_T) = (m^1_t + m^2_t)dt + \sigma^+_t dw^+_t, \]
(21)
and
\[ d\ln(S^1_T/S^2_T) = (m^1_t - m^2_t)dt + \sigma^-_t dw^-_t, \]
(22)
where

\[(\sigma^\pm_t)^2 := (\sigma^1_t)^2 + 2\rho_t\sigma^1_t\sigma^2_t + (\sigma^2_t)^2,\quad (23)\]

and

\[dw^\pm_t := \frac{1}{\sigma^\pm_t}(\sigma^1_t dw^1_t \pm \sigma^2_t dw^2_t).\quad (24)\]

Processes \(w^\pm_t\) in (24) are standard Wiener processes by Levi-Kunita-Watanabe theorem and \(\sigma^\pm_t\) are defined in (23).

In this way, from (21) and (22) we obtain that

\[
\ln(S_1^T S_2^T) = \int_0^t (\sigma^+_s)^2 ds = \int_0^t ((\sigma^+_s)^2 + 2\rho_t\sigma^1_s\sigma^2_s + (\sigma^2_s)^2) ds,
\]

and

\[
\ln(S_1^T / S_2^T) = \int_0^t (\sigma^-_s)^2 ds = \int_0^t ((\sigma^-_s)^2 - 2\rho_t\sigma^1_s\sigma^2_s + (\sigma^2_s)^2) ds.
\]

From (20), (25) and (26) we have directly formula (20):

\[
Cov_R(S^1, S^2) = \frac{1}{4} (\ln(S_1^T S_2^T) - \ln(S_1^T / S_2^T)). \quad (27)
\]

Thus, from (27) we obtain that (see (20) and section 4.1)

\[
Cov_R(S^1, S^2) = 1/4(\sigma^R_2(S^1 S^2) - \sigma^R_2(S^1 / S^2)).
\]

Returning to the valuation of the covariance swap we have

\[
P = E\{e^{-rT}(Cov(S^1, S^2) - K_{cov})\} = \frac{1}{4} e^{-rT}(E\sigma^R_2(S^1 S^2) - E\sigma^R_2(S^1 / S^2) - 4K_{cov}).
\]

The problem now has reduced to the same problem as in the Section 3, but instead of \(\sigma^2_t\) we need to take \((\sigma^+_t)^2\) for \(S^1 S^2\) and \((\sigma^-_t)^2\) for \(S^1 / S^2\) (see (23)), and proceed with the similar calculations as in Section 3.

**Remark 2.** The results of the Sections 2-4 were first presented on the Sixth Annual Financial Econometrics Conference "Estimation of Diffusion Processes in Finance", Friday, March 19, 2004, Centre for Advanced Studies in Finance, University of Waterloo, Waterloo, Canada (Abstract on-line: http://arts.uwaterloo.ca/ACCT/finance/fec6.htm).

## 5 Numerical Example: S&P60 Canada Index

In this section, we apply the analytical solutions from Section 3 to price a swap on the volatility of the S&P60 Canada index for five years (January 1997-February 2002).

These data were kindly presented to author by Raymond Théoret (Université du Québec à Montréal, Montréal, Québec, Canada) and Pierre Rostan (Analyst at the R&D Department of Bourse de Montréal and Université du
Québec à Montréal, Montréal, Québec, Canada). They calibrated the GARCH parameters from five years of daily historic S&P60 Canada Index (from January 1997 to February 2002) (see working paper "Pricing volatility swaps: Empirical testing with Canadian data" by R. Theoret, L. Zabre and P. Ros-tan (2002)).

In the end of February 2002, we wanted to price the fixed leg of a volatility swap based on the volatility of the S&P60 Canada index. The statistics on log returns S&P60 Canada Index for 5 year (January 1997-February 2002) is presented in Table 1:

Table 1

<table>
<thead>
<tr>
<th>Statistics on Log Returns</th>
<th>S&amp;P60 Canada Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Series:</td>
<td>LOG RETURNS S&amp;P60 CANADA INDEX</td>
</tr>
<tr>
<td>Sample:</td>
<td>1 1300</td>
</tr>
<tr>
<td>Observations:</td>
<td>1300</td>
</tr>
<tr>
<td>Mean</td>
<td>0.000235</td>
</tr>
<tr>
<td>Median</td>
<td>0.000593</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.051983</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.101108</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.013567</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.665741</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>7.787327</td>
</tr>
</tbody>
</table>

From the histogram of the S&P60 Canada index log returns on a 5-year historical period (1,300 observations from January 1997 to February 2002) it may be seen leptokurtosis in the histogram. If we take a look at the graph of the S&P60 Canada index log returns on a 5-year historical period we may see volatility clustering in the returns series. These facts indicate about the conditional heteroscedasticity. A GARCH(1,1) regression is applied to the series and the results is obtained as in the next Table 2:

Table 2
**Estimation of the GARCH(1,1) process**

*Dependent Variable: Log returns of S&P60 Canada Index Prices*

*Method: ML-ARCH*

*Included Observations: 1300*

Convergence achieved after 28 observations

<table>
<thead>
<tr>
<th>-</th>
<th>Coefficient</th>
<th>Std. error:</th>
<th>z-statistic:</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.000617</td>
<td>0.000338</td>
<td>1.824378</td>
<td>0.0681</td>
</tr>
</tbody>
</table>

**Variance Equation**

<table>
<thead>
<tr>
<th>-</th>
<th>Coefficient</th>
<th>Std. error:</th>
<th>z-statistic:</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>2.58E-06</td>
<td>3.91E-07</td>
<td>6.597337</td>
<td>0</td>
</tr>
<tr>
<td>ARCH(1)</td>
<td>0.060445</td>
<td>0.007336</td>
<td>8.238968</td>
<td>0</td>
</tr>
<tr>
<td>GARCH(1)</td>
<td>0.927264</td>
<td>0.006554</td>
<td>141.4812</td>
<td>0</td>
</tr>
<tr>
<td>R-squared</td>
<td>-0.000791</td>
<td>Mean dependent var</td>
<td>-</td>
<td>0.000235</td>
</tr>
<tr>
<td>Adjusted R-squared</td>
<td>-0.003108</td>
<td>S.D. dependent var</td>
<td>-</td>
<td>0.013567</td>
</tr>
<tr>
<td>S.E. of regression</td>
<td>0.013588</td>
<td>Akaike info criterion</td>
<td>-</td>
<td>-5.928474</td>
</tr>
<tr>
<td>Sum squared resid</td>
<td>0.239283</td>
<td>Schwartz criterion</td>
<td>-</td>
<td>-5.912566</td>
</tr>
<tr>
<td>Log likelihood</td>
<td>3857.508</td>
<td>Durbin-Watson stat</td>
<td>-</td>
<td>1.886028</td>
</tr>
</tbody>
</table>

This table allows to generate different input variables to the volatility swap model.

We use the following relationship:

\[
\theta = \frac{V}{dt},
\]

\[
k = \frac{1 - \alpha - \beta}{dt},
\]

\[
\gamma = \alpha \sqrt{\frac{\xi - 1}{dt}}
\]

to calculate the following discrete GARCH(1,1) parameters:

- ARCH(1) coefficient \( \alpha = 0.060445 \);
- GARCH(1,1) coefficient \( \beta = 0.927264 \);
- the Pearson kurtosis (fourth moment of the drift-adjusted stock return) \( \xi = 7.787327 \);
- long volatility \( \theta = 0.05289724 \);
- \( k = 3.09733 \);
- \( \gamma = 2.499827486 \);
- a short volatility \( \sigma_0 \) equals to 0.01;
- Parameter \( V \) may be found from the expression \( V = \frac{C}{1-\alpha-\beta} \), where \( C = 2.58 \times 10^{-6} \) is defined in Table 2. Thus, \( V = 0.00020991 \);
- \( dt = 1/252 = 0.003968254 \).
Now, applying the analytical solutions (9) and (13) for a swap maturity $T$ of 0.91 year, we find the following values:

$$E\{V\} = \frac{1 - e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2 = .3364100835,$$

and

$$Var(V) = \frac{\gamma^2 e^{-2kT}}{2k^4 T^2} \left[ \left( 2e^{2kT} - 4e^{kT}kT - 2 \right)(\sigma_0^2 - \theta^2) + \left( 2e^{2kT}kT - 3e^{2kT} + 4e^{kT} - 1 \right)\theta^2 \right] = .0005516049969.$$

The convexity adjustment $\frac{Var(V)}{E(V)^{3/2}}$ is equal to .0003533740855.

If the non-adjusted strike is equal to 18.7751%, then the adjusted strike is equal to

$$18.7751\% - 0.03533740855\% = 18.73976259\%.$$

This is the fixed leg of the volatility swap for a maturity $T = 0.91$.

Repeating this approach for a series of maturities up to 10 years we obtain the following plot (see Appendix, Figure 2) of S&P60 Canada Index Volatility Swap.

Figure 1 (see Appendix) illustrates the non-adjusted and adjusted volatility for the same series of maturities.

6 Appendix: Figures.
Figure 1: Convexity Adjustment.

Figure 2: S&P60 Canada Index Volatility Swap.
References


Chapter 5: CTM and Delayed Heston Model: Pricing and Hedging of Variance and Volatility Swaps

'Better three hours too soon than a minute too late',-William Shakespeare.

1 Introduction

In this chapter, we present a variance drift adjusted version of the Heston model which leads to a significant improvement of the market volatility surface fitting (compared to Heston). The numerical example we performed with recent market data shows a significant reduction of the average absolute calibration error \(^1\) (calibration on 12 dates ranging from Sep. 19\(^{th}\) to Oct. 17\(^{th}\) 2011 for the FOREX underlying EURUSD). Our model has two additional parameters compared to the Heston model, can be implemented very easily and was initially introduced for volatility derivatives pricing purpose. The main idea behind our model is to take into account some past history of the variance process in its (risk-neutral) diffusion. Using a change of time method for continuous local martingales, we derive a closed formula for the Brockhaus&Long approximation of the volatility swap price in this model. We also consider dynamic hedging of volatility swaps using a portfolio of variance swaps.

The volatility process is an important concept in financial modeling as it quantifies at each time \(t\) how likely the modeled asset log-return is to vary significantly over some short immediate time period \([t, t + \epsilon]\). This process can be stochastic or deterministic, e.g. local volatility models in which the (deterministic) volatility depends on time and spot price level. In quantitative finance, we often consider the volatility process \(\sqrt{V_t}\) (where \(V_t\) is the variance process) to be stochastic as it allows to fit the observed vanilla option market prices with an acceptable bias as well as to model the risk linked with the future evolution of the volatility smile (which deterministic model cannot), namely the forward smile. Many derivatives are known to be very sensitive to the forward smile, one of the most popular example being the cliquet options (options on future asset performance, see Kruse and Nögel [18] for example).

Heston model (Heston [11]; Heston and Nandi [12]) is one of the most popular stochastic volatility models in the industry as semi-closed formulas for vanilla option prices are available, few (five) parameters need to be calibrated, and

\(^1\)The average absolute calibration error is defined to be the average of the absolute values of the differences between market and model implied Black & Scholes volatilities.
it accounts for the mean-reverting feature of the volatility.

One might be willing, in the variance diffusion, to take into account not only its current state but also its past history over some interval \([t - \tau, t]\), where \(\tau > 0\) is a constant and is called the delay. Starting from the discrete-time GARCH(1,1) model of Bollerslev [4], a first attempt in this direction was made in Kazmerchuk et al. [16], where a non-Markov delayed continuous-time GARCH model was proposed (\(S_t\) being the asset price at time \(t\), and \(\gamma, \theta, \alpha\) some positive constants). The dynamics considered had the form

\[
\frac{dV_t}{dt} = \gamma \theta^2 + \frac{\alpha}{\tau} \ln \left( \frac{S_t}{S_{t-\tau}} \right) - (\alpha + \gamma)V_t. \tag{1}
\]

This model was inherited from its discrete-time analogue (where \(L\) is a positive integer):

\[
\sigma_n^2 = \tilde{\gamma} \tilde{\theta}^2 + \frac{\tilde{\alpha}}{L} \ln \left( \frac{S_{n-1}}{S_{n-1-L}} \right) + (1 - \tilde{\alpha} - \tilde{\gamma})\sigma_{n-1}^2. \tag{2}
\]

The parameter \(\theta^2\) (resp. \(\gamma\)) can be interpreted as the value of the long-range variance (resp. variance mean-reversion speed) when the delay is equal to 0 (we will see that introducing a delay modifies the value of these two model features). \(\alpha\) is a continuous-time equivalent of the variance ARCH(1,1) autoregressive coefficient. In fact, we can interpret the right-hand side of the diffusion equation (2) as the sum of two terms:

- the delay-free term \(\gamma(\theta^2 - V_t)\), which accounts for the mean-reverting feature of the variance process
- \(\alpha \left( \frac{1}{\tau} \ln \left( \frac{S_t}{S_{t-\tau}} \right) - V_t \right)\) which is a purely (noisy) delay term, i.e. one that vanishes when \(\tau \to 0\) and takes into account the past history of the variance (via the term \(\ln \left( \frac{S_t}{S_{t-\tau}} \right)\)). The autoregressive coefficient \(\alpha\) can be seen as the amplitude of this purely delay term.

In Swishchuk [23] and Swishchuk and Li [21], the authors point out the importance of incorporating the real world \(\mathbb{P}-\)drift \(d_{\mathbb{P}}(t, \tau) := \int_{t-\tau}^t (\mu - \frac{1}{2}V_u)du\) of \(\ln \left( \frac{S_t}{S_{t-\tau}} \right)\) in the model, where \(\mu\) stands for the real world \(\mathbb{P}-\)drift of the stock price \(S_t\), transforms the variance dynamics into:

\[
\frac{dV_t}{dt} = \gamma \theta^2 + \frac{\alpha}{\tau} \left[ \ln \left( \frac{S_t}{S_{t-\tau}} \right) - d_{\mathbb{P}}(t, \tau) \right]^2 - (\alpha + \gamma)V_t. \tag{3}
\]

The latter diffusion (3) was introduced in Swishchuk [23] and Kazmerchuk et al. [15], and the proposed model was proved to be complete and to account for the mean-reverting feature of the volatility process. This model is also
non Markov as the past history \((V_u)_{u \in [t-\tau, t]}\) of the variance appears in its diffusion equation via the term \(\ln \left( \frac{S_t}{S_{t-\tau}} \right)\), as shown in Swishchuk [23]. Following this approach, a series of papers was published by one of the authors [23] focusing on the pricing of variance swaps in this delayed framework: one-factor stochastic volatility with delay has been presented in Swishchuk [23]; multifactor stochastic volatility with delay in Swishchuk [24]; one-factor stochastic volatility with delay and jumps in Swishchuk and Li [21]; and finally local Levy-based stochastic volatility with delay in Swishchuk and Malenfant [26].

Other papers related to the concept of delay are also of interest. For example, Kind et al. [17] obtained a diffusion approximation result for processes satisfying some equations with past dependent coefficients, with application to option pricing. Arriojas et al. [1] derived a Black&Scholes formula for call options assuming the stock price follows a Stochastic Delay Differential Equation (SDDE). Mohammed and Bell have also published a series of papers in which they investigate various properties of SDDE (see e.g. [2], [3]).

Unfortunately, the model (3) doesn’t lead to (semi-)closed formulas for the vanilla options, making it difficult to use for practitioners willing to calibrate on vanilla market prices. Nevertheless, one can notice that the Heston model and the delayed continuous-time GARCH model (3) are very similar in the sense that the expected values of the variances are the same - when we make the delay tend to 0 in (3). As mentioned before, the Heston framework is very convenient, and therefore it is naturally tempting to adjust the Heston dynamics in order to incorporate the delay introduced in (3). In this way, we considered in a first approach adjusting the Heston drift by a deterministic function of time so that the expected value of the variance under the delayed Heston model is equal to the one under the delayed GARCH model (3). In addition to making our delayed Heston framework coherent with (3), this construction makes the variance process diffusion dependent not on its past history \((V_u)_{u \in [t-\tau, t]}\), but on the past history of its risk-neutral expectation \((E_Q^0(V_u))_{u \in [t-\tau, t]}\), preserving the Markov feature of the Heston model (where we denote \(E_Q^0(\cdot) := E_Q(\cdot | \mathcal{F}_t)\) for some filtration \((\mathcal{F}_t)_{t \geq 0}\)). The purpose of sections 2 and 3 is to present the Delayed Heston model as well as some calibration results on call option prices, with a comparison to the Heston model. In sections 4 and 5, we will consider the pricing and hedging of volatility and variance swaps in this model.

Volatility and variance swaps are contracts whose payoff depend (respectively convexly and linearly) on the realized variance of the underlying asset over some specified time interval. They provide pure exposure to volatility, and therefore make it a tradable market instrument. Variance Swaps are even considered by some practitioners to be vanilla derivatives. The most commonly traded variance swaps are discretely sampled and have a payoff \(P_n^V(T)\)
at maturity $T$ of the form:

$$P_n^V(T) = N \left[ \frac{252}{n} \sum_{i=0}^{n} \ln^2 \left( \frac{S_{i+1}}{S_i} \right) - K_{\text{var}} \right],$$

where $S_i$ is the asset spot price on fixing time $t_i \in [0,T]$ (usually there is one fixing time each day, but there could be more, or less), $N$ the notional amount of the contract (in currency per unit of variance) and $K_{\text{var}}$ the strike specified in the contract. The corresponding volatility swap payoff $P_n^\nu(T)$ is given by:

$$P_n^\nu(T) = N \left[ \sqrt{\frac{252}{n} \sum_{i=0}^{n} \ln^2 \left( \frac{S_{i+1}}{S_i} \right) - K_{\text{vol}}} \right].$$

One can also consider continuously sampled volatility and variance swaps (on which we will focus in this article), which payoffs are respectively defined as the limit when $n \to +\infty$ of their discretely sampled versions. Formally, if we denote $(V_t)_{t \geq 0}$ the stochastic volatility process of our asset, adapted to some brownian filtration $(\mathcal{F}_t)_{t \geq 0}$, then the continuously-sampled realized variance $V_R$ from initiation date of the contract $t = 0$ to maturity date $t = T$ is given by $V_R = \frac{1}{T} \int_0^T V ds$. The fair variance strike $K_{\text{var}}$ is calculated such that the initial value of the contract is 0, and therefore is given by:

$$\mathbb{E}_0^Q \left[ e^{-rT} (V_R - K_{\text{var}}) \right] = 0 \Rightarrow K_{\text{var}} = \mathbb{E}_0^Q(V_R).$$

In the same way, the fair volatility strike $K_{\text{vol}}$ is given by:

$$\mathbb{E}_0^Q \left[ e^{-rT} \left( \sqrt{V_R} - K_{\text{vol}} \right) \right] = 0 \Rightarrow K_{\text{vol}} = \mathbb{E}_0^Q(\sqrt{V_R}).$$

The volatility swap fair strike might be difficult to compute explicitly as we have to compute the expectation of a square-root. In Brockhaus and Long [7], the following approximation - based on a Taylor expansion - was proposed to compute the expected value of the square-root of an almost surely non negative random variable $Z$:

$$\mathbb{E}(\sqrt{Z}) \approx \sqrt{\mathbb{E}(Z)} - \frac{\text{Var}(Z)}{8 \mathbb{E}(Z)^{3/2}}. \quad (4)$$

We will refer to this approximation in our paper as the Brockhaus&Long approximation.

There exists a vast literature on volatility and variance swaps. We provide in the following lines a selection of papers covering important topics. Carr and Lee [8] provides an overview of the current market of volatility derivatives. They survey the early literature on the subject. They also provide relatively simple proofs of some fundamental results related to variance swaps and
volatility swaps. Pricing of variance swaps for one-factor stochastic volatility is presented in Swishchuk [22]. Variance and volatility swaps in energy markets are considered in Swishchuk [25]. Broadie and Jain [6] covers pricing and dynamic hedging of volatility derivatives in the Heston model. Moreover, various papers deal with the VIX Index - the Chicago Board Options Exchange Market Volatility Index - which is a popular measure of the one month implied volatility on the S&P 500 index (see e.g. Zhang and Zhu [28], Hao and Zhang [10] or Filipovic [9]).

The paper is organized as follows: in section 2, we present the Delayed Heston model; in section 3, we present calibration results (for underlying EURUSD on 12 dates ranging from Sep. 19th to Oct. 17th 2011) as well as a comparison with the Heston model. In section 4, we compute the price process $X_t(T) := \mathbb{E}^Q_t(V_R)$ of the floating leg of the variance swap of maturity $T$, as well as the Brockhaus&Long approximation of the price process $Y_t(T) := \mathbb{E}^Q_t(\sqrt{V_R})$ of the floating leg of the volatility swap of maturity $T$. This leads in particular to closed formulas for the fair volatility and variance strikes. In section 5, we consider - in this model - dynamic hedging of volatility swaps using variance swaps.

## 2 Presentation of the Delayed Heston model

Throughout this paper, we will assume constant risk-free rate $r$, dividend yield $q$ and finite time-horizon $T$. We fix $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and we consider a stock whose price process is denoted by $(S_t)_{t \geq 0}$. We let $\mathbb{Q}$ be a risk-neutral measure and we let $(Z^Q_t)_{t \geq 0}$ and $(W^Q_t)_{t \geq 0}$ be two correlated standard brownian motions on $(\Omega, \mathcal{F}, \mathbb{Q})$. We let the natural filtration associated to these brownian motions $\mathcal{F}_t := \sigma(Z^Q_t, W^Q_t)$ and we denote $\mathbb{E}^Q_t(\cdot) := \mathbb{E}^{\mathbb{Q}}(\cdot|\mathcal{F}_t)$ and $\text{Var}^Q_t(\cdot) := \text{Var}^{\mathbb{Q}}(\cdot|\mathcal{F}_t)$.

We assume the following risk-neutral $\mathbb{Q}$– stock price dynamics :

$$dS_t = (r - q)S_t dt + S_t \sqrt{V_t} dZ^Q_t.$$  \hfill (5)

The well-known Heston model has the following $\mathbb{Q}$–dynamics for the variance $V_t$:

$$dV_t = \gamma(\theta^2 - V_t) dt + \delta \sqrt{V_t} dW^Q_t,$$ \hfill (6)

where $\theta^2$ is the long-range variance, $\gamma$ the variance mean-reversion speed, $\delta$ the volatility of the variance and $\rho$ the brownian correlation coefficient ($\langle W^Q_t, Z^Q_t \rangle_t = \rho t$). We also assume $S_0 = s_0$ a.e. and $V_0 = v_0$ a.e., for some positive constants $v_0, s_0$. 

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As explained in the introduction, the following delayed continuous-time GARCH dynamics have been introduced for the variance in Swishchuk [23]:

\[
\frac{dV_t}{dt} = \gamma \theta^2 + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^t \sqrt{V_s} dZ_s^Q - (\mu - r) \tau \right]^2 - (\alpha + \gamma) V_t, \tag{7}
\]

where \( \mu \) stands for the real world \( \mathbb{P} \)-drift of the stock price \( S_t \). We notice that \( \theta^2 \) (resp. \( \gamma \)) has been defined in introduction for the delayed continuous-time GARCH model as the value of the long-range variance (resp. variance mean-reversion speed) when \( \tau = 0 \), therefore it has the same meaning as the Heston long-range variance (resp. variance mean-reversion speed). That is why we use the same notations in both models.

We can see that the two models are very similar. Indeed, they both give the same expected value for \( V_t \) as the delay goes to 0 in (7), namely \( \theta^2 + (V_0 - \theta^2)e^{-\gamma t} \). The idea here is to adjust the Heston dynamics (6) in order to account for the delay introduced in (7). Our approach is to adjust the drift by a deterministic function of time so that the expected value of \( V_t \) under the adjusted Heston model is the same as under (7). This approach can be seen as a correction by a pure delay term of amplitude \( \alpha \) of the Heston drift by a deterministic function in order to account for the delay.

Namely, we assume the adjusted Heston dynamics:

\[
dV_t = \left[ \gamma (\theta^2 - V_t) + \epsilon_\tau(t) \right] dt + \delta \sqrt{V_t} dW_t^Q, \tag{8}
\]

\[
\epsilon_\tau(t) := \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^t v_s ds - \alpha v_t, \tag{9}
\]

with \( v_t := \mathbb{E}_0^Q(V_t) \). It was shown in Swishchuk [23] that \( v_t \) solves the following equation:

\[
\frac{dv_t}{dt} = \gamma \theta^2 + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^t v_s ds - (\alpha + \gamma) v_t, \tag{10}
\]

and that we have the following expression for \( v_t \):

\[
v_t = \theta^2_\tau + (V_0 - \theta^2_\tau)e^{-\gamma \tau t}, \tag{11}
\]

with:

\[
\theta^2_\tau := \theta^2 + \frac{\alpha \tau (\mu - r)^2}{\gamma}. \tag{12}
\]

By (11) and (15) (see below), we have \( \lim_{t \to \infty} v_t = \theta_0^2 \) and therefore the parameter \( \theta^2_\tau \) can be interpreted as the adjusted value of the limit towards \( v_t \) tends to as \( t \to \infty \), that has been (positively) shifted from its original value \( \theta^2 \) because of the introduction of delay. We have \( \theta^2_\tau \to \theta^2 \) when \( \tau \to 0 \), which is coherent. We will see below that we can interpret the parameter \( \gamma_\tau > 0 \)
as the adjusted mean-reversion speed. This parameter is given in Swishchuk [23] by a (nonzero) solution to the following equation:

\[
\gamma_\tau = \alpha + \gamma + \frac{\alpha}{\gamma_\tau\tau}(1 - e^{\gamma_\tau\tau}).
\]  

(13)

By (9), (11) and (13) we get an explicit expression for the drift adjustment:

\[
\epsilon_\tau(t) = \alpha\tau(\mu - r)^2 + (V_0 - \theta^2_\tau)(\gamma - \gamma_\tau)e^{-\gamma_\tau t}.
\]  

(14)

The following simple property gives us some information about the correction term \(\epsilon_\tau(t)\) and the parameter \(\gamma_\tau\), that will be useful for interpretation purpose and in the derivation of the semi-closed formulas for call options in Appendix A. Indeed, given (15) and (11), the parameter \(\gamma_\tau\) can be interpreted as the adjusted variance mean-reversion speed because it quantifies the speed at which \(v_t\) tends to \(\theta^2\) as \(t \to \infty\), and we have by using a Taylor expansion in (13) that \(\gamma_\tau \to \gamma\) when \(\tau \to 0\), which is coherent.

**Property 1:** \(\gamma_\tau\) is the unique solution to (13) and:

\[
0 < \gamma_\tau < \gamma, \quad \lim_{\tau \to 0} \sup_{t \in \mathbb{R}^+} |\epsilon_\tau(t)| = 0.
\]  

(15)

**Proof:** Let's show \(\gamma_\tau \geq 0\). If \(\gamma_\tau < 0\) then by (13) we have \(\alpha + \gamma + \frac{\alpha}{\gamma_\tau\tau}(1 - e^{\gamma_\tau\tau}) < 0\), i.e. \(1 - e^{\gamma_\tau\tau} + \gamma_\tau\tau > -\frac{\alpha}{\gamma_\tau}\gamma_\tau\tau\). But \(\tau > 0\) so \(\exists x_0 > 0\) s.t. \(1 - e^{x_0} < x_0 > \frac{\alpha}{\gamma_\tau}\). A simple study shows that is impossible whenever \(x_0 \geq 0\), which is what we have by assumption. Therefore \(\gamma_\tau \geq 0\), and in fact \(\gamma_\tau > 0\) since it is a nonzero solution of (13). If \(\gamma \leq \gamma_\tau\) then by (13) \(\gamma_\tau\tau + 1 - e^{\gamma_\tau\tau} \geq 0\). But \(\gamma_\tau\tau > 0\) therefore \(\exists x_0 > 0\) s.t. \(x_0 + 1 - e^{x_0} \geq 0\). A simple study shows that is impossible. The uniqueness comes from a similar simple study. Now, because \(\gamma_\tau > 0\), we have \(\sup_{t \in \mathbb{R}^+} |\epsilon_\tau(t)| \leq \alpha\tau(\mu - r)^2 + |(V_0 - \theta^2_\tau)(\gamma - \gamma_\tau)| \) and \((V_0 - \theta^2_\tau)(\gamma - \gamma_\tau) = o(1)\) by (13).

So \(\lim_{\tau \to 0} \alpha\tau(\mu - r)^2 + |(V_0 - \theta^2_\tau)(\gamma - \gamma_\tau)| = 0\).

Using (14) and (12), we can rewrite (8) as a time-dependent Heston model with time-dependent long-range variance \(\hat{\theta}^2_t\):

\[
dV_t = \gamma(\hat{\theta}^2_t - V_t)dt + \delta\sqrt{V_t}dW_t^Q,
\]  

(16)

\[
\hat{\theta}^2_t := \theta^2 + (V_0 - \theta^2_\tau)(\gamma - \gamma_\tau)e^{-\gamma_\tau t}.
\]  

(17)

The parameter \(\theta^2_t\) is - as we mentioned above - the adjusted value of the limit towards which \(v_t\) tends to as \(t \to \infty\). For this reason, it is coherent that it is also the limiting value of the time-dependent long-range variance \(\hat{\theta}^2_t\) as \(t \to \infty\) (by (17) and (15)).
3 Calibration on call option prices and comparison to the Heston model

Following Kahl and Jäckel [13] and Mikhailov and Noegel [20], it is possible to get semi-closed formulas for call options in our delayed Heston model. Indeed, our model is a time-dependent Heston model with time-dependent long-range variance $\tilde{\theta}_t^2$. We refer to Appendix A for the procedure to derive such semi-closed formulas.

We perform our calibration on September 30th 2011 for underlying EURUSD on the whole volatility surface (maturities from 1M to 10Y, strikes ATM, 25D Call/Put, 10D Call/Put). The implied volatility surface, the Zero Coupon curves EUR Vs. Euribor 6M and USD Vs. Libor 3M and the spot price are taken from Bloomberg (mid prices). The drift $\mu = 0.0188$ is estimated from 7.5Y of daily close prices (source: www.forexrate.co.uk).

The calibration procedure is a least-squares minimization procedure that we perform via MATLAB (function *lsqnonlin* that uses a trust-region-reflective algorithm). The Heston integral (83) is computed via the MATLAB function *quadl* that uses a recursive adaptive Lobatto quadrature. The integral $\int_0^t e^{-\gamma s} D(s, u)ds$ in (81) is computed via a composite Simpson’s rule with 100 points.

The calibrated parameters for delayed Heston are:

$$(V_0, \gamma, \theta^2, \delta, \rho, \alpha, \tau) = (0.0343, 3.9037, 10^{-8}, 0.808, -0.5057, 71.35, 0.7821),$$

and for Heston, they are:

$$(V_0, \gamma, \theta^2, \delta, \rho) = (0.0328, 0.5829, 0.0256, 0.3672, -0.4824).$$

We notice that we cannot compare straightforwardly the parameters $\theta^2$ of both models. Indeed, as mentioned above, the Delayed Heston model has a time-dependent long range variance $\tilde{\theta}_t^2$ which has been shifted away from its original value $\theta^2$ because of the introduction of the delay $\tau$. When $\tau = 0$, $\tilde{\theta}_t^2 = \theta^2$ but when $\tau > 0$, $\tilde{\theta}_t^2$ and $\theta^2$ differ. Therefore, to be coherent, one should compare the Heston long range-variance $\theta^2 = 0.0256$ with the Delayed Heston time-dependent long-range variance $\tilde{\theta}_t^2$. Below we give the value of $\tilde{\theta}_t^2$ for different maturities $t$:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>$\tilde{\theta}_t^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1M</td>
<td>0.0325</td>
</tr>
<tr>
<td>2M</td>
<td>0.0322</td>
</tr>
<tr>
<td>3M</td>
<td>0.0319</td>
</tr>
<tr>
<td>6M</td>
<td>0.0331</td>
</tr>
<tr>
<td>1Y</td>
<td>0.0294</td>
</tr>
<tr>
<td>2Y</td>
<td>0.0364</td>
</tr>
<tr>
<td>5Y</td>
<td>0.0184</td>
</tr>
<tr>
<td>10Y</td>
<td>0.0102</td>
</tr>
</tbody>
</table>
We remark that the short and medium term values (less than 2Y) of \( \tilde{\theta}_t^2 \) are similar to the value of \( \theta^2 \) in the Heston model, but that for long maturities, the value of \( \tilde{\theta}_t^2 \) decreases significantly. Allowing this time-dependence of the the long-range variance could be an explanation why the Delayed Heston model outperforms the Heston model especially for long maturities (see the discussion below). Similarly, the Heston mean-reversion speed \( \gamma = 0.58 \) has to be compared with the Delayed Heston adjusted mean-reversion speed \( \gamma^\tau \), which is given by (13) and is approximately equal to 0.12 on our calibration date. Focusing on the delay parameters \( \alpha \) and \( \tau \), they were expected to be significantly non zero because as we will see below, the Delayed Heston model significantly outperforms the Heston model in terms of calibration error (and standard deviation of the calibration errors): if \( \alpha \) and \( \tau \) were close to 0, the calibration errors would have been approximately the same for both models, because again, the Delayed Heston model reduces to the Heston model when the delay term vanishes, i.e. when \( \tau = 0 \) or \( \alpha = 0 \).

The calibration errors for all call options (expressed as the absolute value of the difference between market and model implied Black & Scholes volatilities, in bp) for the Heston model and our Delayed Heston model are given below. The results show a 44% reduction of the average absolute calibration error (46bp for delayed Heston, 81bp for Heston). It is to be noted that we didn’t use any weight matrix in our calibration procedure, i.e. the calibration aims at minimizing the sum of the (squares of the) errors of each call option, equally weighted. In practice, one might be willing to give more importance to ATM options for instance, or options of a certain range of maturities. The optimization algorithm aims at minimizing the sum of the squares of the errors: in other words, it aims at minimizing the average absolute calibration error. For this reason, it might be the case that for some specific option (e.g. ATM 6M, see table below), the Heston model has a lower model error than the Delayed Heston model. But the total calibration error for the Delayed Heston model is always expected to be lower than for the Heston model.

On our calibration date, the Delayed Heston model seems to outperform the Heston model specifically for long maturities (\( \geq 3Y \)): if we consider only these options, the average absolute error is of 79bp for the Heston model and 33bp for the Delayed Heston model, which represents a 58% reduction of the calibration error. We can also note that for ATM options only, the improvement is significant too (43bp Vs. 92bp, i.e. an error reduction of 54%). For medium maturity options (6M to 2Y), the Delayed Heston model still outperforms the Heston model but less significantly (53bp Vs. 75bp, i.e. an error reduction of 30%), and we have the same observation for very out of the money options (10 Delta Call and Put, 51bp Vs. 79bp, i.e. an error
Another very interesting observation we can make is that the standard deviation of the calibration errors is much lower for the Delayed Heston model compared to the Heston model (34bp Vs. 52bp, which represents a 35% reduction of the standard deviation): in addition to improving the average absolute calibration error, it also improves the distance of the individual errors to the average error, which is highly appreciable in practice because it means that you won’t face the case where some options are priced really poorly by the model whereas some others are priced almost perfectly.

Table 2: Heston Absolute Calibration Error (in bp of the Black & Scholes volatility).

<table>
<thead>
<tr>
<th></th>
<th>ATM</th>
<th>25D Call</th>
<th>25D Put</th>
<th>10D Call</th>
<th>10D Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>1M</td>
<td>152</td>
<td>192</td>
<td>41</td>
<td>193</td>
<td>67</td>
</tr>
<tr>
<td>2M</td>
<td>114</td>
<td>139</td>
<td>13</td>
<td>136</td>
<td>81</td>
</tr>
<tr>
<td>3M</td>
<td>89</td>
<td>109</td>
<td>3</td>
<td>110</td>
<td>92</td>
</tr>
<tr>
<td>4M</td>
<td>48</td>
<td>61</td>
<td>17</td>
<td>67</td>
<td>101</td>
</tr>
<tr>
<td>6M</td>
<td>5</td>
<td>15</td>
<td>34</td>
<td>29</td>
<td>85</td>
</tr>
<tr>
<td>9M</td>
<td>59</td>
<td>42</td>
<td>63</td>
<td>2</td>
<td>85</td>
</tr>
<tr>
<td>1Y</td>
<td>107</td>
<td>83</td>
<td>102</td>
<td>31</td>
<td>96</td>
</tr>
<tr>
<td>1.5Y</td>
<td>141</td>
<td>116</td>
<td>111</td>
<td>42</td>
<td>73</td>
</tr>
<tr>
<td>2Y</td>
<td>166</td>
<td>137</td>
<td>127</td>
<td>54</td>
<td>68</td>
</tr>
<tr>
<td>3Y</td>
<td>145</td>
<td>124</td>
<td>77</td>
<td>52</td>
<td>0</td>
</tr>
<tr>
<td>4Y</td>
<td>96</td>
<td>95</td>
<td>18</td>
<td>37</td>
<td>66</td>
</tr>
<tr>
<td>5Y</td>
<td>29</td>
<td>47</td>
<td>52</td>
<td>7</td>
<td>138</td>
</tr>
<tr>
<td>7Y</td>
<td>39</td>
<td>10</td>
<td>112</td>
<td>28</td>
<td>186</td>
</tr>
<tr>
<td>10Y</td>
<td>100</td>
<td>67</td>
<td>168</td>
<td>58</td>
<td>225</td>
</tr>
</tbody>
</table>

Table 3: Delayed Heston Absolute Calibration Error (in bp of the Black & Scholes volatility).

<table>
<thead>
<tr>
<th></th>
<th>ATM</th>
<th>25D Call</th>
<th>25D Put</th>
<th>10D Call</th>
<th>10D Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>1M</td>
<td>116</td>
<td>91</td>
<td>109</td>
<td>128</td>
<td>115</td>
</tr>
<tr>
<td>2M</td>
<td>44</td>
<td>24</td>
<td>59</td>
<td>54</td>
<td>88</td>
</tr>
<tr>
<td>3M</td>
<td>14</td>
<td>3</td>
<td>32</td>
<td>36</td>
<td>60</td>
</tr>
<tr>
<td>4M</td>
<td>18</td>
<td>28</td>
<td>1</td>
<td>5</td>
<td>29</td>
</tr>
<tr>
<td>6M</td>
<td>31</td>
<td>37</td>
<td>23</td>
<td>19</td>
<td>3</td>
</tr>
<tr>
<td>9M</td>
<td>45</td>
<td>45</td>
<td>56</td>
<td>37</td>
<td>57</td>
</tr>
<tr>
<td>1Y</td>
<td>51</td>
<td>47</td>
<td>82</td>
<td>50</td>
<td>104</td>
</tr>
<tr>
<td>1.5Y</td>
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<td>30</td>
<td>79</td>
<td>49</td>
<td>129</td>
</tr>
<tr>
<td>2Y</td>
<td>24</td>
<td>23</td>
<td>83</td>
<td>47</td>
<td>139</td>
</tr>
<tr>
<td>3Y</td>
<td>11</td>
<td>9</td>
<td>29</td>
<td>30</td>
<td>90</td>
</tr>
<tr>
<td>4Y</td>
<td>41</td>
<td>28</td>
<td>14</td>
<td>17</td>
<td>38</td>
</tr>
<tr>
<td>5Y</td>
<td>76</td>
<td>55</td>
<td>59</td>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>7Y</td>
<td>71</td>
<td>49</td>
<td>58</td>
<td>1</td>
<td>14</td>
</tr>
<tr>
<td>10Y</td>
<td>26</td>
<td>8</td>
<td>18</td>
<td>47</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 4: Standard Deviation of the calibration errors (bp). The reduction of this error is indicated in brackets.

In order to i) check that our calibration on September 30th 2011 was not an exception and ii) investigate the stability of the calibrated parameters, we
performed calibrations on 11 additional dates evenly spaced around September 30th 2011, ranging from September 19th 2011 to October 17th 2011. We chose a one month window because from the past experience of the authors in the financial industry, it can happen that the parameters are recalibrated by financial institutions every month only, and not every day (because it would be too time-consuming) and therefore the choice of a one month window seems reasonable to investigate the stability of the parameters. We summarize the findings in the tables below. We find that the Delayed Heston model always outperforms significantly the Heston model (average calibration error reduction varying from 29% to 56%), and that the Delayed Heston model is performant especially for long maturities ($\geq 3Y$, calibration error reduction varying from 40% to 66%) and ATM options (calibration error reduction varying from 42% to 67%). Finally, the standard deviation of the calibrations errors is always reduced significantly by the Delayed Heston model (reduction varying from 23% to 49%).

<table>
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<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Error Reduction (%)</td>
<td>44</td>
<td>45</td>
<td>56</td>
<td>47</td>
<td>38</td>
<td>51</td>
<td>42</td>
<td>37</td>
<td>38</td>
<td>29</td>
<td>39</td>
</tr>
<tr>
<td>Long Maturity Error Reduction (%)</td>
<td>58</td>
<td>63</td>
<td>65</td>
<td>55</td>
<td>53</td>
<td>66</td>
<td>55</td>
<td>51</td>
<td>48</td>
<td>40</td>
<td>59</td>
</tr>
<tr>
<td>ATM Error Reduction (%)</td>
<td>57</td>
<td>55</td>
<td>67</td>
<td>62</td>
<td>55</td>
<td>65</td>
<td>56</td>
<td>51</td>
<td>50</td>
<td>42</td>
<td>53</td>
</tr>
</tbody>
</table>

*Table 5: Summary of the calibration error reductions.*

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibration errors St. Dev. Reduction (%)</td>
<td>43</td>
<td>45</td>
<td>49</td>
<td>46</td>
<td>31</td>
<td>49</td>
<td>40</td>
<td>29</td>
<td>29</td>
<td>29</td>
<td>29</td>
</tr>
</tbody>
</table>

*Table 6: Summary of the calibration errors St. Dev reductions.*

In order to investigate the stability of the model parameters, we present below the calibrated parameters for the Heston model and the Delayed Heston model from September 19th 2011 to October 17th 2011.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0$</td>
<td>0.0313</td>
<td>0.0337</td>
<td>0.0384</td>
<td>0.0354</td>
<td>0.0354</td>
<td>0.0368</td>
<td>0.0344</td>
<td>0.0295</td>
<td>0.0279</td>
<td>0.0271</td>
<td>0.0283</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>3.99</td>
<td>3.72</td>
<td>3.82</td>
<td>3.72</td>
<td>4.52</td>
<td>3.47</td>
<td>3.86</td>
<td>3.71</td>
<td>3.13</td>
<td>3.08</td>
<td>3.39</td>
</tr>
<tr>
<td>$\theta^2$</td>
<td>5 e-4</td>
<td>7 e-6</td>
<td>2 e-4</td>
<td>1 e-8</td>
<td>1 e-5</td>
<td>3 e-4</td>
<td>2 e-3</td>
<td>1 e-3</td>
<td>5 e-3</td>
<td>4 e-3</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.79</td>
<td>0.75</td>
<td>0.82</td>
<td>0.81</td>
<td>0.89</td>
<td>0.78</td>
<td>0.81</td>
<td>0.76</td>
<td>0.68</td>
<td>0.67</td>
<td>0.80</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.51</td>
<td>-0.51</td>
<td>-0.52</td>
<td>-0.50</td>
<td>-0.49</td>
<td>-0.51</td>
<td>-0.51</td>
<td>-0.51</td>
<td>-0.51</td>
<td>-0.51</td>
<td>-0.49</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>82.2</td>
<td>77.5</td>
<td>64.5</td>
<td>160.7</td>
<td>124</td>
<td>66.6</td>
<td>76.5</td>
<td>90.2</td>
<td>125.1</td>
<td>83.7</td>
<td>77.3</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.86</td>
<td>0.77</td>
<td>0.71</td>
<td>0.32</td>
<td>0.59</td>
<td>0.72</td>
<td>0.78</td>
<td>0.90</td>
<td>0.67</td>
<td>1.00</td>
<td>0.81</td>
</tr>
</tbody>
</table>

*Table 7: Calibrated Parameters for the Delayed Heston model.*
We can see that in average, the parameters stay relatively stable throughout this 1 month time window. In this case, it would be reasonable to use the same parameters throughout the 1 month time window as some financial institutions do (from the past experience of the authors in the financial industry). Of course, there are some periods of high volatility in which not recalibrating the model parameters often enough might lead to a significant mispricing of the call options by the model.

4 Pricing Variance and Volatility Swaps

In this section, we derive a closed formula for the Brockhaus&Long approximation of the volatility swap price using the change of time method introduced in Swishchuk [22], as well as the price of the variance swap. Precisely, in Brockhaus and Long [7], the following approximation was presented to compute the expected value of the square-root of an almost surely non negative random variable $Z$: $E(\sqrt{Z}) \approx \sqrt{E(Z)} - \frac{\text{Var}(Z)}{2E(Z)}$. We denote $V_R := \frac{1}{T}\int_0^T V_s ds$ the realized variance on $[0, T]$.

We let $X_t(T) := E^Q_t(V_R)$ (resp. $Y_t(T) := E^Q_t(\sqrt{V_R})$) the price process of the floating leg of the variance swap (resp. volatility swap) of maturity $T$.

**Theorem 1:** The price process $X_t(T)$ of the floating leg of the variance swap of maturity $T$ in the delayed Heston model (5)-(8) is given by:

\[
X_t(T) = \frac{1}{T} \int_0^T V_s ds + \frac{T-t}{T} \theta^2 + (V_t - \theta^2) \left( 1 - e^{-\gamma(T-t)} \right) \gamma T + (V_0 - \theta^2) e^{-\gamma t} \left( \frac{1 - e^{-\gamma(T-t)}}{\gamma T} - 1 - e^{-\gamma(T-t)} \right). \tag{18}
\]

**Proof:** By definition, $X_t(T) = E^Q_t(\frac{1}{T} \int_0^T V_s ds) = \frac{1}{T} \int_0^T E^Q_t(V_s) ds + \frac{1}{T} \int_0^T E^Q_t(\sqrt{V_s}) ds$. In the previous integral, the interchange between the expectation and the integral is justified by the use of Tonelli’s theorem, as the variance process $(t, \omega) \rightarrow V_t(\omega)$ is a.e. non-negative and measurable. Let $s \geq t$. Then we have by (8) that $E^Q_t(V_s - V_t) = E^Q_t(V_s) - V_t = \int_t^s \gamma(\theta^2 - E^Q_t(V_u)) + \epsilon_t(u) du + E^Q_t(\int_t^s \sqrt{V_u} dW^Q_u)$. Again, the interchange of the expectation and the integral $E^Q_t(\int_t^s \gamma(\theta^2 - V_u) + \epsilon_t(u) du) = \int_t^s \gamma(\theta^2 - V_u) + \epsilon_t(u) du$.
\[
\int_t^s \gamma(\theta^2 - \mathbb{E}_t^Q(V_u)) + \epsilon_r(u) \, du \text{ is obtained the following way:}
\]

\[
\mathbb{E}_t^Q(\int_t^s \gamma(\theta^2 - V_u) + \epsilon_r(u) \, du) = \int_t^s \gamma \theta^2 + \epsilon_r(u) \, du - \gamma \mathbb{E}_t^Q(\int_t^s V_u \, du). \tag{19}
\]

Then again, by Tonelli’s theorem we get \(\mathbb{E}_t^Q(\int_t^s V_u \, du) = \int_t^s \mathbb{E}_t^Q(V_u) \, du\), which justifies the interchange.

Now, \((\sqrt{V_t})_{t \geq 0}\) is an adapted process to our filtration \((\mathcal{F}_t)_{t \geq 0}\) s.t. \(\mathbb{E}_t^Q(\int_0^T V_u \, du) = \int_0^T \mathbb{E}_t^Q(V_u) \, du < +\infty\) (by Tonelli’s theorem), therefore \(\int_0^t \sqrt{V_u} dW_u^Q\) is a martingale and we have \(\mathbb{E}_t^Q(\int_t^s \sqrt{V_u} dW_u^Q) = 0\). Therefore \(\forall s \geq t \geq 0\), the function \(s \to \mathbb{E}_t^Q(V_s)\) is a solution of \(\gamma \theta^2 - V_s + \epsilon_r(s)\) with initial condition \(y_t = V_t\). This gives us \(\mathbb{E}_t^Q(V_s) = \theta_\tau^2 + (V_t - \theta_\tau^2) e^{-\gamma (s-t)} + (V_0 - \theta_\tau^2) e^{-\gamma \tau} (e^{-\gamma (s-t)} - e^{-\gamma (s-t)})\). Integrating the latter in the variable \(s\) via \(\int_t^T \mathbb{E}_t^Q(V_s) \, ds\) completes the proof.

**Corollary 1:** The price \(K_{\text{var}}\) of the variance swap of maturity \(T\) at initiation of the contract \(t = 0\) in the delayed Heston model (5)-(8) is given by:

\[
K_{\text{var}} = \theta_\tau^2 + (V_0 - \theta_\tau^2) \left(1 - e^{-\gamma \tau T}/\gamma \tau T\right). \tag{20}
\]

**Proof:** By definition, \(K_{\text{var}} = X_0(T)\).

Now, let:

\[
x_t := -(V_0 - \theta_\tau^2) e^{(\gamma - \gamma \tau) t} + e^{\gamma t}(V_t - \theta_\tau^2). \tag{21}
\]

Then by Ito’s Lemma we get:

\[
dx_t = \delta e^{\gamma t} \sqrt{(x_t + (V_0 - \theta_\tau^2) e^{(\gamma - \gamma \tau) t} e^{-\gamma t} + \theta_\tau^2)} dW_t^Q. \tag{22}
\]

Which is of the form \(dx_t = f(t, x_t) dW_t^Q\) with:

\[
f(t, x) := \delta e^{\gamma t} \sqrt{(x + (V_0 - \theta_\tau^2) e^{(\gamma - \gamma \tau) t} e^{-\gamma t} + \theta_\tau^2}. \tag{23}
\]

Indeed, since \(x_t = g(t, V_t)\) with \(g(t, x) := -(V_0 - \theta_\tau^2) e^{(\gamma - \gamma \tau) t} + e^{\gamma t} (x - \theta_\tau^2)\), the multidimensional version of Ito’s lemma reads:

\[
dx_t = dg(t, V_t) = g_t(t, V_t) dt + g_x(t, V_t) dV_t + \frac{1}{2} g_{xx}(t, V_t) d\langle V, V \rangle_t, \tag{24}
\]
where \( \langle V, V \rangle_t \) is the quadratic variation of the process \( (V_t)_{t \geq 0} \) (see e.g. [14], theorem 3.6. of section 3.3). Since \( g_{x,t}(s,x) = 0 \), \( g_t(t,V_t) = - (\gamma - \gamma^\tau)(V_0 - \theta^2_\tau) e^{(\gamma - \gamma^\tau)t} + \gamma e^{\gamma t}(V_t - \theta^2_\tau) \) and \( g_x(t,V_t) = e^{\gamma t} \), we get, using (8), (12) and (14):

\[
d x_t = g_t(t,V_t)dt + g_x(t,V_t)dV_t
\]

(25)

\[
= - (\gamma - \gamma^\tau)(V_0 - \theta^2_\tau) e^{(\gamma - \gamma^\tau)t} dt + \gamma e^{\gamma t}(V_t - \theta^2_\tau)dt + e^{\gamma t}dV_t
\]

(26)

\[
= - e^{\gamma t}(\epsilon_r(t) - \gamma(\theta^2_\tau - \theta_\tau^2))dt + \gamma e^{\gamma t}(V_t - \theta^2_\tau)dt
\]

(27)

\[
+ e^{\gamma t}[\gamma(\theta^2_\tau - V_t) + \epsilon_r(t)] dt + e^{\gamma t}\sqrt{V_t}dW^Q_t
\]

(28)

\[
e^{\gamma t}\sqrt{V_t}dW^Q_t.
\]

(29)

The fact that \( V_t = (x_t + (V_0 - \theta^2_\tau)e^{(\gamma - \gamma^\tau)t})e^{-\gamma t} + \theta^2_\tau \) by definition of \( x_t \) (21) completes the proof.

Because \( dx_t = f(t,x_t)dW^Q_t \), the process \( (x_t)_{t \geq 0} \) is a continuous local martingale, and even a true martingale since \( E^Q(\int_0^T f^2(s,x_s)ds) = \int_0^T E^Q(f^2(s,x_s))ds < \infty \) (again, the interchange between expectation and integral follows from Tonelli’s theorem). We can use the change of time method introduced in Swishchuk [22] and we get \( x_t = \tilde{W}_{\phi_t} \), where \( \tilde{W}_t \) is a \( \mathcal{F}_{\phi_t} \)- adapted Q–Brownian motion, which is based on the fact that every continuous local martingale can be represented as a time-changed brownian motion. The process \( (\phi_t)_{t \geq 0} \) is a.e. increasing, non negative, \( \mathcal{F}_t \)- adapted and is called the change of time process. This process is also equal to the quadratic variation \( \langle x \rangle_t \) of the (square-integrable) continuous martingale \( x_t \) (see [14], section 3.2, Proposition 2.10.).

Expressions of \( \phi_t \), \( \phi_t^{-1} \) and \( \tilde{W}_t \) are given by:

\[
\phi_t = \langle x \rangle_t = \int_0^t f^2(s,x_s)ds,
\]

(30)

\[
\tilde{W}_t = \int_0^{\phi_t^{-1}} f(s,x_s)dW^Q_s,
\]

(31)

\[
\phi_t^{-1} = \int_0^t \frac{1}{f^2(\phi_s^{-1},x_{\phi_s^{-1}})}ds.
\]

(32)

To see that \( \phi_t^{-1} \) has the following form, observe that:

\[
\phi_t^{-1} = \int_0^{\phi_t} \frac{1}{f^2(\phi_s^{-1},x_{\phi_s^{-1}})}ds.
\]

(33)

Now make the change of variable \( s = \phi_u \), so that \( ds = d\phi_u = f^2(u,x_u)du \).

We get:

\[
\phi_u^{-1} = \int_0^t \frac{f^2(u,x_u)}{f^2(\phi_u^{-1},x_{\phi_u^{-1}})}du = \int_0^t \frac{f^2(u,x_u)}{f^2(u,x_u)}du = t.
\]

(34)
This immediately yields:

\[ V_t = \theta^2 \tau + (V_0 - \theta^2) e^{-\gamma t} + e^{-\gamma t} \tilde{W}_t. \]  

(35)

**Lemma 1:** For \( s, t \geq 0 \) we have:

\[ \mathbb{E}_t^Q(\tilde{W}_{\phi_s}) = \tilde{W}_{\phi_t}, \]  

(36)

and for \( s, u \geq t \):

\[
\mathbb{E}_t^Q(\tilde{W}_{\phi_s} \tilde{W}_{\phi_u}) = x_t^2 + \delta^2 \left[ \theta^2 \left( \frac{e^{2\gamma (s \wedge u)} - e^{2\gamma t}}{2\gamma} \right) + (V_0 - \theta^2) \left( \frac{e^{2(\gamma - \gamma \tau) (s \wedge u)} - e^{2(\gamma - \gamma \tau) t}}{2\gamma - \gamma \tau} \right) + x_t \left( \frac{e^{\gamma (s \wedge u)} - e^{\gamma t}}{\gamma} \right) \right].
\]  

(37)

**Proof:** (36) comes from the fact that \( x_t = \tilde{W}_{\phi_t} \) is a martingale. Let \( s \geq u \geq t \). Then by iterated conditioning:

\[
\mathbb{E}_t^Q(\tilde{W}_{\phi_s} \tilde{W}_{\phi_u}) = \mathbb{E}_t^Q(\tilde{W}_{\phi_s}) = \mathbb{E}_t^Q(\tilde{W}_{\phi_u}) = \mathbb{E}_t^Q(\tilde{W}_{\phi_s} \tilde{W}_{\phi_u}) = \mathbb{E}_t^Q(\tilde{W}_{\phi_u} \tilde{W}_{\phi_s}),
\]

because \( x_t = \tilde{W}_{\phi_t} \) is a martingale. Now, by definition of the quadratic variation, \( x_u^2 - \langle x \rangle_u \) is a martingale and therefore \( \mathbb{E}_t^Q(\tilde{W}_{\phi_u}^2) = x_t^2 - \langle x \rangle_t + \mathbb{E}_t^Q(\langle x \rangle_u) = x_t^2 - \phi_t + \mathbb{E}_t^Q(\phi_u) = x_t^2 - \phi_t + \mathbb{E}_t^Q(\int_t^u f^2(s,x_s) ds). \)

We can again interchange expectation and integral by Tonelli’s theorem. By definition of \( f^2(s,x_s) \) (the latter is a linear function of \( x_s \)) and since \( x_t \) martingale, then we have (for \( s \geq t \)) \( \mathbb{E}_t^Q(\int_t^u f^2(s,x_s) ds) = f^2(s,x_t) \), and therefore \( \mathbb{E}_t^Q(\tilde{W}_{\phi_s} \tilde{W}_{\phi_u}) = x_t^2 + \int_t^u f^2(s,x_s) ds. \) We use the fact that, by definition of \( f \) in (23):

\[ f^2(s,x_t) = \delta^2 e^{2\gamma s}[x_t + (V_0 - \theta^2) e^{(\gamma - \gamma \tau)s} e^{-\gamma s} + \theta^2], \]  

(38)

to integrate the latter expression with respect to \( s \) to complete the proof.

The following theorem gives the expression of the Brockhaus&Long approximation of the volatility swap floating leg price process \( Y_t(T) \).

**Theorem 2:** The Brockhaus&Long approximation of the price process \( Y_t(T) \) of the floating leg of the volatility swap of maturity \( T \) in the delayed Heston model (5)-(8) is given by:

\[ Y_t(T) \approx \sqrt{X_t(T)} - \frac{\text{Var}_t^Q(V_R)}{8X_t(T)^2}. \]  

(39)
We can interchange expectation and integral in the latter expression by Tonelli’s following way: by definition of \( \tilde{\varphi} \)

\[
\text{Var}_t^Q(V_R) = \frac{x_t \delta^2}{\gamma^2 T^2} \left[ e^{-\gamma t} \left( 1 - e^{-2\gamma(T-t)} \right) - 2(T-t)\gamma e^{-\gamma T} \right] + \frac{\delta^2}{2\gamma^2 T^2} \left[ 2\theta_\tau^2 \gamma(T-t) + 2(V_0 - \theta_\tau^2) \frac{\gamma}{\gamma T} e^{-\gamma T} + 4\theta_\tau^2 e^{-\gamma(T-t)} - \theta_\tau^2 e^{-2\gamma(T-t)} - 3\theta_\tau^2 \right]
- \frac{\delta^2(V_0 - \theta_\tau^2)}{\gamma^2 T^2(\gamma^2 + 2\gamma^2 - 3\gamma \gamma_t)} \left[ 2(\gamma_t - 2\gamma) e^{-\gamma(T-t)} - \gamma T \right] + \gamma_t (\gamma - \gamma_t) e^{-2\gamma(T-t)} - \gamma_t T .
\]

(40)

**Proof:** The (conditioned) Brockhaus\&Long approximation gives us:

\[
Y_t(T) = \mathbb{E}_t^Q(\sqrt{V_R}) \approx \sqrt{\mathbb{E}_t^Q(V_R) - \text{Var}_t^Q(V_R)} = \sqrt{X_t(T)} - \frac{\text{Var}_t^Q(V_R)}{8X_t(T)^2}.
\]

Furthermore:

\[
\text{Var}_t^Q(V_R) = \mathbb{E}_t^Q((V_R - \mathbb{E}_t^Q(V_R))^2)
= \frac{1}{T^2} \mathbb{E}_t^Q \left( \left( \int_0^T (V_s - \mathbb{E}_t^Q(V_s)) ds \right)^2 \right).
\]

(41)

From (35) we have \( V_t = \theta_\tau^2 + (V_0 - \theta_\tau^2) e^{-\gamma T_t} + e^{-\gamma T} \tilde{W}_{\phi_t} \), and since \( \tilde{W}_{\phi} \) is a martingale, \( V_s - \mathbb{E}_t^Q(V_s) = 0 \) if \( s \leq t \), and \( V_s - \mathbb{E}_t^Q(V_s) = e^{-\gamma s}(\tilde{W}_{\phi_s} - x_t) \) if \( s > t \).

Therefore:

\[
\text{Var}_t^Q(V_R) = \frac{1}{T^2} \mathbb{E}_t^Q \left( \left( \int_t^T e^{-\gamma s}(\tilde{W}_{\phi_s} - x_t) ds \right)^2 \right)
= \frac{1}{T^2} x_t^2 \left( \int_t^T e^{-\gamma s} ds \right)^2 + \frac{1}{T^2} \mathbb{E}_t^Q \left( \left( \int_t^T e^{-\gamma s} \tilde{W}_{\phi_s} ds \right)^2 \right)
- \frac{2}{T^2} x_t \left( \int_t^T e^{-\gamma s} \mathbb{E}_t^Q(\tilde{W}_{\phi_s}) ds \right) \left( \int_t^T e^{-\gamma s} ds \right).
\]

(42)

The interchange of expectation and integral in the last equation is justified the following way: by definition of \( \tilde{W}_{\phi_s} = x_s \) in (21), we get:

\[
\mathbb{E}_t^Q(\int_t^T e^{-\gamma s} \tilde{W}_{\phi_s} ds) = \mathbb{E}_t^Q(\int_t^T (V_0 - \theta_\tau^2) e^{-\gamma s} + V_s - \theta_\tau^2 ds)
= \int_t^T - (V_0 - \theta_\tau^2) e^{-\gamma s} - \theta_\tau^2 ds + \mathbb{E}_t^Q(\int_t^T V_s ds).
\]

(43)

We can interchange expectation and integral in the latter expression by Tonelli’s
theorem, which gives:

\[ E_t^Q \left( \int_t^T e^{-\gamma s} \tilde{W}_\phi ds \right) = \int_t^T (V_0 - \theta_\tau^2)e^{-\gamma s} - \theta_t^2 ds + \int_t^T E_t^Q (V_s) ds \] (45)

\[ = \int_t^T e^{-\gamma s} E_t^Q (\tilde{W}_\phi_s) ds. \] (46)

Now we continue our computation to get:

\[ Var_t^Q (V_R) = \frac{1}{T^2} \left( \int_t^T e^{-\gamma s} ds \right)^2 + \frac{1}{T^2} E_t^Q \left( \left( \int_t^T e^{-\gamma s} \tilde{W}_\phi ds \right)^2 \right) \]

\[ = \frac{1}{T^2} \int_t^T \int_t^T e^{-\gamma(s+u)} E_t^Q (\tilde{W}_\phi_s \tilde{W}_\phi_u) ds du - \frac{1}{T^2} x_t^2 e^{-2\gamma t} \left( \frac{1 - e^{-\gamma(T-t)}}{\gamma} \right)^2. \] (47)

The interchange expectation-integral:

\[ E_t^Q \left( \int_t^T \int_t^T e^{-\gamma(s+u)} \tilde{W}_\phi_s \tilde{W}_\phi_u ds du \right) = \int_t^T \int_t^T e^{-\gamma(s+u)} E_t^Q (\tilde{W}_\phi_s \tilde{W}_\phi_u) ds du \] (48)

is justified the same way as above, using the definition of \( \tilde{W}_\phi_t = x_t \) in (21) together with Tonelli’s theorem. Finally, we use equation (37) of Lemma 1 and integrate the expression with respect to \( s \) and \( u \) to complete the proof.

**Corollary 2:** The Brockhaus&Long approximation of the volatility swap price \( K_{\text{vol}} \) of maturity \( T \) at initiation of the contract \( t = 0 \) in the delayed Heston model (5)-(8) is given by:

\[ K_{\text{vol}} \approx \sqrt{K_{\text{var}}} - \frac{Var_0^Q (V_R)}{8K_{\text{var}}^\frac{3}{2}}, \] (49)

where \( K_{\text{var}} \) is given by formula (20) of Corollary 1 and:

\[ Var_0^Q (V_R) = \frac{\delta^2 e^{-2\gamma T}}{2T^2 \gamma^3} \left[ \theta_\tau^2 \left( 2\gamma T e^{2\gamma T} + 4e^{\gamma T} - 3e^{2\gamma T} - 1 \right) + \frac{\gamma}{2\gamma - \gamma_\tau} (V_0 - \theta_t^2) \right. \]

\[ \left. \left( 2e^{2\gamma T} \left( 2\gamma \frac{\gamma}{\gamma_\tau} - 1 \right) - 4\gamma e^{\gamma T} \left( \frac{e^{(\gamma-\gamma_\tau)T}}{\gamma - \gamma_\tau} - 1 \right) + 4e^{\gamma T} \left( \frac{\gamma - \gamma_\tau \gamma}{\gamma_\tau} e^{(\gamma-\gamma_\tau)T} - 2 \right) \right) \right]. \] (50)

We notice that letting \( \tau \to 0 \) (and therefore \( \gamma_\tau \to \gamma \)) we get the formula of Swishchuk [22].

**Proof:** We have by definition \( K_{\text{vol}} = Y_0(T) \), therefore the result is obtained from equation (40) of Theorem 2.
5 Volatility Swap Hedging

In this section, we consider dynamic hedging of volatility swaps using variance swaps, as the latter are a fairly liquid, easy to trade derivatives. In the spirit of Broadie and Jain [6], we consider a portfolio containing at time \( t \) one unit of volatility swap and \( \beta_t \) units of variance swaps, both of maturity \( T \). Therefore the value \( \Pi_t \) of the portfolio at time \( t \) is:

\[
\Pi_t = e^{-r(T-t)} [Y_t(T) - K_{vol} + \beta_t(X_t(T) - K_{var})].
\]  

(51)

The portfolio is self-financing, therefore:

\[
d\Pi_t = r\Pi_t dt + e^{-r(T-t)} [dY_t(T) + \beta_t dX_t(T)].
\]  

(52)

The price processes \( X_t(T) \) and \( Y_t(T) \) can be expressed, denoting \( I_t := \int_t^T V_s ds \) the accumulated variance at time \( t \) (known at this time):

\[
X_t(T) = \mathbb{E}_t^Q \left[ \frac{1}{T} I_t + \frac{1}{T} \int_t^T V_s ds \right] = g(t, I_t, V_t),
\]  

(53)

\[
Y_t(T) = \mathbb{E}_t^Q \left[ \sqrt{\frac{1}{T} I_t + \frac{1}{T} \int_t^T V_s ds} \right] = h(t, I_t, V_t).
\]  

(54)

Remembering that \( \tilde{\theta}^2 = \theta^2 + (V_0 - \theta^2) e^{-\gamma T} \) and noticing that \( dI_t = V_t dt \), by Ito’s lemma we get:

\[
dX_t(T) = \left[ \frac{\partial g}{\partial t} + \frac{\partial g}{\partial I_t} V_t + \frac{\partial g}{\partial V_t} \gamma (\tilde{\theta}^2 - V_t) + \frac{1}{2} \frac{\partial^2 g}{\partial V_t^2} \delta^2 V_t \right] dt + \frac{\partial g}{\partial V_t} \delta \sqrt{V_t} dW_t^Q,
\]  

(55)

\[
dY_t(T) = \left[ \frac{\partial h}{\partial t} + \frac{\partial h}{\partial I_t} V_t + \frac{\partial h}{\partial V_t} \gamma (\tilde{\theta}^2 - V_t) + \frac{1}{2} \frac{\partial^2 h}{\partial V_t^2} \delta^2 V_t \right] dt + \frac{\partial h}{\partial V_t} \delta \sqrt{V_t} dW_t^Q.
\]  

(56)

As conditional expectations of cashflows at maturity of the contract, the price processes \( X_t(T) \) and \( Y_t(T) \) are by construction martingales, and therefore we should have:

\[
\frac{\partial g}{\partial t} + \frac{\partial g}{\partial I_t} V_t + \frac{\partial g}{\partial V_t} \gamma (\tilde{\theta}^2 - V_t) + \frac{1}{2} \frac{\partial^2 g}{\partial V_t^2} \delta^2 V_t = 0,
\]  

(57)

\[
\frac{\partial h}{\partial t} + \frac{\partial h}{\partial I_t} V_t + \frac{\partial h}{\partial V_t} \gamma (\tilde{\theta}^2 - V_t) + \frac{1}{2} \frac{\partial^2 h}{\partial V_t^2} \delta^2 V_t = 0.
\]  

(58)

The second equation, combined with some appropriate boundary conditions, was used in Broadie and Jain [6] to compute the value of the price process \( Y_t(T) \), whereas we focus on its Brockhaus&Long approximation.
Therefore we get:

\[ dX_t(T) = \frac{\partial g}{\partial V_t} \delta \sqrt{V_t} dW_t^Q, \]  
(59)

\[ dY_t(T) = \frac{\partial h}{\partial V_t} \delta \sqrt{V_t} dW_t^Q. \]  
(60)

and so:

\[ d\Pi_t = r \Pi_t dt + e^{-r(T-t)} \left[ \frac{\partial h}{\partial V_t} \delta \sqrt{V_t} dW_t^Q + \beta_t \frac{\partial g}{\partial V_t} \delta \sqrt{V_t} dW_t^Q \right]. \]  
(61)

In order to dynamically hedge a volatility swap of maturity \( T \), one should therefore hold \( \beta_t \) units of variance swap of maturity \( T \), with:

\[ \beta_t = -\frac{\partial h}{\partial V_t} = -\frac{\partial Y_t(T)}{\partial V_t}. \]  
(62)

Remembering that \( Var^Q_0(V_R), K_{\text{var}} \) are given respectively in Corollary 2 and 1, the initial hedge ratio \( \beta_0 \) is given by:

\[ \beta_0 = -\frac{\partial Y_0(T)}{\partial X_0(T)} \frac{\partial X_0(T)}{\partial V_0}, \]  
(63)

\[ \frac{\partial X_0(T)}{\partial V_0} = \frac{1 - e^{-\gamma T}}{\gamma T}, \]  
(64)

\[ \frac{\partial Y_0(T)}{\partial V_0} \approx \frac{\partial X_0(T)}{\partial V_0} - \frac{K_{\text{var}} \frac{\partial Var^Q_0(V_R)}{\partial V_0} - \frac{3}{2} \frac{\partial X_0(T)}{\partial V_0} Var^Q_0(V_R)}{2 \sqrt{2 \times 5^2}}. \]  
(65)

\[ \frac{\partial Var^Q_0(V_R)}{\partial V_0} = \frac{\delta^2 e^{-2 \gamma T}}{T^2 \gamma^3} \left[ \frac{\gamma}{2 \gamma - \gamma_T} e^{2 \gamma T} \left( \frac{2}{\gamma_T} - 1 \right) + 2 \gamma T e^{(\gamma-\gamma_T)T} - 1 \right]. \]  
(66)

Remembering that \( Var^Q_t(V_R), X_t(T) \) are given respectively in Theorems 2 and 1, the hedge ratio \( \beta_t \) for \( t > 0 \) is given by:

\[ \beta_t = -\frac{\partial Y_t(T)}{\partial X_t(T)} \frac{\partial X_t(T)}{\partial V_t}, \]  
(67)

\[ \frac{\partial X_t(T)}{\partial V_t} = \frac{1 - e^{-\gamma(T-t)}}{\gamma T}, \]  
(68)

\[ \frac{\partial Y_t(T)}{\partial V_t} \approx \frac{\partial X_t(T)}{\partial V_t} - \frac{X_t(T) \frac{\partial Var^Q_t(V_R)}{\partial V_t} - \frac{3}{2} \frac{\partial X_t(T)}{\partial V_t} Var^Q_t(V_R)}{8 X_t(T)^{\frac{3}{2}}}. \]  
(69)

\[ \frac{\partial Var^Q_t(V_R)}{\partial V_t} = \frac{\delta^2}{\gamma^3 T^2} \left[ 1 - e^{-2 \gamma(T-t)} - 2(T-t) e^{-\gamma(T-t)} \right]. \]  
(70)
We take the parameters that have been calibrated in section 3 on September 30th 2011 and we plot the naive Volatility Swap strike $\sqrt{K_{\text{var}}}$ together with the adjusted Volatility Swap strike $\sqrt{K_{\text{var}}} - \frac{\text{Var}^Q(V_R)}{8K_{\text{var}}^2}$ along the maturity dimension, as well as the convexity adjustment $\frac{\text{Var}^Q(V_R)}{8K_{\text{var}}^2}$:

![Figure 1: Naive Volatility Swap Strike Vs. Adjusted Volatility Swap Strike](image1)

![Figure 2: Convexity Adjustment](image2)

The naive Volatility Swap Strike represents the initial fair value of the volatility swap contract obtained without taking into account the convexity adjustment $\frac{\text{Var}^Q(V_R)}{8K_{\text{var}}^2}$ linked to the Brockhaus&Long approximation, whereas the adjusted Volatility Swap Strike represents this initial fair value when we do take into account the convexity adjustment. The difference between the former and the latter is quantified by the convexity adjustment and is represented on the second graphic. We see that neglecting the convexity adjustment leads to an overpricing of the volatility swap. On this example, the overpricing is especially significant for maturities less than 2Y, with a peak difference of more than 2% between the naive and adjusted strikes for maturities around 6M. The position of this local extremum (here, around 6M) is linked to the values of the calibrated parameters and therefore varies depending on the
date we perform the calibration at.

We also plot the initial hedge ratio $\beta_0$ along the maturity dimension:

![Figure 3: Initial Hedge Ratio](image)

This initial hedge ratio $\beta_0$ represents the quantity of variance swap contracts we need to buy (if $\beta_0 > 0$) or sell (if $\beta_0 < 0$) to hedge our position on one volatility swap contract of the same maturity. Of course, in order to cancel the risk, $\beta_0$ has to be negative if we buy a volatility swap contract, and positive if we sell one. Here we have assumed that we hold a long position on a volatility swap contract, i.e. that we have bought one such contract. The plot tells us that for one volatility swap contract bought, we need to sell approximately 3 variance swap contracts of the same maturity (depending of the maturity of the contract) to hedge our position on the volatility swap, i.e. to cancel the risk inherent to our position. We say that we hold a short position on the variance swap contracts. The trend is that the higher the maturity of the volatility swap contract, the more variance swap contracts we need to sell in order to hedge our position. This was to be expected because for such pure volatility contracts, the longer the maturity, the higher the probability that the volatility varies significantly, i.e. the higher the risk.

6 Conclusions

In this Chapter, we introduced a variance drift adjusted version of the Heston model based on the concept of delay, the Delayed Heston model (section 2). As explained in the introduction, this model makes a bridge between the popular Heston model and the delayed stochastic volatility model considered by Swishchuk in [23]. Our model has two additional parameters compared to the Heston model and since it can be seen as a time-dependent Heston model with time-dependent long-range variance $\tilde{\theta}_t^2$, it can be implemented very easily, for both Monte Carlo simulation and pricing of call options via
the semi-closed formulas which can be derived (see Appendix A). We cali-
bred our model on 12 dates ranging from Sep. 19th to Oct. 17th 2011 for
the FOREX underlying EURUSD (section 3). Our findings were twofold: the
Delayed Heston model always outperformed significantly the Heston model
in terms of average (absolute) calibration error (especially for long maturi-
ties and ATM options), but also in terms of the standard deviation of the
calibration errors. The latter is highly desirable in practice as we do not
want to face the case of very poorly priced options on one hand, and almost
perfectly priced options on the other: it is better that each individual cali-
bration error corresponding to each call option is close to the average calibration
error, and therefore that the standard deviation of the calibration errors is
low. In sections 4 and 5, we considered respectively the pricing of variance
and volatility swaps, and the dynamic hedging of a position on a volatility
swap by a position on variance swaps, the latter being very liquid financial
derivatives. We obtained a closed formula for both the price process of the
variance swap and the Brockhaus&Long approximation (which is a 2nd order
approximation) of the price process of the volatility swap. Finally, to illus-
trate these last sections, we displayed 3 graphics showing the importance of
taking into account the convexity adjustment (corresponding to the Brock-
haus&Long approximation) when pricing a volatility swap, and that taking
naively the volatility swap strike \(K_{vol}\) to be \(\sqrt{K_{var}}\) may lead to a significant
mispricing of the volatility swap.

A Semi-closed formulas for call options in the
Delayed Heston Model

From Kahl&Jäckel [13], we get equations (71) to (74) for the price of a
call option with maturity \(T\) and strike \(K\) in the time-dependent long-range
variance Heston model:

\[
C_0 = e^{-rT} \left[ \frac{1}{2} (F - K) + \frac{1}{\pi} \int_0^\infty (Fh_1(u) - Kh_2(u)) du \right], \tag{71}
\]

\[
h_1(u) = \Re \left( \frac{e^{-iu \ln(K)} \varphi(u - i)}{iuF} \right), \tag{72}
\]

\[
h_2(u) = \Re \left( \frac{e^{-iu \ln(K)} \varphi(u)}{iu} \right), \tag{73}
\]

with \(F = S_0 e^{(r-q)T}\) and:

\[
\varphi(u) = e^{C(T,u)+V_0D(T,u)+iu \ln(F)}. \tag{74}
\]
By Michailov & Nöegel [20], we have that $C(t, u)$ and $D(t, u)$ solve the following differential equations:

$$\frac{dC(t, u)}{dt} = \gamma \bar{\theta}^2 t D(t, u),$$  \hspace{1cm} (75)

$$\frac{dD(t, u)}{dt} - \frac{\delta^2}{2} D^2(t, u) + (\gamma - iu\rho \delta)D(t, u) + \frac{1}{2}(u^2 + iu) = 0,$$

$$C(0, u) = D(0, u) = 0.$$ \hspace{1cm} (77)

The Riccati equation for $D(t, u)$ doesn’t depend on $\bar{\theta}^2 t$, therefore its solution is just the solution of the classical Heston model given in Kahl & Jäckel [13]:

$$D(t, u) = \frac{\gamma - i\rho \delta u + d}{\gamma - i\rho \delta u - d} \left[ \frac{1 - e^{dt}}{1 - ge^{dt}} \right],$$ \hspace{1cm} (78)

$$g = \frac{\gamma - i\rho \delta u + d}{\gamma - i\rho \delta u - d},$$ \hspace{1cm} (79)

$$d = \sqrt{(\gamma - i\rho \delta u)^2 + \delta^2(iu + u^2)}.$$ \hspace{1cm} (80)

Given $D(t, u)$ and the definition of $\bar{\theta}^2 t$, we can compute $C(t, u)$ from (75) and (77):

$$C(t, u) = \gamma \theta^2 f(t, u) + (V_0 - \theta^2 \tau) (\gamma - \gamma_\tau) \int_0^t e^{-\gamma s} D(s, u) ds.$$ \hspace{1cm} (81)

Where $f(t, u) = \int_0^t D(s, u) ds$ is given in Kahl & Jäckel [13]:

$$f(t, u) = \frac{1}{\delta^2} \left( (\gamma - i\rho \delta u + d)t - 2 \ln \left( \frac{1 - ge^{dt}}{1 - g} \right) \right).$$ \hspace{1cm} (82)

Unfortunately, the integral $\int_0^t e^{-\gamma s} D(s, u) ds$ in (81) cannot be computed directly as $\int_0^t D(s, u) ds$. The logarithm in $f(t, u)$ can be handled as suggested in Kahl & Jäckel [13], as well as the integration of the Heston integral, namely:

$$C_0 = e^{-rT} \int_0^1 y(x) dx,$$

$$y(x) = \frac{1}{2}(F - K) + \frac{F h_1(\frac{\ln(x)}{C_\infty}) - K h_2(\frac{\ln(x)}{C_\infty})}{x \pi C_\infty},$$ \hspace{1cm} (84)
where \( C_\infty > 0 \) is an integration constant.

The following limit conditions are given in Kahl\&Jäckel [13]:

\[
\lim_{x \to 0} y(x) = \frac{1}{2}(F - K), \tag{85}
\]

\[
\lim_{x \to 1} y(x) = \frac{1}{2}(F - K) + \frac{FH_1 - KH_2}{\pi C_\infty}, \tag{86}
\]

\[
H_j = \lim_{u \to 0} h_j(u) = \ln \left( \frac{F}{K} \right) + \tilde{c}_j(T) + V_0 \tilde{d}_j(T), \tag{87}
\]

where:

\[
\tilde{d}_1(t) = \Im \left( \frac{\partial D}{\partial u}(t, -i) \right), \tag{88}
\]

\[
\tilde{c}_1(t) = \Im \left( \frac{\partial C}{\partial u}(t, -i) \right), \tag{89}
\]

\[
\tilde{d}_2(t) = \Im \left( \frac{\partial D}{\partial u}(t, 0) \right), \tag{90}
\]

\[
\tilde{c}_2(t) = \Im \left( \frac{\partial C}{\partial u}(t, 0) \right). \tag{91}
\]

Expressions for \( \tilde{d}_1(t) \) and \( \tilde{d}_2(t) \) are the same as in Kahl\&Jäckel [13] as \( \tilde{\theta}_t^2 \) doesn’t play any role in them. Given (75) and (77), we compute \( \tilde{c}_1(T) \) and \( \tilde{c}_2(T) \) in our time-dependent long-range variance Heston model by:

\[
\tilde{c}_j(T) = \gamma \int_0^T \tilde{\theta}_t^2 \tilde{d}_j(t) dt. \tag{92}
\]

After computing the integrals we get:

If \( \gamma - \rho \delta \neq 0 \) and \( \gamma - \rho \delta + \gamma_\tau \neq 0 \):

\[
\tilde{d}_1(T) = \frac{1 - e^{-(\gamma - \rho \delta)T}}{2(\gamma - \rho \delta)}, \tag{93}
\]

\[
\tilde{c}_1(T) = \gamma \theta^2 \frac{e^{-(\gamma - \rho \delta)T} - 1 + (\gamma - \rho \delta)T}{2(\gamma - \rho \delta)^2} \tag{94}
\]

\[
+ \frac{(V_0 - \theta^2)(\gamma - \gamma_\tau)}{2(\gamma - \rho \delta)} \left( -\frac{e^{-\gamma_\tau T} - 1}{\gamma_\tau} + \frac{e^{-(\gamma - \rho \delta + \gamma_\tau)T} - 1}{\gamma - \rho \delta + \gamma_\tau} \right). \tag{95}
\]
If $\gamma - \rho \delta \neq 0$ and $\gamma - \rho \delta + \gamma_{\tau} = 0$:

$$
\tilde{d}_1(T) = \frac{1 - e^{-(\gamma - \rho \delta)T}}{2(\gamma - \rho \delta)},
$$
(96)

$$
\tilde{c}_1(T) = \gamma \theta_{\tau}^2 \frac{e^{-(\gamma - \rho \delta)T} - 1 + (\gamma - \rho \delta)T}{2(\gamma - \rho \delta)^2} + \frac{(V_0 - \theta_{\tau}^2)(\gamma - \gamma_{\tau})}{2(\gamma - \rho \delta)} \left( -\frac{e^{-\gamma_{\tau}T} - 1}{\gamma_{\tau}} - T \right).
$$
(97)

If $\gamma - \rho \delta = 0$:

$$
\tilde{d}_1(T) = \frac{T}{2},
$$
(99)

$$
\tilde{c}_1(T) = \gamma \theta_{\tau}^2 \frac{T^2}{4} + \frac{(V_0 - \theta_{\tau}^2)(\gamma - \gamma_{\tau})}{2} \left( -\frac{T e^{-\gamma_{\tau}T}}{\gamma_{\tau}} + 1 - e^{-\gamma_{\tau}T} \right),
$$
(100)

and:

$$
\tilde{d}_2(T) = \frac{e^{-\gamma T} - 1}{2\gamma},
$$
(101)

$$
\tilde{c}_2(T) = \gamma \theta_{\tau}^2 \frac{1 - e^{-\gamma T} - \gamma T}{2\gamma^2} + \frac{(V_0 - \theta_{\tau}^2)(\gamma - \gamma_{\tau})}{2\gamma} \left( -\frac{1 - e^{-\gamma_{\tau}T}}{\gamma_{\tau}} - \frac{e^{(-\gamma_{\tau} - \gamma)T} - 1}{\gamma_{\tau} + \gamma} \right).
$$
(103)

### References


Chapter 6: CTM and Explicit Option Pricing

Formula for a Mean-reverting Asset in Energy Markets

"Equations are more important to me, because politics is for the present, but an equation is something for eternity," - Albert Einstein.

1 Introduction

Some commodity prices, like oil and gas, exhibit the mean reversion, unlike stock price. It means that they tend over time to return to some long-term mean. In this paper we consider a risky asset $S_t$ following the mean-reverting stochastic process given by the following stochastic differential equation

$$dS_t = a(L - S_t)dt + \sigma S_t dW_t,$$

where $W$ is a standard Wiener process, $\sigma > 0$ is the volatility, the constant $L$ is called the 'long-term mean' of the process, to which it reverts over time, and $a > 0$ measures the 'strength' of mean reversion.

This mean-reverting model is a one-factor version of the two-factor model made popular in the context of energy modelling by Piliopovic (1997). Black’s model (1976) and Schwartz’s model (1997) have become a standard approach to the problem of pricing options on commodities. These models have the advantage of mathematical convenience, in that they give rise to closed-form solutions for some types of options (See Wilmott (2000)).

Bos, Ware and Pavlov (2002) presented a method for evaluation of the price of a European option based on $S_t$, using a semi-spectral method. They did not have the convenience of a closed-form solution, however, they showed that values for certain types of options may nevertheless be found extremely efficiently. They used the following partial differential equation (see, for example, Wilmott, Howison and Dewynne (1995))

$$C_t' + R(S,t)C_s'' + \sigma^2 S^2 C_{ss}''/2 = rC$$

for option prices $C(S,t)$, where $R(S,t)$ depends only on $S$ and $t$, and corresponds to the drift induced by the risk-neutral measure, and $r$ is the risk-free interest rate. Simplifying this equation to the singular diffusion equation they were able to calculate numerically the solution.

The aim of this paper is to obtain an explicit expression for a European option price, $C(S,t)$, based on $S_t$, using a change of time method (see Swishchuk (2007)). This method was once applied by the author to price variance, volatility, covariance and correlation swaps for the Heston model (see Swishchuk (2004)).
2 Mean-Reverting Asset Model (MRAM)

Let \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) be a probability space with a sample space \(\Omega\), \(\sigma\)-algebra of Borel sets \(\mathcal{F}\) and probability \(P\). The filtration \(\mathcal{F}_t, \quad t \in [0, T]\), is the natural filtration of a standard Brownian motion \(W_t, \quad t \in [0, T]\), such that \(\mathcal{F}_T = \mathcal{F}\).

Some commodity prices, like oil and gas, exhibit the mean reversion, unlike stock price. It means that they tend over time to return to some long-term mean. In this paper we consider a risky asset \(S_t\) following the mean-reverting stochastic process given by the following stochastic differential equation

\[
dS_t = a(L - S_t)dt + \sigma S_t dW_t, \tag{1}
\]

where \(W_t\) is an \(\mathcal{F}_t\)-measurable one-dimensional standard Wiener process, \(\sigma > 0\) is the volatility, constant \(L\) is called the 'long-term mean' of the process, to which it reverts over time, and \(a > 0\) measures the 'strength' of mean reversion.

3 Explicit Option Pricing Formula for European Call Option for MRAM under Physical Measure

In this section, we are going to obtain an explicit expression for a European option price, \(C(S, t)\), based on \(S_t\), using a change of time method and physical measure.

3.1 Explicit Solution of MRAM.

Let

\[
V_t := e^{at}(S_t - L). \tag{2}
\]

Then, from (2) and (1) we obtain

\[
dV_t = ae^{at}(S_t - L)dt + e^{at}dS_t = \sigma(V_t + e^{at}L)dW_t. \tag{3}
\]

Using change of time approach to the equation (3) (see Ikeda and Watanabe (1981) or Elliott (1982)) we obtain the following solution of the equation (3)

\[
V_t = S_0 - L + \tilde{W}(\phi_t^{-1}),
\]

or (see (2)),

\[
S_t = e^{-at}[S_0 - L + \tilde{W}(\phi_t^{-1})] + L, \tag{4}
\]
where $\tilde{W}(t)$ is an $\mathcal{F}_t$-measurable standard one-dimensional Wiener process, $\phi_t^{-1}$ is an inverse function to $\phi_t$:

$$\phi_t = \sigma^{-2} \int_0^t (S_0 - L + \tilde{W}(s) + e^{a\phi_s}L)^2 ds. \tag{5}$$

We note that

$$\phi_t^{-1} = \sigma^2 \int_0^t (S_0 - L + \tilde{W}^{-1}(s) + e^{as}L)^2 ds, \tag{6}$$

which follows from (5) and the following transformations:

$$d\phi_t = \sigma^{-2}(S_0 - L + \tilde{W}(t) + e^{a\phi_t}L)^2 dt \Rightarrow \sigma^2(S_0 - L + \tilde{W}(t) + e^{a\phi_t}L)^2 d\phi_t = dt \Rightarrow$$

$$t = \sigma^2 \int_0^t (S_0 - L + \tilde{W}(s) + e^{a\phi_s}L)^2 ds \Rightarrow$$

$$\phi_t^{-1} = \sigma^2 \int_0^{\phi_t^{-1}} (S_0 - L + \tilde{W}(s) + e^{a\phi_s}L)^2 ds \phi_s$$

$$= \sigma^2 \int_0^{\phi_t^{-1}} (S_0 - L + \tilde{W}(\phi_s^{-1}) + e^{as}L)^2 ds.$$

### 3.2 Some Properties of the Process $\tilde{W}(\phi_t^{-1})$

We note that process $\tilde{W}(\phi_t^{-1})$ is $\bar{\mathcal{F}}_t := \mathcal{F}_{\phi_t^{-1}}$-measurable and $\bar{\mathcal{F}}_t$-martingale.

Then

$$E\tilde{W}(\phi_t^{-1}) = 0. \tag{7}$$

Let’s calculate the second moment of $\tilde{W}(\phi_t^{-1})$ (see (6)):

$$E\tilde{W}^2(\phi_t^{-1}) = E < \tilde{W}(\phi_t^{-1}) > = E\phi_t^{-1}$$

$$= \sigma^2 \int_0^t E(S_0 - L + \tilde{W}(\phi_s^{-1}) + e^{as}L)^2 ds$$

$$= \sigma^2 \left[(S_0 - L)^2 + \frac{2L(S_0 - L)}{a} \left(e^{at} - 1\right) + \frac{L^2}{2a} \left(e^{2at} - 1\right) \right] \tag{8}$$

From (8), solving this linear ordinary nonhomogeneous differential equation with respect to $E\tilde{W}^2(\phi_t^{-1})$,

$$\frac{dE\tilde{W}^2(\phi_t^{-1})}{dt} = \sigma^2 [(S_0 - L)^2 + 2L(S_0 - L)e^{at} + L^2e^{2at} + E\tilde{W}^2(\phi_t^{-1})],$$

we obtain

$$E\tilde{W}^2(\phi_t^{-1}) = \sigma^2 \left[(S_0 - L)^2 \frac{e^{\sigma^2t} - 1}{\sigma^2} + \frac{2L(S_0 - L)}{a - \sigma^2} \left(e^{at} - e^{\sigma^2t}\right) + \frac{L^2}{2a - \sigma^2} \left(e^{2at} - e^{2\sigma^2t}\right) \right]. \tag{9}$$
3.3 Explicit Expression for the Process \( \tilde{W}(\phi_t^{-1}) \).

It is turns out that we can find the explicit expression for the process \( \tilde{W}(\phi_t^{-1}) \).

From the expression (see Section 3.1)

\[
V_t = S_0 - L + \tilde{W}(\phi_t^{-1}),
\]

we have the following relationship between \( W(t) \) and \( \tilde{W}(\phi_t^{-1}) \):

\[
d\tilde{W}(\phi_t^{-1}) = \sigma \int_0^t [S(0) - L + Le^{at} + \tilde{W}(\phi_s^{-1})]dW(t).
\]

It is a linear SDE with respect to \( \tilde{W}(\phi_t^{-1}) \) and we can solve it explicitly. The solution has the following look:

\[
\tilde{W}(\phi_t^{-1}) = S(0)(e^{\sigma W(t) - \frac{\sigma^2 t}{2}} - 1) + L(1 - e^{at}) + aLe^{\sigma W(t)} - \frac{\sigma^2 t}{2} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds.
\]

(10)

It is easy to see from (10) that \( \tilde{W}(\phi_t^{-1}) \) can be presented in the form of a linear combination of two zero-mean martingales \( m_1(t) \) and \( m_2(t) \):

\[
\tilde{W}(\phi_t^{-1}) = m_1(t) + Lm_2(t),
\]

where

\[
m_1(t) := S(0)(e^{\sigma W(t) - \frac{\sigma^2 t}{2}} - 1)
\]

and

\[
m_2(t) = (1 - e^{at}) + aLe^{\sigma W(t)} - \frac{\sigma^2 t}{2} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds.
\]

Indeed, process \( \tilde{W}(\phi_t^{-1}) \) is a martingale (see Section 3.2), also it is well-known that process \( e^{\sigma W(t) - \frac{\sigma^2 t}{2}} \) and, hence, process \( m_1(t) \) is a martingale. Then the process \( m_2(t) \), as the difference between two martingales, is also martingale. In this way, we have

\[
Em_1(t) = 0,
\]

since

\[
Ee^{\sigma W(t) - \frac{\sigma^2 t}{2}} = 1.
\]

As for \( m_2(t) \) we have

\[
Em_2(t) = 0,
\]

since from Itô’s formula we have

\[
d(ae^{\sigma W(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds) = a\sigma e^{\sigma W(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds dW(t) + ae^{\sigma W(t) - \frac{\sigma^2 t}{2}} e^{at} e^{-\sigma W(t) + \frac{\sigma^2 t}{2}} dt + a\sigma e^{\sigma W(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds dW(t) + ae^{at} dt.
\]
and, hence,

\[ Eae^{\sigma W(t)} - \frac{\sigma^2 t}{2} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{a^2 s}{2} ds = e^{at} - 1. \]

It is interesting to see that the last expression, the first moment for

\[ \eta(t) := ae^{\sigma W(t)} - \frac{\sigma^2 t}{2} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{a^2 s}{2} ds, \]

does not depend on \( \sigma \).

It is true not only for the first moment but for all the moments of the process \( \eta(t) = ae^{\sigma W(t)} - \frac{\sigma^2 t}{2} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{a^2 s}{2} ds \).

Indeed, using Itô’s formula for \( \eta^n(t) \) we obtain

\[ d\eta^n(t) = nae^{\sigma W(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{a^2 s}{2} ds W(t) + an(\eta(t))^{n-1} e^{at} dt, \]

and

\[ dE\eta^n(t) = nac^at E\eta^{n-1}(t) dt, \quad n \geq 1. \]

This is a recursive equation with initial function \( n = 1 \) \( E\eta(t) = e^{at} - 1 \).

After calculations we obtain the following formula for \( E\eta^n(t) \):

\[ E\eta^n(t) = (e^{at} - 1)^n. \]

### 3.4 Some Properties of the Mean-Reverting Asset \( S_t \)

From (4) we obtain the mean value of the first moment for mean-reverting asset \( S_t \):

\[ ES_t = e^{-at}[S_0 - L] + L. \]

It means that \( ES_t \to L \) when \( t \to +\infty \).

Using formulae (4) and (9) we can calculate the second moment of \( S_t \):

\[ ES_t^2 = (e^{-at}(S_0 - L) + L)^2 + \sigma^2 e^{-2at}[(S_0 - L)^2 \frac{e^{2at} - 1}{\sigma^2} + \frac{2L(S_0 - L)(e^{at} - e^{\sigma^2 t})}{a - \sigma^2} + \frac{L^2(e^{2at} - e^{2\sigma^2 t})}{2a - \sigma^2}]. \]

Combining the first and the second moments we have the variance of \( S_t \):

\[ Var(S_t) = ES_t^2 - (ES_t)^2 = \sigma^2 e^{-2at}[(S_0 - L)^2 \frac{e^{2at} - 1}{\sigma^2} + \frac{2L(S_0 - L)(e^{at} - e^{\sigma^2 t})}{a - \sigma^2} + \frac{L^2(e^{2at} - e^{2\sigma^2 t})}{2a - \sigma^2}]. \]

From the expression for \( W(\phi_t^{-1}) \) (see (10)) and for \( S(t) \) in (4) we can find the explicit expression for \( S(t) \) through \( W(t) \):
\[ S(t) = e^{-at}[S_0 - L + \tilde{W}(\phi_t^{-1})] + L \\
= e^{-at}[S_0 - L + m_1(t) + Lm_2(t)] + L \\
= S(0)e^{-at}e^{\sigma W(t) - \frac{\sigma^2}{2}t} + aLe^{-at}e^{\sigma W(t) - \frac{\sigma^2}{2}t} \int_0^t e^{as}e^{-\sigma W(s) + \frac{\sigma^2}{2}s} ds, \]

(11)

where \( m_1(t) \) and \( m_2(t) \) are defined as in Section 3.3.

### 3.5 Explicit Option Pricing Formula for European Call Option for MRAM under Physical Measure.

The payoff function \( f_T \) for European call option equals

\[ f_T = (S_T - K)^+ := \max(S_T - K, 0), \]

where \( S_T \) is an asset price defined in (4), \( T \) is an expiration time (maturity) and \( K \) is a strike price.

In this way (see (11)),

\[ f_T = [e^{-aT}(S_0 - L + \tilde{W}(\phi_T^{-1})) + L - K]^+ \\
= [S(0)e^{-aT}e^{\sigma W(T) - \frac{\sigma^2}{2}T} + aLe^{-aT}e^{\sigma W(T) - \frac{\sigma^2}{2}T} \int_0^T e^{as}e^{-\sigma W(s) + \frac{\sigma^2}{2}s} ds - K]^+. \]

(12)

To find the option pricing formula we need to calculate

\[ C_T = e^{-rT}F_T \\
= e^{-rT}E[e^{-aT}(S_0 - L + \tilde{W}(\phi_T^{-1})) + L - K]^+ \\
= \frac{1}{\sqrt{2\pi}}e^{-RT} \int_{-\infty}^{+\infty} \max[S(0)e^{-aT}e^{\sigma y\sqrt{T} - \frac{\sigma^2}{2}T} + aLe^{-aT}e^{\sigma y\sqrt{T} - \frac{\sigma^2}{2}T} \int_0^T e^{as}e^{-\sigma y\sqrt{s} + \frac{\sigma^2}{2}s} ds - K, 0]e^{-\frac{y^2}{2}} dy. \]

Let \( y_0 \) be a solution of the following equation:

\[ S(0) \times e^{-aT}e^{\sigma y_0\sqrt{T} - \frac{\sigma^2}{2}T} + aLe^{-aT}e^{\sigma y_0\sqrt{T} - \frac{\sigma^2}{2}T} \int_0^T e^{as}e^{-\sigma y_0\sqrt{s} + \frac{\sigma^2}{2}s} ds = K \]

(13)

or

\[ y_0 = \frac{\ln\left(\frac{K}{S(0)}\right) + \left(\frac{\sigma^2}{2} + a\right)T}{\sigma \sqrt{T}} - \frac{\ln\left(1 + \frac{aL}{S(0)} \int_0^T e^{as}e^{-\sigma y_0\sqrt{s} + \frac{\sigma^2}{2}s} ds\right)}{\sigma \sqrt{T}}. \]

(14)
From (12)-(13) we have:

\[
C_T = \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{-\infty}^{+\infty} \max[S(0)e^{-aT}e^{\sigma y T - \frac{y^2}{2}}
+ aLe^{-aT}e^{\sigma y \sqrt{T} - \frac{y^2}{2}} \int_{0}^{T} e^{-\sigma y \sqrt{s} + \frac{y^2}{2}} ds - K, 0]e^{-t^2} dy
\]

\[
= \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{-\infty}^{+\infty} [S(0)e^{-aT}e^{\sigma y \sqrt{T} - \frac{y^2}{2}}
+ aLe^{-aT}e^{\sigma y \sqrt{T} - \frac{y^2}{2}} \int_{0}^{T} e^{-\sigma y \sqrt{s} + \frac{y^2}{2}} ds - K]e^{-t^2} dy
\]

\[
= \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{y_0}^{+\infty} (a e^{\sigma y \sqrt{T} - \frac{y^2}{2}} \int_{0}^{T} e^{-\sigma y \sqrt{s} + \frac{y^2}{2}} ds) e^{-t^2} dy
\]

\[
= BS(T) + A(T),
\]

where

\[
BS(T) := \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{y_0}^{+\infty} [S(0)e^{-aT}e^{\sigma y \sqrt{T} - \frac{y^2}{2}} e^{-t^2} dy - e^{-rT} K[1 - \Phi(y_0)]
\]

\[
A(T) := Le^{-(r+a)T}
\times \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{y_0}^{+\infty} (a e^{\sigma y \sqrt{T} - \frac{y^2}{2}} \int_{0}^{T} e^{-\sigma y \sqrt{s} + \frac{y^2}{2}} ds) e^{-t^2} dy,
\]

and

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2} dt.
\]

After calculation of \(BS(T)\) we obtain

\[
BS(T) = e^{-(r+a)T} S(0) \Phi(y_+) - e^{-rT} K \Phi(y_-),
\]

where

\[
y_+ := \sigma \sqrt{T} - y_0 \quad \text{and} \quad y_- := -y_0
\]

and \(y_0\) is defined in (14).

Consider \(A(T)\) in (17).

Let \(F_T(dz)\) be a distribution function for the process

\[
\eta(T) = a e^{\sigma W(T) - \frac{a^2}{2}} \int_{0}^{T} e^{-\sigma W(s) + \frac{a^2}{2}} ds,
\]

which is a part of the integrand in (17).

As M. Yor [16, 17] mentioned there is still no closed form probability density function for time integral of an exponential Brownian motion, while the best result is a function with a double integral.

We can use Yor’s result [16] to get \(F_T(dz)\) above. Using the scaling property of Wiener process and change of variables, we can rewrite our expression for \(S(t)\) in (11) in the following way

\[
S(T) = S(0) e^{-2B_{y_0}} + \frac{4}{\sigma^2} a L e^{-2B_{y_0}} A_{y_0},
\]

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where $T_0 = \frac{\sigma^2}{4}T, v = \frac{2}{\sigma^2}a + 1, B_t = -\frac{\sigma^2}{2}W(\frac{4}{\sigma^2}t), B^v_{T_0} = vT_0 + B_{T_0}, A^v_{T_0} = \int_0^{T_0} e^{2B^v_s}ds$.

Also, the process $\eta(T)$ may be presented in the following way using these transformations

$$\eta(T) = \frac{4ae^{-aT}}{\sigma^2}e^{-\frac{2B^v_{T_0}}{4}A^{v^2}_{T}}.$$

We state here the result obtained by Yor [16] for the joint probability density function of $A^v_{T_0}$ and $B^v_{T_0}$.

**Theorem 4.3.-1.** (M. Yor [15]). *The joint probability density function of $A^v_{T_0}$ and $B^v_{T_0}$ satisfies

$$P(A^v_{T_0} \in du, B^v_{T_0} \in dx) = e^{v^2u^2/2} \exp -\frac{1+e^{2x}}{2u} - \theta(\frac{e^x}{u}, t) \frac{du}{u},$$

where $t > 0, u > 0, x \in \mathbb{R}$ and

$$\theta(r, t) = \frac{r}{(2\pi t)^{1/2}} \int_0^{+\infty} e^{-s^2/2t} \cosh(r) \sinh(s) \sin(\frac{\pi s}{t}) ds.$$*

Using this result we can write the distribution function for $\eta(T)$ in the following way

$$P(\eta(T) \leq u) = P(\frac{4ae^{-aT}}{\sigma^2}e^{-\frac{2B^v_{T_0}}{4}A^{v^2}_{T}} \leq u)$$

$$= P(\frac{e^{-2B^v_{T_0}}}{4}A^{v^2}_{T} \leq \frac{\sigma^2 e^{-aT}u}{4a})$$

$$= F_T(u).$$

In this way, $A(T)$ in (17) may be presented in the following way:

$$A(T) = Le^{-(r+a)T} \int_{y_0}^{+\infty} zF_T(dz).$$

After calculation of $A(T)$ we obtain the following expression for $A(T)$:

$$A(T) = Le^{-(r+a)T}[(e^{aT} - 1) - \int_{0}^{y_0} zF_T(dz)],$$

since $E\eta(T) = e^{aT} - 1$.

Finally, summarizing (12)-(21), we have obtained the following Theorem.

**Theorem 3.1.** *Option pricing formula for European call option for mean-reverting asset under physical measure has the following look:

$$C_T = e^{-(r+a)T}S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-)$$

$$+ Le^{-(r+a)T}[(e^{aT} - 1) - \int_{0}^{y_0} zF_T(dz)],$$

(22)

where $y_0$ is defined in (14), $y_+$ and $y_-$ in (20), $\Phi(y)$ in (18), and $F_T(dz)$ is a distribution function in (21).
Remark. From (21)-(22) we find that European Call Option Price $C_T$ for mean-reverting asset lies between the following boundaries:

$$BS(T) \leq C_T \leq BS(T) + L e^{-(r+a)T} [e^{aT} - 1],$$

or (see (19)),

$$e^{-(r+a)T} S(0) \Phi(y_+) - e^{-rT} K \Phi(y_-) \leq C_T \leq e^{-(r+a)T} S(0) \Phi(y_+) - e^{-rT} K \Phi(y_-) + L e^{-(r+a)T} [e^{aT} - 1].$$

4 Mean-Reverting Risk-Neutral Asset Model (MRRNAM)

Consider our model (1)

$$dS_t = a(L - S_t) dt + \sigma S_t dW_t. \quad (23)$$

We want to find a probability $P^*$ equivalent to $P$, under which the process $e^{-rt} S_t$ is a martingale, where $r > 0$ is a constant interest rate. The hypothesis we made on the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ allows us to express the density of the probability $P^*$ with respect to $P$. We denote this density by $L_T$.

It is well-known (see Lamperton and Lapeyre (1996), Proposition 6.1.1, p. 123), that there is an adopted process $(q(t))_{t \in [0,T]}$ such that, for all $t \in [0,T]$,

$$L_t = \exp\left[ \int_0^t q(s) dW_s - \frac{1}{2} \int_0^t q^2(s) ds \right] \text{ a.s.}$$

In this case,

$$\frac{dP^*}{dP} = \exp\left[ \int_0^T q(s) dW_s - \frac{1}{2} \int_0^T q^2(s) ds \right] = L_T.$$

In our case, with model (17), the process $q(t)$ is equal to

$$q(t) = -\lambda S_t, \quad (24)$$

where $\lambda$ is the market price of risk and $\lambda \in R$. Hence, for our model

$$L_T = \exp[ -\lambda \int_0^T S(u) du - \frac{1}{2} \lambda \int_0^T S^2(u) du].$$
Under probability $P^*$, the process $(W_t^*)$ defined by

$$W_t^* := W_t + \lambda \int_0^t S(u)du$$

is a standard Brownian motion (Girsanov theorem) (see Elliott and Kopp (1999)).

Therefore, in a risk-neutral world our model (23) takes the following look:

$$dS_t = (aL - (a + \lambda \sigma)S_t)dt + \sigma S_t dW_t^*,$$

or, equivalently,

$$dS_t = a^*(L^* - S_t)dt + \sigma S_t dW_t^*,$$  \(26\)

where

$$a^* := a + \lambda \sigma, \quad L^* := \frac{aL}{a + \lambda \sigma};$$  \(27\)

and $W_t^*$ is defined in (25).

Now, we have the same model in (26) as in (1), and we are going to apply our method of changing of time to this model (26) to obtain the explicit option pricing formula.

5 Explicit Option Pricing Formula for European Call Option for MRRNAM

In this section, we are going to obtain explicit option pricing formula for European call option under risk-neutral measure $P^*$, using the same arguments as in sections 3-7, where in place of $a$ and $L$ we are going to take $a^*$ and $L^*$

$$a \rightarrow a^* := a + \lambda \sigma, \quad L \rightarrow L^* := \frac{aL}{a + \lambda \sigma},$$

where $\lambda$ is a market price of risk (See section 3).

5.1 Explicit Solution for the Mean-Reverting Risk-Neutral Asset Model.

Applying (2)-(6) to our model (26) we obtain the following explicit solution for our risk-neutral model (26):

$$S_t = e^{-a^* t}[S_0 - L^* + \tilde{W}^*((\phi_t^*)^{-1})] + L,$$  \(28\)

where $\tilde{W}^*(t)$ is an $\mathcal{F}_t$-measurable standard one-dimensional Wiener process under measure $P^*$ and $(\phi_t^*)^{-1}$ is an inverse function to $\phi_t^*$:

$$\phi_t^* = \sigma^{-2} \int_0^t (S_0 - L^* + \tilde{W}^*(s) + e^{a^* \phi_s^* L^*})^{-2} ds.$$  \(29\)
We note that

\[(\phi^*_t)^{-1} = \sigma^2 \int_0^t (S_0 - L + \bar{W}^*((\phi^*_t)^{-1}) + e^{\sigma^*_s L^*})^2 ds, \quad (30)\]

where \(a^*\) and \(L^*\) are defined in (27).

### 5.2 Some Properties of the Process \(\tilde{W}^*((\phi^*_t)^{-1})\).

Using the same argument as in Section 4, we obtain the following properties of the process \(\tilde{W}^*((\phi^*_t)^{-1})\) in (25). This is a zero-mean \(P^*\)-martingale and

\[
E^{*}[\tilde{W}^*((\phi^*_t)^{-1})] = 0,
\]

\[
E^{*}[(\tilde{W}^*((\phi^*_t)^{-1}))^2] = \sigma^2 [(S_0 - L)^2 e^{\sigma^*_2 t} + 2L^*(S_0 - L^*)(e^{\sigma^*_2 t} - e^{\sigma^*_2 t})]
\]

\[
+ \frac{(L^*)^2(e^{2\sigma^*_2 t} - e^{\sigma^*_2 t})}{2a^* - \sigma^*}, \quad (31)
\]

where \(E^*\) is the expectation with respect to the probability \(P^*\) and \(a^*, L^*\) and \((\phi^*_t)^{-1}\) are defined in (27) and (30), respectively.

### 5.3 Explicit Expression for the Process \(\tilde{W}^*(\phi_t^{-1})\).

It turns out that we can find the explicit expression for the process \(\tilde{W}^*(\phi_t^{-1})\).

From the expression

\[V_t = S_0 - L + \tilde{W}^*(\phi_t^{-1}),\]

we have the following relationship between \(W(t)\) and \(\tilde{W}(\phi_t^{-1})\):

\[d\tilde{W}^*(\phi_t^{-1}) = \sigma \int_0^t [S(0) - L + Le^{at} + \tilde{W}^*(\phi_{s}^{-1})] dW^*(t).\]

It is linear SDE with respect to \(\tilde{W}^*(\phi_t^{-1})\) and we can solve it explicitly. The solution has the following look:

\[
\tilde{W}^*(\phi_t^{-1}) = S(0)(e^{\sigma W^*(t)} - \frac{\sigma^2 t}{2} - 1) + L(1 - e^{at})
\]

\[
+ aLe^{\sigma W^*(t)} - \frac{\sigma^2 t}{2} \int_0^t e^{as} e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds,
\]

\[
(32)
\]

It is easy to see from (32) that \(\tilde{W}^*(\phi_t^{-1})\) can be presented in the form of a linear combination of two zero-mean \(P^*\)-martingales \(m_1^*(t)\) and \(m_2^*(t)\):

\[\tilde{W}^*(\phi_t^{-1}) = m_1^*(t) + L^* m_2^*(t),\]

where

\[m_1^*(t) := S(0)(e^{\sigma W^*(t)} - \frac{\sigma^2 t}{2} - 1)\]
and 
\[ m_2^*(t) = (1 - e^{a^* t}) + a^* e^{a^* W^*(t) - \frac{a^*}{2} t} \int_0^t e^{a^* s} e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds. \]

Indeed, process $\tilde{W}^*(\phi_t^{-1})$ is a martingale (see Section 5.2), also it is well-known that process $e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}}$ and, hence, process $m_1^*(t)$ is a martingale. Then the process $m_2^*(t)$, as the difference between two martingales, is also martingale. In this way, we have

\[ E_P m_1^*(t) = 0, \]

since

\[ E_P e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} = 1. \]

As for $m_2(t)$ we have

\[ E_P m_2(t) = 0, \]

since from Itô’s formula we have

\[ d \left( a^* e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{a^* s} e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds \right) = a^* \sigma e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{a^* s} e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds dW^*(t) + a^* e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} e^{a^* t} e^{-\sigma W^*(t) + \frac{\sigma^2 t}{2}} dt \]

and, hence,

\[ E_P a^* e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{a^* s} e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds = e^{a^* t} - 1. \]

It is interesting to see that in the last expression, the first moment for

\[ \eta^*(t) := a^* e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{a^* s} e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds, \]

does not depend on $\sigma$.

This is true not only for the first moment but for all the moments of the process $\eta^*(t) = a^* e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{a^* s} e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds$.

Indeed, using the Itô’s formula for $(\eta^*(t))^n$ we obtain

\[ d(\eta^*(t))^n = n(a^*)^n \sigma e^{a^* W^*(t) - \frac{na^*}{2} t} (\int_0^t e^{a^* s} e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds)^n dW^*(t) + a^* n(\eta_2(t))^{n-1} e^{a^* t} dt, \]

and

\[ dE(\eta^*(t))^n = na^* e^{a^* t} E(\eta^*(t))^{n-1} dt, \quad n \geq 1. \]

This is a recursive equation with initial function $(n = 1)$ $E\eta^*(t) = e^{a^* t} - 1$.

After calculations we obtain the following formula for $E(\eta^*(t))^n$:

\[ E(\eta^*(t))^n = (e^{a^* t} - 1)^n. \]
5.4 Some Properties of the Mean-Reverting Risk-Neutral Asset $S_t$.

Using the same argument as in Section 5, we obtain the following properties of the mean-reverting risk-neutral asset $S_t$ in (18):

\begin{align}
E^*S_t &= e^{-a^*t}[S_0 - L^*] + L^* \\
Var^*(S_t) &= E^*S_t^2 - (E^*S_t)^2
\end{align}

\begin{align*}
&= \sigma^2 e^{-2a^*t}[(S_0 - L^*)^2 \frac{e^{a^*t} - 1}{a^* - \sigma^2} + \frac{2L^*(S_0 - L^*)}{a^* - \sigma^2}]
& \quad + \frac{(L^*)^2(e^{a^*t} - e^{\sigma^2t})}{2a^* - \sigma^2},
\end{align*}

where $E^*$ is the expectation with respect to the probability $P^*$ and $a^*, L^*$ and $(\phi_t^*)^{-1}$ are defined in (27) and (30), respectively.

From the expression for $\tilde{W}^*(\phi_t^{-1})$ (see (32)) and for $S(t)$ in (28) (see also (29)-(30)) we can find the explicit expression for $S(t)$ through $W^*(t)$:

\begin{align}
S(t) &= e^{-a^*t}[S_0 - L^* + \tilde{W}^*(\phi_t^{-1})] + L^* \\
&= e^{-a^*t}[S_0 - L^* + m_1^*(t) + L^*m_2^*(t)] + L^* \\
&= S(0)e^{-at}e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} + aLe^{-at}e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as}e^{-\sigma W^*(s) + \frac{\sigma^2 s}{2}} ds,
\end{align}

where $m_1^*(t)$ and $m_2^*(t)$ are defined as in section 5.3.

5.5 Explicit Option Pricing Formula for European Call Option for MRAM under Risk-Neutral Measure.

Proceeding with the same calculations (15)-(22) as in Section 3, where in place of $a$ and $L$ we take $a^*$ and $L^*$ in (27), we obtain the following Theorem.

**Theorem 5.1.** Explicit option pricing formula for European call option under risk-neutral measure has the following look:

\begin{align}
C^*_T &= e^{-(r+a^*)T} S(0) \Phi(y_+) - e^{-rT} K \Phi(y_-) \\
& \quad + L^*e^{-(r+a^*)T}[(e^{a^*T} - 1) - \int_0^{y_0} zF^*_T(dz)],
\end{align}

where $y_0$ is the solution of the following equation

\begin{align}
y_0 &= \frac{\ln\left(\frac{K}{S(0)}\right) + \frac{\sigma^2}{2} + a^*T}{\sigma\sqrt{T}} \\
& \quad - \frac{\ln(1 + \frac{a^*L^*}{S(0)} \int_0^T e^{a^*s}e^{-\sigma y_0\sqrt{T} + \frac{\sigma^2 s}{2}} ds)}{\sigma\sqrt{T}},
\end{align}

\begin{align}
y_+ := \sigma\sqrt{T} - y_0 \quad \text{and} \quad y_- := -y_0,
\end{align}
\[ a^* := a + \lambda \sigma, \quad L^* := \frac{aL}{a + \lambda \sigma}, \]

and \( F^*_T(dz) \) is the probability distribution as in (21), where instead of \( a \) we have to take \( a^* = a + \lambda \sigma \).

Remark. From (35) we can find that European Call Option Price \( C^*_T \) for mean-reverting asset under risk-neutral measure lies between the following boundaries:

\[
e^{-r(a^*)T} S(0) \Phi(y_+) - e^{-rT} K \Phi(y_-) \leq C_T \leq e^{-r(a^*)T} S(0) \Phi(y_+) - e^{-rT} K \Phi(y_-) + L^* e^{-r(a^*)T} [e^{a^*T} - 1],
\]

where \( y_0, y_-, y_+ \) are defined in (36)-(37).

5.6 Black-Scholes Formula Follows: \( L^* = 0 \) and \( a^* = -r \).

If \( L^* = 0 \) and \( a^* = -r \) we obtain from (35)

\[ C_T = S(0) \Phi(y_+) - e^{-rT} K \Phi(y_-), \quad (39) \]

where

\[ y_+ := \sigma \sqrt{T} - y_0 \quad \text{and} \quad y_- := -y_0, \quad (40) \]

and \( y_0 \) is the solution of the following equation (see (36))

\[ S(0)e^{-rT} e^{\sigma y_0 \sqrt{T} - \frac{a^2 T}{2}} = K \]

or

\[ y_0 = \frac{\ln \left( \frac{K}{S(0)} \right) + \left( \frac{a^2}{2} - r \right) T}{\sigma \sqrt{T}}. \quad (41) \]

But (39)-(41) is exactly the well-known Black-Scholes result!

6 Numerical Example: AECO Natural GAS Index (1 May 1998-30 April 1999)

We shall calculate the value of a European call option on the price of a daily natural gas contract. To apply our formula for calculating this value we need to calibrate the parameters \( a, \ L, \ \sigma \) and \( \lambda \). These parameters may be obtained from futures prices for the AECO Natural Gas Index for the period 1 May 1998 to 30 April 1999 (see Bos, Ware and Pavlov (2002), p.340). The parameters pertaining to the option are the following:
From this table we can calculate the values for $a^*$ and $L^*$:

$$a^* = a + \lambda \sigma = 4.9337,$$

and

$$L^* = \frac{aL}{a + \lambda \sigma} = 2.5690.$$

For the value of $S_0$ we can take $S_0 \in [1, 6]$.

Figure 1 (see Appendix) depicts the dependence of mean value $ES_t$ on the maturity $T$ for AECO Natural Gas Index (1 May 1998 to 30 April 1999).

Figure 2 (see Appendix) depicts the dependence of mean value $ES_t$ on the initial value of stock $S_0$ and maturity $T$ for AECO Natural Gas Index (1 May 1998 to 30 April 1999).

Figure 3 (see Appendix) depicts the dependence of variance of $S_t$ on the initial value of stock $S_0$ and maturity $T$ for AECO Natural Gas Index (1 May 1998 to 30 April 1999).

Figure 4 (see Appendix) depicts the dependence of volatility of $S_t$ on the initial value of stock $S_0$ and maturity $T$ for AECO Natural Gas Index (1 May 1998 to 30 April 1999).

Figure 5 (see Appendix) depicts the dependence of European Call Option Price for MRRNAM on the maturity (months) for AECO Natural Gas Index (1 May 1998 to 30 April 1999) with $S(0) = 1$ and $K = 3$.

Appendix: Figures
Fig. 1. Dependence of $ES_t$ on $T$ (AECO Natural Gas Index (1 May 1998-30 April 1999))

Fig. 2. Dependence of $ES_t$ on $S_0$ and $T$ (AECO Natural Gas Index (1 May 1998-30 April 1999))

Fig. 3. Dependence of variance of $S_t$ on $S_0$ and $T$ (AECO Natural Gas Index (1 May 1998-30 April 1999))

Fig. 4. Dependence of volatility of $S_t$ on $S_0$ and $T$ (AECO Natural Gas Index (1 May 1998-30 April 1999))

Fig. 5. Dependence of European Call Option Price on Maturity (months) ($S(0) = 1$ and $K = 3$) (AECO Natural Gas Index (1 May 1998-30 April 1999))
References


Chapter 7: CTM and Multi-Factor Lévy Models for Pricing Financial and Energy Derivatives

'The energy of the mind is the essence of life',-Aristotle.

1 Introduction

In this chapter, we introduce one-factor and multi-factor $\alpha$-stable Lévy-based models to price financial and energy derivatives. These models include, in particular, as one-factor models, the Lévy-based geometric motion model, the Ornstein-Uhlenbeck (1930), the Vasicek (1977), the Cox-Ingersoll-Ross (1985), the continuous-time GARCH, the Ho-Lee (1986), the Hull-White (1990) and the Heath-Jarrow-Morton (1992) models, and, as multi-factor models, various combinations of the previous models. For example, we introduce new multi-factor models such as the Lévy-based Heston model, the Lévy-based SABR/LIBOR market models, and Lévy-based Schwartz-Smith and Schwartz models. Using the change of time method for SDEs driven by $\alpha$-stable Lévy processes we present the solutions of these equations in simple and compact forms. We then apply this method to price many financial and energy derivatives such as variance swaps, options, forward and futures contracts.

In this first section, we first review in brief literature on the change of time method (CTM) for Lévy-based models, and give an overview of the multi-factor Gaussian, LIBOR and SABR models, swaps and energy derivatives.

1.1 Change of Time Method (CTM) for Lévy-based Models: Short Literature History


Lévy processes can also be used as a time change for other Lévy processes (subordinators). Madan & Seneta (1990) introduced Variance Gamma (VG) process (Brownian motion with drift time changed by a gamma process). Geman, Madan & Yor (2001) considered time changes (‘business times’) for Lévy processes. Carr, Geman, Madan & Yor (2003) used a change of time to introduce stochastic volatility into a Lévy model to achieve leverage effect and a long-term skew. Kallsen & Shiryaev (2001) showed that the Rosiński-Woyczyński-Kallenberg result can not be extended to any other Lévy processes other than the symmetric $\alpha$-stable processes. Swishchuk (2004, 2007) applied a change of time method for options and swaps pricing for Gaussian models.
The book ‘Change of Time and Change of Measure’ by Barndorff-Nielsen and Shiryaev (2010) states the main ideas and results of the stochastic theory of ‘change of time and change of measure’ in the semimartingale setting.

### 1.2 Stochastic Differential Equations (SDEs) Driven by Lévy Processes

Girsanov (1960) used the change of time method to construct a weak solution to a specific SDE driven by Brownian motion.

The existence and uniqueness of solutions for SDEs driven by Lévy processes have been studied in Applebaum (2003). The existence and uniqueness of solutions for SDEs driven by general semimartingale with jumps have been studied in Protter (2005) and Jacod (1979).

Janicki, Michna & Weron (1996) proved that there exists a unique solution of the SDE for continuous drift $b$ and diffusion coefficient $\sigma$ and $\alpha$-stable Lévy process $S_{\alpha}((t-s)^{1/\alpha}, \beta, \delta)$, $\beta \in [-1, +1]$.

Zanzotto (1997) had also considered solutions of one-dimensional SDEs driven by stable Lévy motion.


### 1.3 Multi-Factor Gaussian Models: Literature Review

Eydeland and Geman (1998) proposed extending the Heston (1993) stochastic volatility model to gas or electricity prices by introducing mean-reversion in the spot price and leaving the CIR model for the variance, resulting in a two-state variable model for commodity prices.

Geman (2000) proposed a three-state variable for oil prices by introducing mean-reversion in the spot price, geometric Brownian motion in the equilibrium (or mean-reverting) price and the CIR model in the variance.

Gibson and Schwartz (1990) note that the convenience yield has been shown to be a key factor driving the relationship between spot and futures prices, and they proposed the two-state variable model for oil-contingent claim pricing.

Continuing to the more complex level of two factor models, keeping the same mean-reverting level $L_t$, the SDEs for spot price $S_t$ and $L_t$ can be generalized by either allowing the long run mean $L_t$ or the volatility $\sigma$ to be governed by an SDE. This leads to two distinct two factor models, with different dynamics. The first model assumes a stochastic long run mean and was introduced by Pilipović (1997). Pilipović (1997) describes a two-factor mean-reverting model where spot prices revert to a long term equilibrium level which is itself a random variable. Pilipovich derives a closed-form solution for forward prices to her model when the spot and long term prices are uncorrelated, but does not discuss option pricing in her two-factor model.
Gibson and Schwartz (1990), Schwartz (1997) and Hilliard and Reis (1998) all analyze versions of the same two-factor model that allows for a stochastic convenience yield and permits a high level of analytical tractability. The first factor is the spot price process which is assumed to follow the geometric Brownian motion (GBM) and the second factor is the instantaneous convenience yield of the spot energy and is assumed to follow the mean reverting process.

Schwartz (1997) extends his two-factor model to include stochastic interest rates. In this three-factor model the short term rate is assumed to follow the Vasicek (1977) mean-reverting process.

Fouque, Papanicolaou and Sircar (2000) considered multi-factor stochastic volatility model that is a function of two processes in the form of geometric Brownian motions.

Fouque and Han (2003) found that two-factor SV models provide a better fit to the term structure of implied volatility than one factor SV models by capturing the behavior at short and long maturities.


Molina et al (2003) found a strong evidence of two-factor SV models with well-separated time scales in foreign exchange data.

Carmona and Ludkovski (2003) reviewed the literature of spot convenience yield models, and analyzed in detail two new extensions. First, they discussed a variant of the Gibson-Schwartz model with time-dependent parameters and second, they described a new three-factor affine model with stochastic convenience yield and stochastic market price of risk.

1.4 Pricing Swaps Overview

Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999) explained the properties and the theory of both variance and volatility swaps. They derived an analytical formula for the theoretical fair value in the presence of realistic volatility skews, and pointed out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap.

Javaheri A, Wilmott, P. and Haug, E. G. (2002) discussed the valuation and hedging of a GARCH(1,1) stochastic volatility model. They used a general and flexible PDE approach to determine the first two moments of the realized variance in a continuous or discrete context. Then they approximate the expected realized volatility via a convexity adjustment.

Brockhaus and Long (2000) provided an analytical approximation for the valuation of volatility swaps and analyzed other options with volatility exposure.

In Swishchuk (2004) we found the values of variance and volatility swaps for financial markets with underlying asset and variance that follow the Heston (1993) model. We also studied covariance and correlation swaps for financial markets. As an application, we provided a numerical example using S&P 60 Canada Index to price swap on the volatility.

Variance swaps for financial markets with underlying asset and one-factor and multi-factor stochastic volatilities with delay are modelled and priced in Swishchuk (2005) and Swishchuk (2006), respectively.

In Elliott and Swishchuk (2007) we found the value of variance swap for financial markets with Markov stochastic volatility. In Swishchuk (2010) we found the values of variance and volatility swaps for semi-Markov volatility.

Swishchuk (2007) contains applications of the change of time method to Gaussian financial models such as geometric Brownian motion, the Heston model and a mean-reverting model to price options and different kinds of swaps.

1.5 Libor Market and SABR Models: Short Literature Review

The basic log-normal Forward Libor (also known as Libor Market, or BGM) model has proved to be an essential tool for pricing and risk-managing interest rate derivatives. First introduced in Brace, Gatarek, Musiela (1996) and Jamshidian (1997), forward Libor models are in the mainstream of interest rate modeling. For general information regarding the model, textbooks such as Brigo and Mercurio (2001) and Rebonato (2002) provide a good starting point. Various extensions of forward Libor models that attempt to incorporate volatility smiles of interest rates have been proposed. Local volatility type extensions were pioneered in Andersen and Andersen (2000). A stochastic volatility extension is proposed in Andersen and Brotherton-Ratcliffe (2001) and further extended in Andersen and Andersen (2002). A different approach to stochastic volatility forward Libor models is described in Rebonato (2002). Jump-diffusion forward Libor models are treated in Glasserman and Merener (2001) and Glasserman and Kou (1999). In Sin (2002), a stochastic volatility/jump diffusion forward Libor model is advocated.

The SABR and the Libor market models (LMM) have become industry standards for pricing plain-vanilla and complex interest rate products, respectively. For a description of the SABR model, see, for example, Hagan et al (2002). Several stochastic-volatility extensions of the LMM exist that do provide a consistent dynamic description of the evolution of the forward rates (see, for example, Andersen and Andersen (2000), Joshi and Rebonato
(2003), Rebonato and Joshi (2002), Rebonato and Kainth (2004)), but these extensions are not equivalent to the SABR model. Piterbarg (2003, 2005) presents an approach based on displaced diffusion that is similar in spirit to a dynamical extension of the SABR model. Henry-Labordère (2007) obtains some interesting exact results, but his attempt to unify the BGM and SABR model using an application of hyperbolic geometry is very complex and the computational issues are daunting. Rebonato (2007) proposed an extension of the LMM that recovers the SABR caplet prices almost exactly for all strikes and maturities. Many smiles and skews are usually managed by using local volatility models by Dupire (1994).

1.6 Energy Derivatives’ Overview

Black’s model (1976) and Schwartz’s model (1997) have become a standard approach to the problem of pricing options on commodities. These models have the advantage of mathematical convenience, in that they give rise to closed-form solutions for some types of options (See Wilmott (2000)).

A drawback of single-factor mean-reverting models lies in the case of options pricing: the fact the long-term rate is fixed results in a model-implied volatility term structure that has the volatilities going to zero as expiration time increases.

Using single-factor non-mean-reverting models has also a drawback: it will impact valuation and hedging. The differences between the distributions are particularly obvious when pricing out-of-the-money options, where the tails of the distribution play a very important role. Thus, if a lognormal model, for example, is used to price a far out-of-the-money option, the price can be very different from a mean-reverting model’s price (see Pilipović (1998)). A popular model used for modeling energy and agricultural commodities and introduced by Schwartz (1990) aims at resembling the geometric Brownian motion while introducing mean-reversion to a long-term value in the drift term (see Schwartz (1997)). This mean-reverting model is a one-factor version of the two-factor model made popular in the context of energy modelling by Pilipović (1997).

Villaplana (2003) proposed the introduction of two sources of risk $X$ and $Y$ representing, respectively, short-term and long-term shocks, and describes the spot price $S_t$. Geman and Roncoroni (2002) introduced a jump-reversion model for electricity prices. The two-factor model for oil-contingent claim pricing was proposed by Gibson and Schwartz (1990). Eydeland and Geman (1998) proposed extending the Heston (1993) stochastic volatility model to gas or electricity prices by introducing mean-reversion in the spot price and proposing two-factor model. Geman (2000) introduced three-factor model for commodity prices taking into account stochastic equilibrium level and stochastic volatility. Björk and Landen (2002) investigated the term structure of forward and futures prices for models where the price processes are allowed to be driven by a general market point process as well as by a mul-
tidimensional Wiener process. Benth et. al. (2008) applied independent increments processes (see Skorokhod (1964), Lévy (1965)) to model and price electricity, gas and temperature derivatives (forwards, futures, swaps, options). Swishchuk (2008) considers a risky asset in energy markets following mean-reverting stochastic process. An explicit expression for a European option price based on this asset, using a change of time method, is derived. A numerical example for the AECO Natural Gas Index (1 May 1998-30 April 1999) is presented.

1.7 Organization of the Chapter

The Chapter is organized as follows. Section 2 introduces one-factor and multi-factor Gaussian models. The change of time method for Gaussian models and solutions of the introduced Gaussian models are presented in Section 3. α-stable Lévy processes and their properties are defined in Section 4. One-factor and multi-factor Lévy-based models are presented in Section 5. The change of time method and solution of these equations using the change of time method are introduced in Section 6. Applications of Lévy-based models in financial and energy markets are presented in Section 7.

2 One-Factor and Multi-Factor Gaussian Models in Finance

In this section, we introduce well-known one-factor and multi-factor models (used in finance) described by SDEs driven by Brownian motion (so-called Gaussian models). The one-factor models have already been considered in Chapter 2, however for the completeness of the description we repeat some models one more time.

2.1 One-Factor Gaussian Models

For one-factor Gaussian models we define the following well-known processes:

1. The Geometric Brownian Motion: $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$.
2. The Continuous-Time GARCH Process: $dS(t) = \mu(b - S(t))dt + \sigma S(t)dW(t)$.
3. The Ornstein-Uhlenbeck (1930) Process: $dS(t) = -\mu S(t)dt + \sigma dW(t)$.
4. The Vasiček (1977) Process: $dS(t) = \mu(b - S(t))dt + \sigma dW(t)$.
5. The Cox-Ingersoll-Ross (1985) Process: $dS(t) = k(\theta - S(t))dt + \gamma \sqrt{S(t)}dW(t)$.
6. The Ho and Lee (1986) Process: $dS(t) = \theta(t)dt + \sigma dW(t)$.
7. The Hull and White (1990) Process: $dS(t) = (a(t) - b(t)S(t))dt + \sigma(t)dW(t)$
8. The Heath, Jarrow and Morton (1987) Process: Define the forward interest rate $f(t,s)$, for $t \leq s$, characterized by the following equality $P(t,u) =$
\[ \exp[-\int_0^u f(t,s)ds] \] for any maturity \( u \). \( f(t,s) \) represents the instantaneous interest rate at time \( s \) as 'anticipated' by the market at time \( t \). It is natural to set \( f(t,t) = r(t) \). The process \( f(u,f(t,s))ds \) represents the instantaneous interest rate at time \( s \) as 'anticipated' by the market at time \( t \).

It is natural to set \( f(t,t) = r(t) \). The process \( f(t,u)_{0 \leq t \leq u} \) satisfies an equation

\[ f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dW(v), \]

where the processes \( a \) and \( b \) are continuous. We note that the last SDE may be written in the following form:

\[ df(t,u) = b(f(t,u))(\int_t^u b(f(t,s)))ds + b(f(t,u))d\hat{W}(t), \]

where \( \hat{W}(t) = W(t) - \int_0^t q(s)ds \)

\[ q(t) = \int_t^u b(f(t,s))ds - \frac{a(t,u)}{b(f(t,u))}. \]

### 2.2 Multi-Factor Gaussian Models

For the multi-factor Gaussian model, we give one example of a two-factor continuous-time GARCH model:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= \mu(b(t) - S(t))dt + \sigma S(t)dW^1(t) \\
\frac{db(t)}{b(t)} &= \xi b(t)dt + \eta b(t) dW^2(t),
\end{align*}
\]

where \( W^1, W^2 \) may be correlated and \( \mu, \xi, \sigma, \eta > 0 \).

Other multi-factor models driven by Brownian motions can be obtained using various combinations of the above-mentioned processes, see Subsection 2.1.

### 3 Change of Time Method (CTM) for SDEs driven by Brownian motion

**Definition 1.** A time change is a right-continuous increasing \([0, +\infty)\)-valued process \((T_t)_{t \in R^+}\) such that \( T_t \) is a stopping time for any \( t \in R^+ \). By \( \hat{F}_t := F_{T_t} \) we define the time-changed filtration \((\hat{F}_t)_{t \in R^+}\). The inverse time change \((\hat{T}_t)_{t \in R^+}\) is defined as \( \hat{T}_t := \inf\{s \in R^+ : T_s > t\} \). (See Ikeda and Watanabe (1983)).

We consider the following SDE driven by a Brownian motion:

\[ dX(t) = a(t,X(t))dW(t), \]

where \( W(t) \) is a Brownian motion and \( a(t,X) \) is a continuous and measurable by \( t \) and \( X \) function on \([0, +\infty) \times R\).

**Theorem.** (Ikeda and Watanabe (1981), Chapter IV, Theorem 4.3) Let \( \hat{W}(t) \) be an one-dimensional \( F_t \)-Wiener process with \( \hat{W}(0) = 0 \), given on a probability space \((\Omega, F, (F_t)_{t \geq 0}, P)\) and let \( X(0) \) be an \( F_0 \)-adapted random variable.

Define a continuous process \( V = V(t) \) by the equality

\[ V(t) = X(0) + \hat{W}(t). \]
Let $T_t$ be the change of time process:

$$T_t = \int_0^t a^{-2}(T_s, X(0) + \dot{W}(s))ds.$$ 

If

$$X(t) := V(\hat{T}_t) = X(0) + \dot{W}(\hat{T}_t)$$

and $\hat{T}_t := \mathcal{F}_{\hat{T}_t}$, then there exists $\hat{T}_t$-adapted Wiener process $W = W(t)$ such that $(X(t), W(t))$ is a solution of $dX(t) = a(t, X(t))dW(t)$ on the probability space $(\Omega, F, \hat{\mathcal{F}}_t, P)$, where $\hat{T}_t$ is the inverse time change $T_t$.

### 3.1 Solutions to One-Factor and Multi-Factor Gaussian Models Using the CTM

#### 3.1.1 Solution of One-Factor Gaussian SIRMs Using the CTM

We use the change of time method (see Ikeda and Watanabe (1981)) to get the solutions to the following equations (see Swishchuk (2007)). $W(t)$ below is an standard Brownian motion, and $\hat{W}(t)$ is a $(\hat{T}_t)_{t \in \mathbb{R}^+}$-adapted standard Brownian motion on $(\Omega, F, (\hat{\mathcal{F}}_t)_{t \in \mathbb{R}^+}, P)$.

1. **The Geometric Brownian Motion:** $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$. Solution $S(t) = e^{\mu t}[S(0) + \dot{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t [S(0) + \dot{W}(\hat{T}_s)]^2 ds$.

2. **The Continuous-Time GARCH Process:** $dS(t) = \mu(b - S(t))dt + \sigma S(t)dW(t)$. Solution $S(t) = e^{-\mu t}(S(0) - b + \dot{W}(\hat{T}_t)) + b$, where $\hat{T}_t = \sigma^2 \int_0^t [S(0) - b + \dot{W}(\hat{T}_s)+ e^{\mu t}b]^2 ds$.

3. **The Ornstein-Uhlenbeck Process:** $dS(t) = -\mu S(t)dt + \sigma dW(t)$. Solution $S(t) = e^{-\mu t}[S(0) + \dot{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t (e^{\mu t}[S(0) - b + \dot{W}(\hat{T}_s)] + b)^2 ds$.

4. **The Vasicek Process:** $dS(t) = \mu(b - S(t))dt + \sigma dW(t)$, solution $S(t) = e^{-\mu t}[S(0) - b + \dot{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t (e^{\mu t}[S(0) - b + \dot{W}(\hat{T}_s)] + b)^2 ds$.

5. **The Cox-Ingersoll-Ross Process:** $dS^2(t) = k(\theta - S^2(t))dt + \gamma S(t)dW(t)$, solution $S^2(t) = e^{-kt}[S^2_0 - \theta^2 + \dot{W}(\hat{T}_t)] + \theta^2$, where $\hat{T}_t = \sigma^2 \int_0^t [e^{\gamma T_s}(S^2_0 - \theta^2 + \dot{W}(s)) + \theta^2e^{2\gamma T_s}]^{-1} ds$.

6. **The Ho and Lee Process:** $dS(t) = \theta(t)dt + \sigma dW(t)$. Solution $S(t) = S(0) + \dot{W}(\sigma^2 t) + \int_0^t \theta(s)ds$.

7. **The Hull and White Process:** $dS(t) = (a(t) - b(t)S(t))dt + \sigma(t)dW(t)$. Solution $S(t) = \exp[-\int_0^t b(s)ds][S(0) - \frac{a(s)}{b(s)} + \dot{W}(\hat{T}_t)]$, where $\hat{T}_t = \int_0^t \sigma^2(s)[S(0) - \frac{a(s)}{b(s)} + \dot{W}(\hat{T}_s)] + \exp[\int_0^s b(u)du]a(s)^2 ds$. 

8. **The Heath, Jarrow and Morton Process:** $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(v, u)dW(v)$. Solution $f(t, u) = f(0, u) + \dot{W}(\hat{T}_t) + \int_0^t a(v, u)dv$, where $\hat{T}_t = \int_0^t b^2(f(0, u) + \dot{W}(\hat{T}_s) + \int_0^s a(v, u)dv)ds$.

#### 3.1.2 Solution of Multi-Factor Gaussian Models Using CTM

Solutions of multi-factor models driven by Brownian motions can be obtained using various combinations of solutions of the above-mentioned processes, see
subsection 3.1, and CTM. We give one example of a two-factor continuous-time GARCH model driven by Brownian motions:

\[
\begin{align*}
    dS(t) & = r(t)S(t)dt + \sigma S(t)dW^1(t) \\
    dr(t) & = a(m - r(t))dt + \sigma_2 r(t)dW^2(t),
\end{align*}
\]

where \( W^1, W^2 \) may be correlated, \( m \in R, \sigma, a > 0 \).

The solution, using the CTM for the first and the second equations is:

\[
S(t) = e^{\int_0^t r(s)ds}[S_0 + \hat{W}^1(\hat{T}_1^1)] = e^{\int_0^t e^{-as}[\gamma_0 - m + \hat{W}^2(\hat{T}_2^1)]ds}[S_0 + \hat{W}^1(\hat{T}_1^1)],
\]

where \( \hat{T}^i \) and \( \hat{W}^i \) are defined in 1. and 2., Section 3.1.1.

4 α-stable Lévy Processes and their Properties

4.1 Lévy Processes

Definition 2. By Lévy process we mean a stochastically continuous process with stationary and independent increments (see Sato (2005), Applebaum (2003), Schoutens (2003)).

Examples of Lévy Processes \( L(t) \) include: a linear deterministic function \( L(t) = \gamma t \); Brownian motion with drift; Poisson process, compound Poisson process; jump-diffusion process; variance-gamma (VG), inverse Gaussian (IG), normal inverse Gaussian (NIG), generalized hyperbolic and α-stable processes (see Sato (2005)).

4.2 Lévy-Khintchine Formula and Lévy-Itô Decomposition for Lévy Processes \( L(t) \)

The characteristic function of the Lévy process follows the following formula (so-called Lévy-Khintchine formula)

\[
E(e^{i(u,L(t))}) = \exp\{t[i(u, \gamma) - \frac{1}{2}(u, Au) + \int_{R^d-\{0\}}[e^{i(u,y)} - 1 - i(u, y)1_{B_1(0)}]\nu(dy)]\}
\]

where \((\gamma, A, \nu)\) is the Lévy-Khintchine triplet.

If \( L \) is a Lévy process, then there exists \( \gamma \in R^d \), a Brownian motion \( B_A \) with covariance matrix \( A \) and an independent Poisson random measure \( N \) on \( R^+ \times (R^d - \{0\}) \) such that, for each \( t \geq 0 \), \( L(t) \) has the following decomposition (Lévy-Itô decomposition)

\[
L(t) = \gamma t + B_A(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq1} xN(t, dx),
\]

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where \( N \) is a Poisson counting measure and \( \tilde{N} \) is a compensated Poisson measure (see Applebaum (2003)).

**Remark (Lévy Processes in Finance).** The most commonly used Lévy processes in finance include Brownian motion with drift (the only continuous Lévy process), the Merton process=Brownian motion+drift+Gaussian jumps, the Kou process=Brownian motion+drift+exponential jumps and variance gamma (VG), inverse Gaussian (IG), normal inverse Gaussian (NIG), generalized hyperbolic (GH) and \( \alpha \)-stable Lévy processes.

### 4.3 \( \alpha \)-Stable Distributions and Lévy Processes

In this section, we introduce \( \alpha \)-stable distributions and Lévy processes, and describe their properties.

#### 4.3.1 Symmetric \( \alpha \)-Stable (S\( \alpha \)S) Distribution

The characteristic function of the S\( \alpha \)S distribution is defined as follows:

\[
\phi(u) = e^{i\delta u - \sigma|u|^\alpha},
\]

where \( \alpha \) is the characteristic exponent \((0 < \alpha \leq 2)\), \( \delta \in (-\infty, +\infty) \) is the location parameter, and \( \sigma > 0 \) is the dispersion.

For values of \( \alpha \in (1, 2] \) the location parameter \( \delta \) corresponds to the mean of the \( \alpha \)-stable distribution, while for \( 0 < \alpha \leq 1 \), \( \delta \) corresponds to its median. The dispersion parameter \( \sigma \) corresponds to the spread of the distribution around its location parameter \( \delta \). The characteristic exponent \( \alpha \) determines the shape of the distribution.

A stable distribution is called standard if \( \delta = 0 \) and \( \sigma = 1 \). If a random variable \( L \) is stable with parameters \( \alpha, \delta, \sigma \), then \((L - \delta)/\sigma^{1/\alpha}\) is standard with characteristic exponent \( \alpha \). By letting \( \alpha \) take the values \( 1/2 \), \( 1 \) and \( 2 \), we get three important special cases: the Lévy \((\alpha = 1/2)\), Cauchy \((\alpha = 1)\) and the Gaussian \((\alpha = 2)\) distributions:

\[
\begin{align*}
    f_{1/2}(\gamma, \delta; x) & = \left(\frac{1}{2\sqrt{\pi}}\right) x^{-3/2} e^{-t^2/(4x)} \\
    f_1(\gamma, \delta; x) & = \frac{1}{\pi \gamma^2 + (x-\delta)^2}, \\
    f_2(\gamma, \delta; x) & = \frac{1}{\sqrt{4\pi \gamma}} \exp\left[-\frac{(x-\delta)^2}{4\gamma}\right].
\end{align*}
\]

Unfortunately, no closed form expression exists for general \( \alpha \)-stable distributions other than the Lévy, the Cauchy and the Gaussian. However, power series expansions can be derived for the density \( f_\alpha(\delta, \sigma; x) \). Its tails (algebraic tails) decay at a lower rate than the Gaussian density tails (exponential tails).

The smaller the characteristic exponent \( \alpha \) is, the heavier the tails of the \( \alpha \)-stable density.

This implies that random variables following \( \alpha \)-stable distribution with small characteristic exponent are highly impulsive, and it is this heavy-tail
characteristic that makes this density appropriate for modeling noise which is impulsive in nature, for example, electricity prices or volatility.

Only moments of order less than \( \alpha \) exist for the non-Gaussian family of \( \alpha \)-stable distribution. The fractional lower order moments with zero location parameter and dispersion \( \sigma \) are given by

\[
E|X|^p = D(p, \alpha)\sigma^{p/\alpha}, \quad \text{for} \quad 0 < p < \alpha,
\]

\[
D(p, \alpha) = \frac{2^{p+1/2}\Gamma(p+1)\Gamma(1-\frac{p}{\alpha})}{\alpha\sqrt{\pi}\Gamma(1-\frac{p}{2})},
\]

where \( \Gamma(\cdot) \) is the Gamma function (Sato (2005)).

Since the \( S_\alpha S \) r.v. has 'infinite variance', the covariation of two jointly \( S_\alpha S \) real r.v. with dispersions \( \gamma_x \) and \( \gamma_y \) defined by

\[
[X,Y]_\alpha = \frac{E[X|Y|^{p-2}Y]}{E[|Y|^p]}\gamma_y
\]

has often been used instead of the covariance (and correlation), where \( \gamma_y = [Y,Y]_\alpha \) is the dispersion of r.v. \( Y \).

4.3.2 \( \alpha \)-stable Lévy Processes

Definition 3. Let \( \alpha \in (0, 2] \). An \( \alpha \)-stable Lévy process \( L \) such that \( L_1 \) (or equivalently any \( L_t \)) has a strictly \( \alpha \)-stable distribution (i.e., \( L_1 \equiv S_\alpha(\sigma, \beta, \delta) \)) for some \( \alpha \in (0, 2) \setminus \{1\} \), \( \sigma \in R_+ \), \( \beta \in [-1, 1] \), \( \delta = 0 \) or \( \alpha = 1 \), \( \sigma \in R_+ \), \( \beta = 0 \), \( \delta \in R \). We call \( L \) a symmetric \( \alpha \)-stable Lévy process if the distribution of \( L_1 \) is even symmetric \( \alpha \)-stable (i.e., \( L_1 \equiv S_\alpha(\sigma, 0, 0) \)) for some \( \alpha \in (0, 2) \), \( \sigma \in R_+ \). A process \( L \) is called \( (T_t)_{t \in R_+} \)-adapted if \( L \) is constant on \([T_{t-}, T_t]\) for any \( t \in R_+ \) (see Sato (2005)).

4.3.3 Properties of \( \alpha \)-stable Lévy Processes

The \( \alpha \)-stable Lévy processes are the only self-similar Lévy processes such that \( L(at) \overset{\text{Law}}{=} a^{1/\alpha}L(t) \), \( a \geq 0 \). They are either Brownian motion or pure jump. They have characteristic exponent and Lévy-Khintchine triplet known in closed form. They also have only 4 parameters, but infinite variance (except for Brownian motion). The \( \alpha \)-stable Lévy Processes are semimartingales (in this way, \( \int_0^t f_s dL_s \) can be defined) and \( \alpha \)-stable Lévy Processes are pure discontinuous Markov processes with generator

\[
Af(x) = \int_{R \setminus \{0\}} [f(x+y) - f(x) - yf'(y)1_{|y|<1}(y)]K_\alpha \frac{dy}{|y|^{1+\alpha}}.
\]

\( E|L(t)|^p \) is finite or infinite according as \( 0 < p < \alpha \) or \( p > \alpha \), respectively. In particular, for an \( \alpha \)-stable process, \( EL(t) = \delta t \) (1 < \( \alpha < 2 \)) (Sato (2005)).
5 Stochastic Differential Equations Driven by \( \alpha \)-Stable Lévy Processes

Consider the following SDE driven by an \( \alpha \)-stable Lévy process \( L(t) \):

\[
dZ_t = b(t, Z_{t-})dt + \sigma(t, Z_{t-})dL(t).
\]

(1)

Janicki et al (1996) proved that this equation has a weak solution for continuous coefficients \( a \) and \( b \).

We consider below one-factor and multi-factor models described by SDEs driven by \( \alpha \)-stable Lévy process \( L(t) \).

5.1 One-Factor \( \alpha \)-stable Lévy Models

\( L(t) \) below is a symmetric \( \alpha \)-stable Lévy process. We define below various processes via SDE driven by \( \alpha \)-stable Lévy process.

1. The Geometric \( \alpha \)-stable Lévy motion: \( dS(t) = \mu S(t-)dt + \sigma S(t-)dL(t) \).

2. The Ornstein-Uhlenbeck Process Driven by \( \alpha \)-stable Lévy motion: \( dS(t) = -\mu S(t-)dt + \sigma dL(t) \).

3. The Vasicek Process Driven by \( \alpha \)-stable Lévy motion: \( dS(t) = \mu(b - S(t-))dt + \sigma dL(t) \).

4. The Continuous-Time GARCH Process Driven by \( \alpha \)-stable Lévy motion: \( dS(t) = \mu(b - S(t-))dt + \sigma S(t-)dL(t) \).

5. The Cox-Ingersoll-Ross Process Driven by \( \alpha \)-stable Lévy motion: \( dS(t) = k(\theta - S(t-))dt + \gamma \sqrt{S(t-)}dL(t) \).

6. The Ho and Lee Process Driven by \( \alpha \)-stable Lévy motion: \( dS(t) = \theta(t-)dt + \sigma dL(t) \).

7. The Hull and White Process Driven by \( \alpha \)-stable Lévy motion: \( dS(t) = (a(t-) - b(t-)S(t-))dt + \sigma(t)dL(t) \).

8. The Heath, Jarrow and Morton Process Driven by \( \alpha \)-stable Lévy motion: Define the forward interest rate \( f(t, s) \), for \( t \leq s \), that represents the instantaneous interest rate at time \( s \) as ‘anticipated’ by the market at time \( t \). The process \( f(t, u)_{0 \leq t \leq u} \) satisfies an equation

\[
f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dL(v),
\]

where the processes \( a \) and \( b \) are continuous.

We note that Eberlein & Raible (1999) considered Lévy-based term structure models.

5.2 Multi-Factor \( \alpha \)-stable Lévy Models

Multi-factor models driven by \( \alpha \)-stable Lévy motions can be obtained using various combinations of the above-mentioned processes. We give one example of a two-factor continuous-time GARCH model driven by \( \alpha \)-stable Lévy
motions.

\[
dS(t) = r(t-)S(t-)dt + \sigma S(t-)dL^1(t)
\]
\[
dr(t) = a(m - r(t-))dt + \sigma_2 r(t-)dL^2(t),
\]

where \(L^1, L^2\) may be correlated, \(m \in R, \sigma_i, a > 0, i = 1, 2\).

Also, we can consider various combinations of models, presented above, i.e., mixed models containing Brownian and Lévy motions. For example,

\[
dS(t) = \mu(b(t-) - S(t-))dt + \sigma S(t-)dL(t)
\]
\[
db(t) = \xi(b(t))dt + \eta(b(t))dW(t),
\]

where the Brownian motion \(W(t)\) and Lévy process \(L(t)\) may be correlated.

6 Change of Time Method (CTM) for SDEs Driven by Lévy Processes

We denote by \(L_{\alpha,a.s.}^\ast\) the family of all real measurable \(\mathcal{F}_t\)-adapted processes \(a\) on \(\Omega \times [0, +\infty)\) such that for every \(T > 0\), \(\int_0^T |a(t, \omega)|^\alpha dt < +\infty\) a.s. We consider the following SDE driven by a Lévy motion:

\[
dX(t) = a(t, X(t-))dL(t),
\]

where \(L(t)\) is an \(\alpha\)-stable Lévy process.

**Theorem.** (Rosinski and Woyczynski (1986), Theorem 3.1., p. 277).

Let \(a \in L_{\alpha,a.s.}^\ast\) be such that \(T(u) := \int_0^u |a|^\alpha dt \rightarrow +\infty\) a.s. as \(u \rightarrow +\infty\). If \(\hat{T}(t) := \inf\{u : T(u) > t\}\) and \(\hat{\mathcal{F}}_t = \mathcal{F}_{\hat{T}(t)}\), then the time-changed stochastic integral \(\hat{L}(t) = \int_0^{\hat{T}(t)} a dL(t)\) is an \(\hat{\mathcal{F}}_t - \alpha\)-stable Lévy process, where \(L(t)\) is \(\mathcal{F}_t\)-adapted and \(\mathcal{F}_t\)-\(\alpha\)-stable Lévy process. Consequently, a.s. for each \(t > 0\) \(\int_0^t a dL = \hat{L}(T(t))\), i.e., the stochastic integral with respect to a \(\alpha\)-stable Lévy process is nothing but another \(\alpha\)-stable Lévy process with randomly changed time scale.

6.1 Solutions of One-Factor Lévy Models using the CTM

Below we give the solutions to the one-factor Lévy models described by SDEs driven by \(\alpha\)-stable Lévy process introduced in section 5.1.

**Proposition 1.** Let \(L(t)\) be a symmetric \(\alpha\)-stable Lévy process, and \(\hat{L}\) is a \((\hat{T}_t)_{t \in R^+}\)-adapted symmetric \(\alpha\)-stable Levy process on \((\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t \in R^+}, P)\). Then, we have the following solutions for the above-mentioned one-factor Lévy models 1-8 (section 6.1):

1. The Geometric \(\alpha\)-stable Lévy Motion: \(dS(t) = \mu S(t-)dt + \sigma S(t-)dL(t)\). Solution \(S(t) = e^{\mu t}[S(0) + L(T_t)]\), where \(T_t = \sigma \int_0^t [S(0) + L(T_s)]^\alpha ds\).

2. The Ornstein-Uhlenbeck Process Driven by \(\alpha\)-stable Lévy Motion: \(dS(t) = -\mu S(t-)dt + \sigma dL(t)\). Solution \(S(t) = e^{-\mu t}[S(0) + \tilde{L}(\hat{T}_t)]\), where \(\tilde{T}_t = \sigma \int_0^t (e^{-\mu s}[S(0) + \tilde{L}(\hat{T}_s)])^\alpha ds\).
3. The Vasiček Process Driven by α-stable Lévy Motion: \( dS(t) = \mu(b-S(t-))dt + \sigma dL(t) \). Solution \( S(t) = e^{-\mu t}[S(0) - b + \hat{L}(\hat{T})] \), where \( \hat{T} = \int_0^t e^{\mu s}[S(0) - b + \hat{L}(\hat{T})] + b \) ds.

4. The Continuous-Time GARCH Process Driven by α-stable Lévy process: \( dS(t) = \mu(b-S(t-))dt + \sigma S(t-)dL(t) ) \). Solution \( S(t) = e^{-\mu t}(S(0) - b + \hat{L}(\hat{T})_t) + b \), where \( \hat{T} = \sigma \int_0^t [S(0) - b + \hat{L}(\hat{T})_t] + b \) ds.

5. The Cox-Ingersoll-Ross Process Driven by α-stable Lévy Motion: \( dS(t) = \kappa(\theta^2 - S(t-))dt + \sqrt{S(t-)}dL(t) \). Solution \( S^2(t) = e^{-kt} [S_0^2 - \theta^2 + \hat{L}(\hat{T})] + \theta^2 \), where \( \hat{T} = \gamma \int_0^t e^{\kappa \hat{T}_t} (S_0^2 - \theta^2 + \hat{L}(\hat{T}_t)) + \theta^2 e^{2kt} \) ds.

6. The Ho and Lee Process Driven by α-stable Lévy Motion: \( dS(t) = \theta(t-)_t + \sigma dL(t) \). Solution \( S(t) = S(0) + \hat{L}(\sigma^2 t) + \int_0^t \theta(s)ds \).

7. The Hull and White Process Driven by α-stable Lévy Motion: \( dS(t) = (a(t-) - b(t-) S(t-))dt + \sigma(t-) dL(t) \). Solution \( S(t) = \exp (- \int_0^t b(s)ds) [S(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T})] \), where \( \hat{T} = \int_0^t \sigma^2(s)[S(0) - \frac{s}{b(s)} + \hat{L}(\hat{T})] + \exp [\int_0^t b(u)du] \frac{s}{b(s)} ds \).

8. The Heath, Jarrow and Morton Process Driven by α-stable Lévy Motion: \( f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dL(v) \). Solution \( f(t, u) = f(0, u) + \hat{L}(\hat{T}) + \int_0^t a(v, u)dv \), where \( \hat{T} = \int_0^t b^2(f(0, u) + \hat{L}(\hat{T}) + \int_0^t a(v, u)dv) ds \).

**Proof.** The approach is to eliminate drifts, reduce obtained SDE to the above-mentioned form \( dX(t) = a(t, X(t-))dL(t) \) and then to use the above-mentioned Rosinski-Woyczynski (1983) result.

### 6.2 Solution of Multi-Factor Lévy Models Using CTM

Solution of multi-factor models driven by α-stable Lévy motions (see Section 6.1) can be obtained using various combinations of solutions of the above-mentioned processes and the CTM. We give one example of two-factor continuous-time GARCH model driven by α-stable Lévy motions.

**Proposition 2.** Let we have the following two-factor Lévy-based model:

\[
\begin{align*}
    dS(t) &= r(t-)S(t-)dt + \sigma_1 S(t-)dL^1(t) \\
    dr(t) &= a(m-r(t-))dt + \sigma_2 r(t-)dL^2(t),
\end{align*}
\]

where \( L^1, L^2 \) may be correlated, \( m \in R, \sigma_i, a > 0, i = 1, 2 \).

Then the solution of the two-factor Lévy model using the CTM is (applying CTM for the first and the second equations, respectively):

\[
\begin{align*}
    S(t) &= e^{\int_0^t r_s ds} [S_0 + \hat{L}^1(\hat{T}^1_t)] \\
        &= e^{\int_0^t e^{-\alpha s}[r_0 + \hat{L}^1(\hat{T}^1_t)]ds} [S_0 + \hat{L}^1(\hat{T}^1_t)],
\end{align*}
\]

where \( \hat{T}^i \) are defined in 1. and 4., respectively, Section 6.1.

**Proof.** The approach is to eliminate drifts in both equations, reduce the obtained SDEs to the above-mentioned form \( dX(t) = a(t, X(t-))dL(t) \) and then to use the above-mentioned Rosinski-Woyczynski (1983) result.
Important Remark. Kallsen & Shiryaev (2002) showed that the Rosiński & Woyczyński (1986) statement cannot be extended to any other Lévy process but \( \alpha \)-stable processes. If one considers only nonnegative integrands \( a \) in 
\[
dX(t) = a(t, X(t-))dL(t),
\]
then we can extend their statement to asymmetric \( \alpha \)-stable Lévy processes.

7 Applications in Financial and Energy Markets

In this section, we consider various applications of the change of time method for Lévy-based SDEs arising in financial and energy markets: swap and option pricing, interest derivatives pricing and forward and futures contracts pricing.

7.1 Variance Swaps for Lévy-Based Heston Model

Assume that in the risk-neutral world the underlying asset \( S_t \) and the variance follow the following model:

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sigma_t dw_t, \\
\frac{d\sigma_t^2}{\sigma_t^2} &= k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dL_t,
\end{align*}
\]

where \( r_t \) is the deterministic interest rate, \( \sigma_0 \) and \( \theta \) are the short and long volatilities, \( k > 0 \) is a reversion speed, \( \gamma > 0 \) is a volatility (of volatility) parameter, \( w_t \) and \( L_t \) are independent standard Wiener and \( \alpha \)-stable Lévy processes (\( \alpha \in (0, 2] \)).

The solution for the second equation has the following form:

\[
\sigma^2(t) = e^{-kt}[\sigma_0^2 - \theta^2 + \hat{L}(\hat{T}_t)] + \theta^2,
\]

where \( \hat{T}_t = \gamma^\alpha \int_0^t [e^{k\hat{T}_s}(\sigma_0^2 - \theta^2 + \hat{L}(\hat{T}_s)) + \theta^2 e^{2k\hat{T}_s}]^{\alpha/2}ds. \)

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

\[
N(\sigma^2_R(S) - K_{var}),
\]

where \( \sigma^2_R(S) \) is the realized stock variance (quoted in annual terms) over the life of the contract,

\[
\sigma^2_R(S) := \frac{1}{T} \int_0^T \sigma^2(s)ds,
\]

\( K_{var} \) is the delivery price for variance, and \( N \) is the notional amount.

Valuing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract \( P \) on future
realized variance with strike price $K_{\text{var}}$ is the expected present value of the future payoff in the risk-neutral world:

$$P_{\text{var}} = E\{e^{-rT}(\sigma_R^2(S) - K_{\text{var}})\},$$

where $r$ is the risk-free discount rate corresponding to the expiration date $T$, and $E$ denotes the expectation.

The realized variance in our case is:

$$\sigma_R^2(S) := \frac{1}{T} \int_0^T \sigma^2(s) ds = \frac{1}{T} \int_0^T \{e^{-ks}[\sigma_0^2 - \theta^2 + \hat{L}(\hat{T}_s)] + \theta^2\} ds.$$

The value of the variance swap then is:

$$P_{\text{var}} = E\{e^{-rT}(\sigma_R^2(S) - K_{\text{var}})\} = E\{e^{-rT}\left(\frac{1}{T} \int_0^T \{e^{-ks}[\sigma_0^2 - \theta^2 + \hat{L}(\hat{T}_s)] + \theta^2\} ds - K_{\text{var}}\right)\}.$$

Thus, for calculating variance swaps we need to know only $E\{\sigma_R^2(S)\}$, namely, the mean value of the underlying variance, or $E\{\hat{L}(\hat{T}_s)\}$.

Only moments of order less than $\alpha$ exist for the non-Gaussian family of $\alpha$-stable distributions. We suppose that $1 < \alpha < 2$ to find $E\{\hat{L}(\hat{T}_s)\}$.

The value of a variance swap for the Lévy-based Heston model is:

$$P_{\text{var}} = e^{-rT}\left[\frac{1 - e^{-kT}}{kT} (\sigma_0^2 - \theta^2) + \theta^2 + \frac{\delta T}{2} - K_{\text{var}}\right],$$

where $\delta$ is a location parameter.

If $\delta = 0$, then the value of a variance swap for the Lévy-based Heston model is:

$$P_{\text{var}} = e^{-rT}\left[\frac{1 - e^{-kT}}{kT} (\sigma_0^2 - \theta^2) + \theta^2 - K_{\text{var}}\right],$$

which coincides with the well-known result by Brockhaus and Long (2000) and Swishchuk (2004).

### 7.2 Volatility Swaps for Lévy-Based Heston Model?

A stock volatility swap is a forward contract on the annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R(S) - K_{\text{vol}}),$$

where $\sigma_R(S)$ is the realized stock volatility (quoted in annual terms) over the life of contract,

$$\sigma_R(S) := \sqrt{\frac{1}{T} \int_0^T \sigma_s^2 ds},$$
\( \sigma_t \) is a stochastic stock volatility, \( K_{\text{vol}} \) is the annualized volatility delivery price, and \( N \) is the notional amount.

To calculate volatility swaps we need more. From Brockhaus-Long (2000) approximation (which used the second order Taylor expansion for the function \( \sqrt{x} \)) we have:

\[
E\{\sqrt{\sigma^2_R(S)}\} \approx \sqrt{E\{V\}} - \frac{\text{Var}\{V\}}{8E\{V\}^{3/2}},
\]

where \( V := \sigma^2_R(S) \) and \( \frac{\text{Var}\{V\}}{8E\{V\}^{3/2}} \) is the convexity adjustment.

Thus, to calculate the value of volatility swaps

\[
P_{\text{vol}} = \{e^{-rT}(E\{\sigma_R(S)\} - K_{\text{vol}})\}
\]

we need both \( E\{V\} \) and \( \text{Var}\{V\} \).

For \( S_{\alpha}S \) processes only the moments of order \( p < \alpha \) exist, \( \alpha \in (0, 2] \).

Since the \( S_{\alpha}S \) r.v. has ‘infinite variance’, the covariation of two jointly \( S_{\alpha}S \) real r.v. with dispersions \( \gamma_x \) and \( \gamma_y \) defined by

\[
[X, Y]_\alpha = \frac{E[X|Y|^{p-2}Y]}{E[|Y|^p]} \gamma_y,
\]

where \( \gamma_y = [Y, Y]_\alpha \) is the dispersion of r.v. \( Y \), has often been used instead of the covariance (and correlation).

One of possible ways to get volatility swaps for the Lévy-based Heston model is to use covariation.

### 7.3 Gaussian- and Lévy-based SABR/LIBOR Market Models

SABR model (see Hagan, Kumar, Lesniewski and Woodward (2002)) and the Libor Market Model (LMM) (Brace, Gatarek and Musiela (BGM, 1996), Piterbarg (2003)) have become industry standards for pricing plain-vanilla and complex interest rate products, respectively.

The Gaussian-based SABR model (Hagan, Kumar, Lesniewski and Woodward (2002)) is a stochastic volatility model in which the forward value satisfies the following SDE:

\[
\begin{align*}
\frac{dF_t}{F_t} &= \sigma_t F_t^\beta dW^1_t, \\
\frac{d\sigma_t}{\sigma_t} &= \nu \sigma_t dW^2_t.
\end{align*}
\]

In a similar way, we introduce the Lévy-based SABR model, a stochastic volatility model in which the forward value satisfies the following SDE:

\[
\begin{align*}
\frac{dF_t}{F_t} &= \sigma_t F_t^\beta dW_t, \\
\frac{d\sigma_t}{\sigma_t} &= \nu \sigma_t dL_t,
\end{align*}
\]

where \( L(t) \) is an \( \alpha \)-stable Lévy process.
The solution of Lévy-based SABR model using a change of time method has the following expression:

\[ F_t = F_0 + \hat{W}(\hat{T}_1^1), \]

\[ T_1^1 = \int_0^t \sigma_{T_2}^{-2}(F_0 + \hat{W}(s))^{-\frac{1}{2}}ds, \]

\[ \sigma_t = \sigma_0 + \hat{L}(\hat{T}_2^2), \]

\[ T_2^2 = \nu^{-\alpha} \int_0^t (\sigma_0 + \hat{L}(s))^{-\alpha}ds. \]

The expressions for \( F_t \) and \( \sigma_t \) give the possibility to calculate many financial derivatives.

### 7.4 Energy Forwards and Futures

Random variables following \( \alpha \)-stable distribution with small characteristic exponent are highly impulsive, and it is this heavy-tail characteristic that makes this density appropriate for modeling noise which is impulsive in nature, for example, energy prices such as electricity. Here, we introduce two Lévy-based models in energy market: two-factor Lévy-based Schwartz-Smith and three-factor Schwartz models. We show how solve them using the change of time method.

#### 7.4.1 Lévy-based Schwartz-Smith Model

We introduce the Lévy-based Schwartz-Smith model:

\[
\begin{align*}
\ln(S_t) &= \kappa_t + \xi_t \\
d\kappa_t &= (-k\kappa_t - \lambda_\kappa)dt + \sigma_\kappa dL_\kappa \\
d\xi_t &= (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dW_\xi,
\end{align*}
\]

where \( S_t \) is the current spot price, \( \kappa_t \) is the short-term deviation in prices, and \( \xi_t \) is the equilibrium price level.

Let \( F_{t,T} \) denote the market price for a futures contract with maturity \( T \), then:

\[ \ln(F_{t,T}) = e^{-k(T-t)}\kappa_t + \xi_t + A(T - t), \]

where \( A(T - t) \) is a deterministic function with explicit expression. We note that \( \kappa_t \), using change of time for \( \alpha \)-stable processes, can be presented in the following form:

\[ \kappa_t = e^{-kt}[\kappa_0 + \frac{\lambda_\kappa}{k} + \hat{L}_\kappa(\hat{T}_t)], \]

\[ \hat{T}_t = \sigma_\kappa^\alpha \int_0^t (e^{-ks}[\kappa_0 + \frac{\lambda_\kappa}{k} + \hat{L}_\kappa(\hat{T}_s)] - \frac{\lambda_\kappa}{k})^\alpha ds. \]

In this way, the market price for a futures contract with maturity \( T \) has the following form:
\[
\ln(F_{t,T}) = e^{-kT}[\kappa_0 + \frac{\lambda_0}{k} + \hat{L}_\kappa(\hat{T}_t)] + \xi_0 + (\mu_\xi - \lambda_\xi)t + \sigma_\xi W_\xi + A(T - t),
\]

where the Lévy process \( \hat{L}_\kappa \) and Wiener process \( W_\xi \) may be correlated.

If \( \alpha \in (1, 2] \), then we can calculate the value of Lévy-based futures contracts.

### 7.4.2 Lévy-based Schwartz Model

We also introduce a Lévy-Based Schwartz model:

\[
\begin{align*}
\frac{d}{dt}\ln(S_t) &= (r_t - \delta_t)S_t dt + S_t \sigma_1 dW_1 \\
\frac{d}{dt}\delta_t &= k(a - \delta_t)dt + \sigma_2 dL \\
\frac{d}{dt}r_t &= a(m - r_t)dt + \sigma_3 dW_2,
\end{align*}
\]

where the Wiener processes \( W_1, W_2 \) and \( \alpha \)-stable Lévy process \( L \) may be correlated. \( \delta_t \) and \( r_t \) are the instantaneous convenience yield and interest rate, respectively.

We note that:

\[
\delta_t = e^{kt}(\delta_0 - a + \hat{L}(\hat{T}_t)), \\
\hat{T}_t = \sigma_2^2 \int_0^t (e^{ks}[\delta_0 - a + \hat{L}(\hat{T}_s)] + a)^\alpha ds
\]

and

\[
\begin{align*}
r_t &= e^{at}(r_0 - m + \hat{W}_2(\hat{T}_t)), \\
\hat{T}_t &= \sigma_2^2 \int_0^t (e^{as}[r_0 - m + \hat{W}_2(\hat{T}_s)] + m)^2 ds.
\end{align*}
\]

The solution for \( \ln[S_t] \):

\[
\ln[S_t] = e^{\int_0^t [e^{as}(r_0 - m + \hat{W}_2(\hat{T}_s^2) - e^{ks}(\delta_0 - a + \hat{L}(\hat{T}_s))] + \sigma_2^2 \int_0^t (e^{as}[r_0 - m + \hat{W}_2(\hat{T}_s)] + m)^2 ds} [\ln S_0 + \hat{W}_1(\hat{T}_1)].
\]

In this way, the futures contract has the following form:

\[
\ln(F_{t,T}) = \frac{1 - e^{-k(T-t)}}{k(T-t)} \delta_t + \frac{1 - e^{-a(T-t)}}{a} r_t + \ln(S_t) + C(T - t)
\]

where \( C(T - t) \) is a deterministic explicit function. If \( \alpha > 1 \), then we can calculate the value of a futures contract.
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Chapter 8: CTM and Variance and Volatility Swaps in Energy Markets

‘Energy and persistence conquer all things,’-Benjamin Franklin.

1 Introduction

This Chapter is devoted to the pricing of variance and volatility swaps in energy market. We found explicit variance swap formula and closed form volatility swap formula (using change of time) for energy asset with stochastic volatility that follows continuous-time mean-reverting GARCH (1,1) model. Numerical example is presented for AECO Natural Gas Index (1 May 1998-30 April 1999).

Variance swaps are quite common in commodity, e.g., in energy market, and they are commonly traded. We consider Ornstein-Uhlenbeck process for commodity asset with stochastic volatility following continuous-time GARCH model or Pilipovic (1998) one-factor model. The classical stochastic process for the spot dynamics of commodity prices is given by the Schwartz’ model (1997). It is defined as the exponential of an Ornstein-Uhlenbeck (OU) process, and has become the standard model for energy prices possessing mean-reverting features.

In this paper, we consider a risky asset in energy market with stochastic volatility following a mean-reverting stochastic process satisfying the following SDE (continuous-time GARCH(1,1) model):

$$d\sigma^2(t) = a(L - \sigma^2(t))dt + \gamma \sigma^2(t)dW_t,$$

where $a$ is a speed of mean reversion, $L$ is the mean reverting level (or equilibrium level), $\gamma$ is the volatility of volatility $\sigma(t)$, $W_t$ is a standard Wiener process. Using a change of time method we find an explicit solution of this equation and using this solution we are able to find the variance and volatility swaps pricing formula under the physical measure. Then, using the same argument, we find the option pricing formula under risk-neutral measure. We applied Brockhaus-Long (2000) approximation to find the value of volatility swap. A numerical example for the AECO Natural Gas Index for the period 1 May 1998 to 30 April 1999 is presented.

Commodities are emerging as an asset class in their own. The range of products offered to investors range from exchange traded funds (ETFs) to sophisticated products including principal protected structured notes on individual commodities or baskets of commodities and commodity range-accrual or variance swap.

More and more institutional investors are including commodities in their asset allocation mix and hedge funds are also increasingly active players in commodities. Example: Amaranth Advisors lost USD 6 billion during
September 2006 from trading natural gas futures contracts, leading to the fund’s demise.

Concurrent with these developments, a number of recent papers have examined the risk and return characteristics of investments in individual commodity futures or commodity indices composed of baskets of commodity futures. See, e.g., Erb and Harvey (2006), Gorton and Rouwenhorst (2006), Ibbotson (2006), Kan and Oomen (2007).

However, since all but the most plain-vanilla investments contain an exposure to volatility, it is equally important for investors to understand the risk and return characteristics of commodity volatilities.

Our focus on energy commodities derives from two reasons:

1) energy is the most important commodity sector, and crude oil and natural gas constitute the largest components of the two most widely tracked commodity indices: the Standard & Poors Goldman Sachs Commodity Index (S&P GSCI) and the Dow Jones-AIG Commodity Index (DJ-AIGCI).

2) existence of a liquid options market: crude oil and natural gas indeed have the deepest and most liquid options marketss among all commodities.

The idea is to use variance (or volatility) swaps on futures contracts.

At maturity, a variance swap pays off the difference between the realized variance of the futures contract over the life of the swap and the fixed variance swap rate.

And since a variance swap has zero net market value at initiation, absence of arbitrage implies that the fixed variance swap rate equals to conditional risk-neutral expectation of the realized variance over the life of swap.

Therefore, e.g., the time-series average of the payoff and/or excess return on a variance swap is a measure of the variance risk premium.

Variance risk premia in energy commodities, crude oil and natural gas, has been considered by A. Trolle and E. Schwartz (2009).

The same methodology as in Trolle & Schwartz (2009) was used by Carr & Wu (2009) in their study of equity variance risk premia. The idea was to use variance swaps on futures contracts.

The study in Trolle & Schwartz (2009) is based on daily data from January 2, 1996 until November 30, 2006-a total of 2750 business days. The source of the data is NYMEX.

Trolle & Schwartz (2009) found that:

1) the average variance risk premia are negative for both energy commodities but more strongly statistically significant for crude oil than for natural gas;

2) the natural gas variance risk premium (defined in dollars terms or in return terms) is higher during the cold months of the year (seasonality and peaks for natural gas variance during the cold months of the year);

3) energy risk premia in dollar terms are time-varying and correlated with the level of the variance swap rate. In contrast, energy variance risk premia in return terms, paerticularly in the case of natural gas, are much less correlated with the variance swap rate.
The S&P GSCI is comprised of 24 commodities with the weight of each commodity determined by their relative levels of world production over the past five years.

The DJ-AIGCI is comprised of 19 commodities with the weight of each component determined by liquidity and world production values, with liquidity being the dominant factor.

Crude oil and natural gas are the largest components in both indices. In 2007, their weight were 51.30% and 6.71%, respectively, in the S&P GSCI and 13.88% and 11.03%, respectively, in the DJ-AIGCI.

The Chicago Board Options Exchange (CBOE) recently introduced a Crude Oil Volatility Index (ticker symbol OVX).

This index also measures the conditional risk-neutral expectation of crude oil variance, but is computed from a cross-section of listed options on the United States Oil Fund (USO), which tracks the price of WTI as closely as possible. The CBOE Crude Oil ETF Volatility Index (“Oil VIX”, Ticker - OVX) measures the market’s expectation of 30-day volatility of crude oil prices by applying the VIX methodology to United States Oil Fund, LP (Ticker - USO) options spanning a wide range of strike prices (see Figures below. Courtesy-CBOE: http://www.cboe.com/micro/oilvix/introduction.aspx).
We have to notice that crude oil and natural gas trade in units of 1,000 barrels and 10,000 British thermal units (mmBtu), respectively. Price are quoted as US dollars and cents per barrel or mmBtu.

The continuous-time GARCH model has also been exploited by Javaheri, Wilmott and Haug (2002) to calculate volatility swap for S&P500 index. They used PDE approach and mentioned (page 8, sec. 3.3) that "it would be interesting to use an alternative method to calculate \( F(v,t) \) and the other above quantities". This paper exactly contains the alternative method, namely, "change of time method", to get varinace and volatility swaps. The change of time method was also applied by Swishchuk (2004) for pricing variance, volatility, covariance and correlation swaps for Heston model. The first paper on pricing of commodity contracts was published by Black (1976).

## 2 Mean-Reverting Stochastic Volatility Model (MRSVM)

In this section we introduce MRSVM and study some properties of this model that we can use later for calculating variance and volatility swaps.

Let \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \) be a probability space with a sample space \( \Omega \), \( \sigma \)-algebra of Borel sets \( \mathcal{F} \) and probability \( P \). The filtration \( \mathcal{F}_t, \ t \in [0,T] \), is the natural filtration of a standard Brownian motion \( W_t, \ t \in [0,T] \), such that \( \mathcal{F}_T = \mathcal{F} \).

We consider a risky asset in energy market with stochastic volatility following a mean-reverting stochastic process the following stochastic differential equation:

\[
\begin{align*}
\text{d}\sigma^2(t) &= a(L - \sigma^2(t))dt + \gamma\sigma^2(t)dW_t, \\
\end{align*}
\]

where \( a > 0 \) is a speed (or 'strength') of mean reversion, \( L > 0 \) is the...
mean reverting level (or equilibrium level, or long-term mean), \( \gamma > 0 \) is the volatility of volatility \( \sigma(t) \), \( W_t \) is a standard Wiener process.

2.1 Explicit Solution of MRSVM

Let
\[
V_t := e^{\alpha t}(\sigma^2(t) - L).
\]

Then, from (2) and (1) we obtain
\[
dV_t = ae^{\alpha t}(\sigma^2(t) - L)dt + e^{\alpha t}d\sigma^2(t) = \sigma(V_t + e^{\alpha t}L)dW_t.
\]

Using change of time approach to the equation (3) (see Ikeda and Watanabe (1981) or Elliott (1982)) we obtain the following solution of the equation (3)
\[
V_t = \sigma^2(0) - L + \bar{W}(\phi_t^{-1}),
\]
or (see (2)),
\[
\sigma^2(t) = e^{-\alpha t}[\sigma^2(0) - L + \bar{W}(\phi_t^{-1})] + L.
\]

2.2 Some Properties of the Process \( \bar{W}(\phi_t^{-1}) \)

We note that process \( \bar{W}(\phi_t^{-1}) \) is \( \bar{F}_t := F_{\phi_t^{-1}} \)-measurable and \( \bar{F}_t \)-martingale. Then
\[
E\bar{W}(\phi_t^{-1}) = 0.
\]

Let’s calculate the second moment of \( \bar{W}(\phi_t^{-1}) \) (see (6)):
\[
E\bar{W}^2(\phi_t^{-1}) = E < \bar{W}(\phi_t^{-1}) >= E\phi_t^{-1}
\]
\[
= \gamma^2 \int_0^t E[\sigma^2(0) - L + \bar{W}(\phi_s^{-1}) + e^{\alpha s}L]^2ds
\]
\[
= \gamma^2[\sigma^2(0) - L]^2t + 2L(\sigma^2(0) - L)(e^{\alpha t} - 1) + \frac{L^2(e^{2\alpha t} - 1)}{2\alpha}
\]
\[
+ \int_0^t E\bar{W}^2(\phi_s^{-1})ds.
\]

From (8), solving this linear ordinary nonhomogeneous differential equation with respect to \( E\bar{W}^2(\phi_t^{-1}) \),
We note, that process \( \tilde{W} \) and the second moment for \( \tilde{W} \) turn out that we can find the explicit expression for the process \( \tilde{W} \). It is a linear SDE with respect to \( \tilde{W} \) and we have the following relationship between \( \tilde{W}(t) \) and \( \tilde{W}(\phi^{-1}_s) \):

\[
d\tilde{W}(\phi^{-1}_s) = \gamma \int_0^t [S(0) - L + \tilde{W}(\phi^{-1}_s)] dW(t).
\]

It is easy to see from (10) that \( \tilde{W}(\phi^{-1}_s) \) can be presented in the form of a linear combination of two zero-mean martingales \( m_1(t) \) and \( m_2(t) \):

\[
\tilde{W}(\phi^{-1}_s) = m_1(t) + Lm_2(t),
\]

where \( m_1(t) := \sigma^2(0)(e^{\gamma W(t) - \frac{x^2}{2} t} - 1) \) and \( m_2(t) = (1 - e^{at}) + ae^{\gamma W(t) - \frac{x^2}{2} t} \int_0^t e^{as} \gamma e^{-\gamma W(s) + \frac{x^2}{2} s} ds \).

Indeed, process \( \tilde{W}(\phi^{-1}_s) \) is a martingale (see Section 3.2), also it is well-known that process \( e^{\gamma W(t) - \frac{x^2}{2} t} \) and, hence, process \( m_1(t) \) is a martingale. Then the process \( m_2(t) \), as the difference between two martingales, is also martingale.
2.4 Some Properties of the Mean-Reverting Stochastic Volatility $\sigma^2(t)$: First Two Moments, Variance and Covariation

From (4) we obtain the mean value of the first moment for mean-reverting stochastic volatility $\sigma^2(t)$:

$$E\sigma^2(t) = e^{-at}[\sigma^2(0) - L] + L.$$  \hfill (11)

It means that $E\sigma^2(t) \to L$ when $t \to +\infty$. We need this moment to value the variance swap.

Using formula (4) and (9) we can calculate the second moment of $\sigma^2(t)$:

$$E\sigma^2(t) = (e^{-at}(\sigma^2(0) - L) + L)^2 + \gamma^2e^{-2at}\left[(\sigma^2(0) - L)^2\gamma^2 + \frac{2L(\sigma^2(0) - L)(e^{at\gamma^2} - 1)}{a - \gamma^2} + \frac{L^2(e^{2at\gamma^2} - e^{2\gamma^2})}{2a - \gamma^2}\right].$$

Combining the first and the second moments we have the variance of $\sigma^2(t)$:

$$Var(\sigma^2(t)) = E\sigma^2(t)^2 - (E\sigma^2(t))^2 = \gamma^2e^{-2at}\left[(\sigma^2(0) - L)^2\gamma^2 + \frac{2L(\sigma^2(0) - L)(e^{at\gamma^2} - 1)}{a - \gamma^2} + \frac{L^2(e^{2at\gamma^2} - e^{2\gamma^2})}{2a - \gamma^2}\right].$$

From the expression for $\tilde{W}(\phi^{-1})$ (see (10)) and for $\sigma^2(t)$ in (4) we can find the explicit expression for $\sigma^2(t)$ through $W(t)$:

$$\sigma^2(t) = e^{-at}[\sigma^2(0) - L + \tilde{W}(\phi^{-1})] + L = e^{-at}[\sigma^2(0) - L + m_1(t) + hm_2(t)] + L = \sigma^2(0)e^{-at}e^{\gamma t} - \frac{\gamma^2}{2} + mL e^{-at}e^{\gamma W(t)} - \frac{\gamma^2}{2} + Le^{as}e^{-\gamma W(s)} + \frac{\gamma^2}{2} ds,$$

(12)

where $m_1(t)$ and $m_2(t)$ are defined as in Section 3.3.

From (12) it follows that $\sigma^2(t) > 0$ as long as $\sigma^2(0) > 0$.

The covariation for $\sigma^2(t)$ may be obtained from (4), (7) and (9):

$$E\sigma^2(t)\sigma^2(s) = e^{-a(t+s)}(\sigma(0) - L)^2 + e^{-a(t+s)}\left[\gamma^2[\sigma^2(0) - L]^2\frac{e^{\gamma^2t} - 1}{\gamma^2} + \frac{2L[\sigma^2(0) - L](e^{at\gamma^2} - 1)}{a - \gamma^2} + \frac{L^2(e^{2at\gamma^2} - e^{2\gamma^2})}{2a - \gamma^2}\right] + e^{-at}(\sigma^2(0) - L)L + e^{-as}(\sigma^2(0) - L)L + L^2.$$  \hfill (13)

We need this covariance to value the volatility swap.

3 Variance Swap for MRSVM

To calculate the variance swap for $\sigma^2(t)$ we need $E\sigma^2(t)$ (see Chapter 2). From (11) it follows that

$$E\sigma^2(t) = e^{-at}[\sigma^2(0) - L] + L.$$
Then $E\sigma^2_R := EV$ takes the following form:

$$E\sigma^2_R := EV := \frac{1}{T} \int_0^T E\sigma^2(t)dt = \frac{(\sigma^2(0) - L)}{aT}(1 - e^{-aT}) + L. \quad (14)$$

Recall, that $V := \frac{1}{T} \int_0^T \sigma^2(t)dt$.

## 4 Volatility Swap for MRSVM

To calculate the volatility swap for $\sigma^2(t)$ we need $E\sqrt{V} = E\sqrt{\sigma_R}$ and it means that we more than just $E\sigma^2(t)$ (see Chapter 2), because the realized volatility $\sigma_R := \sqrt{V} = \sqrt{\sigma^2_R}$. Using Brockhaus-Long approximation we then get

$$E\sqrt{V} \approx \sqrt{EV} - \frac{Var(V)}{8(EV)^{3/2}}. \quad (15)$$

We have $EV$ calculated in (14). We need

$$Var(V) = EV^2 - (EV)^2. \quad (16)$$

From (14) it follows that $(EV)^2$ has the form:

$$(EV)^2 = \frac{(\sigma^2(0) - L)^2}{a^2T^2}(1 - e^{-aT})^2 + 2\frac{(\sigma^2(0) - L)}{aT}(1 - e^{-aT})L + L^2. \quad (17)$$

Let us calculate $EV^2$ using (9) and (13):

$$EV^2 = \frac{1}{T^2} \int_0^T \int_0^T E\sigma^2(t)\sigma^2(s)dtds$$

$$= \frac{1}{T^2} \int_0^T \int_0^T [e^{-a(t+s)}(\sigma^2(0) - L)^2$$

$$+ e^{-a(t+s)}\gamma^2[2[(\sigma^2(0) - L)^2e^{\gamma^2(t+s)}1$$

$$+ 2L(\sigma^2(0) - L)[e^{\gamma^2(t+s)}e^{-\gamma^2(t+s)}] + L^2[e^{2a(t+s)} - e^{\gamma^2(t+s)}]]}$$

$$+ e^{-a(t)}(\sigma^2(0) - L)L + e^{-as}(\sigma^2(0) - L)L + L^2]dtds \quad (18)$$

After calculating the interals in the second, forth and fifth lines in (18) we have:

$$EV^2 = \frac{1}{T^2} \int_0^T \int_0^T E\sigma^2(t)\sigma^2(s)dtds$$

$$= \frac{(\sigma^2(0) - L)^2}{a^2T^2}(1 - e^{-aT})^2$$

$$+ \frac{1}{T^2} \int_0^T \int_0^T e^{-a(t+s)}\gamma^2[2[(\sigma^2(0) - L)^2e^{\gamma^2(t+s)}1$$

$$+ 2L(\sigma^2(0) - L)[e^{\gamma^2(t+s)}e^{-\gamma^2(t+s)}] + L^2[e^{2a(t+s)} - e^{\gamma^2(t+s)}]]]dtds$$

$$+ \frac{(\sigma^2(0) - L)L}{aT}(1 - e^{-aT}) + \frac{(\sigma^2(0) - L) L}{aT}(1 - e^{-aT}) + L^2. \quad (19)$$
Taking into account (16), (17) and (19) we arrive at the following expression for $\text{Var}(V)$:

$$
\text{Var}(V) = EV^2 - (EV)^2
= \frac{1}{T^2} \int_0^T \int_0^T e^{-a(t+s)} \left\{ \gamma^2 \left[ (\sigma^2(0) - L)^2 e^{\gamma^2(t<s)} - \frac{1}{\gamma^2} \right] + \frac{2L(\sigma^2(0)-L)(e^{\gamma^2(t<s)}-e^{\gamma^2(t<s)})}{a-\gamma^2} \right\} dt ds
+ \frac{2L(\sigma^2(0)-L)(e^{\gamma^2(t<s)}-e^{\gamma^2(t<s)})}{a-\gamma^2} dt ds
+ \frac{2L^2 e^{\gamma^2(t<s)}(e^{\gamma^2(t<s)}-e^{\gamma^2(t<s)})}{a-\gamma^2} dt ds.
$$

After calculating the three integrals in (20) we obtain:

$$
\text{Var}(V) = EV^2 - (EV)^2
= \frac{1}{T^2} \int_0^T \int_0^T e^{-a(t+s)} \left\{ \gamma^2 \left[ (\sigma^2(0) - L)^2 e^{\gamma^2(t<s)} - \frac{1}{\gamma^2} \right] + \frac{2L(\sigma^2(0)-L)(e^{\gamma^2(t<s)}-e^{\gamma^2(t<s)})}{a-\gamma^2} \right\} dt ds
+ \frac{2L^2 e^{\gamma^2(t<s)}(e^{\gamma^2(t<s)}-e^{\gamma^2(t<s)})}{a-\gamma^2} dt ds.
$$

From (15) and (21) we get the volatility swap:

$$
EV^\sqrt{\text{Var}} \approx \sqrt{EV} - \frac{\text{Var}(V)}{8(EV)^{3/2}}.
$$

5 Mean-Reverting Risk-Neutral Stochastic Volatility Model

In this section, we are going to obtain the values of variance and volatility swaps under risk-neutral measure $P^*$, using the same arguments as in sections 3-7, where in place of $a$ and $L$ we are going to take $a^*$ and $L^*$

$$
a \rightarrow a^* := a + \lambda \sigma, \; \; L \rightarrow L^* := \frac{aL}{a + \lambda \sigma},
$$

where $\lambda$ is a market price of risk (See section 3).

5.1 Risk Neutral Stochastic Volatility Model (SVM)

Consider our model (1)

$$
d\sigma^2(t) = a(L - \sigma^2(t))dt + \gamma \sigma^2(t)dW_t.
$$

Let $\lambda$ be 'market price of risk' and defind the following constants:

$$
a^* := a + \lambda \sigma, \; \; L^* := \frac{aL}{a^*}.
$$

Then, in the risk-neutral world, the drift parameter in (23) has the following form:

$$
a^*(L^* - \sigma^2(t)) = a(L - \sigma^2(t)) - \lambda \gamma \sigma^2(t).
$$
If we define the following process \((W^*_t)\) by
\[
W^*_t := W_t + \lambda t,
\] (25)
where \(W_t\) is a standard Brownian motion, then the risk neutral stochastic
volatility model has the following form
\[
d\sigma^2(t) = (aL - (a + \lambda \gamma)\sigma^2(t))dt + \gamma \sigma^2(t)dW^*_t,
\]
or, equivalently,
\[
d\sigma^2(t) = a^*(L^* - \sigma^2(t))dt + \gamma \sigma^2(t)dW^*_t,
\] (26)
where
\[
a^* := a + \lambda \gamma, \quad L^* := \frac{aL}{a + \lambda \gamma},
\] (27)
and \(W^*_t\) is defined in (25).

Now, we have the same model in (26) as in (1), and we are going to apply
our change of time method to this model (26) to obtain the values of variance
and volatility swaps.

5.2 Variance and Volatility Swaps for Risk-Neutral SVM

Using the same arguments as in the previous section (where in place of (4) we
have to take (26)) we get the following expressions for variance and volatility
swaps taking into account (27).

For the variance swaps we have (see (14) and (27)):
\[
E^*\sigma^2_R := EV := \frac{1}{T} \int_0^T E\sigma^2(t)dt = \frac{(\sigma^2(0) - L^*)}{a^*T} (1 - e^{-a^*T}) + L^*.
\] (28)

For the volatility swap we obtain (see (22) and (27))
\[
E^*\sqrt{V} \approx \sqrt{E^*V} - \frac{\text{Var}^*(V)}{8(E^*V)^{3/2}}
\] (29)

5.9. Numerical Example: AECO Natural GAS Index (1 May 1998-30 April 1999)

We shall calculate the value of variance and volatility swaps prices of a daily
natural gas contract. To apply our formula for calculating these values we
need to calibrate the parameters \(a, \quad L, \quad \sigma^2_0\) and \(\gamma\) (\(T\) is monthly). These
parameters may be obtained from futures prices for the AECO Natural Gas
Index for the period 1 May 1998 to 30 April 1999 (see Bos, Ware and Pavlov (2002)). The parameters are the following:

<table>
<thead>
<tr>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
</tr>
<tr>
<td>4.6488</td>
</tr>
</tbody>
</table>
For variance swap we use formula (14) and for volatility swap we use formula (22).

From this table we can calculate the values for risk adjusted parameters $a^*$ and $L^*$:

$$a^* = a + \lambda \gamma = 4.9337,$$

and

$$L^* = \frac{aL}{a + \lambda \gamma} = 2.5690.$$

For the value of $\sigma^2(0)$ we can take $\sigma^2(0) = 2.25$.

For variance swap and for volatility swap with risk adjusted parameters we use formula (28) and (29), respectively.

**Appendix: Figures**

Figure 1 depicts variance swap (price vs. maturity) for AECO Natural Gas Index (1 May 1998 to 30 April 1999), using formula (14).

Figure 2 depicts volatility swap (price vs. maturity) for AECO Natural Gas Index (1 May 1998 to 30 April 1999), using formula (22).

Figure 3 depicts variance swap with risk adjusted parameters (price vs. maturity) for AECO Natural Gas Index (1 May 1998 to 30 April 1999), using formula (28).

Figure 4 depicts volatility swap with risk adjusted parameters (price vs. maturity) for AECO Natural Gas Index (1 May 1998 to 30 April 1999), using formula (29).

Figure 5 depicts comparison of adjusted (green line) and non-adjusted price (red line) (naive strike vs. adjusted strike).

Figure 6 depicts convexity adjustment. It’s decreasing with swap maturity (the volatility of volatility over a long period of time is low).
Fig. 1: Variance Swap
Fig. 2: Volatility Swap
Fig. 3: Variance Swap (Risk Adjusted Parameters)
Fig. 4: Volatility Swap (Risk Adjusted Parameters)
Fig. 5: Comparison: Adjusted and Non-Adjusted Price
Fig. 6: Convexity Adjustment
References


Epilogue

'Time you enjoy wasting, was not wasted’,-John Lennon.

The present book was devoted to the history of change of time methods (CTM), connection of CTM with stochastic volatilities and finance, and many applications of CTM. As a reader may noticed, this book is a brief introduction to the theory of CTM and may be considered as a handbook in this area.

I hope that you enjoyed your time while reading this book, and thank you for getting to this very end.