Modeling and Pricing of Variance Swaps for Local Stochastic Volatilities with Delay and Jumps*

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Introduction: Stock Price with Local Stochastic Volatility

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t), \quad t > 0,$$

where $\mu \in R$ is the mean rate of return, the volatility term $\sigma > 0$ is a continuous and bounded function and $W(t)$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$. We also let $r > 0$ be the risk-free rate of return of the market.

We denote $S_t = S(t - \tau), \quad t > 0$ and the initial data of $S(t)$ is defined by $S(t) = \varphi(t)$, where $\varphi(t)$ is a deterministic function with $t \in [-\tau, 0], \quad \tau > 0$, $\varphi(t) > 0$.

Introduction: Stochastic Volatility with Delay and Jumps

$$\begin{aligned}\frac{d\sigma^2(t,S_t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sigma(u,S_u)dW(u) + \int_{t-\tau}^{t}\sigma(u,S_u)d\tilde{N}(u)\right]^2 \\ &- (\alpha+\gamma)\sigma^2(t,S_t)\end{aligned}$$

where $N(t)$ is a Poisson process independent of $W(t)$ with intensity $\lambda > 0$ and $\tilde{N}(t) := N(t) - \lambda t$.

Introduction: Stochastic Volatility with Delay and Jumps (cntd)

Our model of stochastic volatility exhibits *jumps* and also *past-dependence*: the behavior of a stock price right after a given time t not only depends on the situation at t, but also on the whole past (history) of the process $S(t)$ up to time t. This draws some similarities with fractional Brownian motion models (see *Mandelbrot (1997)*) due to a long-range dependence property.

Another advantage of this model is *mean-reversion*. This model is also a continuous-time version of GARCH(1,1) model (see *Bollerslev (1986)*) with jumps.

Motivation: Why Delay?

Some statistical studies of stock prices indicate the dependence on past returns:

- *Sheinkman and LeBaron (1989)*,
- *Akgiray (1989)*
- *Kind, Liptser and Runggaldier (1991)*
- *Hobson and Rogers (1998)*
- *Chang and Yoree (1999)*
- *Mohammed, Arriojas and Pap (2001)*

Motivation: Why Delay? (cntd)

Our work is also based on the GARCH(1,1) model (see *Bollerslev (1986)*)

$$\sigma_n^2 = \gamma V + \alpha \ln^2(S_{n-1}/S_{n-2}) + (1-\alpha-\gamma)\sigma_{n-1}^2$$

or, more general,

$$\sigma_n^2 = \gamma V + \frac{\alpha}{l} \ln^2(S_{n-1}/S_{n-1-l}) + (1-\alpha-\gamma)\sigma_{n-1}^2$$

Motivation: Why Delay? (cntd)

If we write down the last equation in differential form we can get the continuos-time GARCH with expectation of log-returns of zero:

$$\frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau}\ln^2\left(\frac{S(t)}{S(t-\tau)}\right) - (\alpha+\gamma)\sigma^2(t)$$

If we incorporate non-zero expectation of log-return (using Itô Lemma for $\ln\frac{S(t)}{S(t-\tau)}$) then we arrive to our continuous-time GARCH model for stochastic volatility with delay:

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sigma(s,S_s)dW(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t).$$

Motivation: Why Jumps?

There is currently fairly compelling evidence for jumps in the level of financial prices. The most convincing evidence comes from recent nonparametric work using high-frequency data as in *Barndorff-Nielsen and Shephard (2007)* and *Aït-Sahalia and Jacod (2008)* among others. Also, paper by *Todorov and Tauchen (2008)* conducts a non-parametric analysis of the market volatility dynamics using high-frequency data on the VIX index compiled by the CBOE and the $S\&P500$ index.

Motivation: Why Jumps? (cntd)

Some attempts have been made to incorporate jumps in stochastic volatility to price variance and volatility swaps (see, for example, *Howison et al. (2004)*).

Motivation: Why Jumps? (cntd)

The key risk factors considered in option pricing models, besides the diffusive price risk of the underlying asset, are stochastic volatility and jumps, both in the asset price and its volatility. Models that include some or all of these factors were developed by *Merton (1976), Heston (1993), Bates (1996), Bakshi et al. (1997) and Duffie et al. (2000)*.

The Model

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t), \quad t > 0,$$

$$\begin{aligned}\frac{d\sigma^2(t,S_t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(u, S_u)dW(u) + \int_{t-\tau}^{t} \sigma(u, S_u)d\tilde{N}(u)\right]^2 \\ &- (\alpha + \gamma)\sigma^2(t, S_t)\end{aligned}$$

Conditions

C1) $\sigma(t, S_t)$ satisfies local Lipschitz and growth conditions;

C2) $\int_0^T E\sigma^2(t, S_t)dt < +\infty$;

C3) $\int_0^T (\frac{r-\mu}{\sigma(t,S_t)})^2 dt < +\infty$ *a.s.*

Condition C1) guarantees the existence and uniqueness of a solution of equations for $S(t)$ and $\sigma^2(t)$ in Section 2 (see *Mohammed (1998)*). Condition C2) guarantees the existence of Itô integral in equation for $\sigma^2(t)$ and C3) guarantees the existence of risk-neutral measure P^* (see below).

Risk-Neutral World (cntd)

$$dS(t) = rS(t)dt + \sigma(t, S_t)S(t)dW^*(t)$$

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(s, S_s)dW^*(s) + \int_{t-\tau}^{t} \sigma(u, S_u)d\tilde{N}(u) - (\mu - r)\tau\right]^2 - (\alpha + \gamma)\sigma^2(t, S_t).$$

where

$$W^*(t) = \int_0^t \theta(s)ds + W(t)$$

and

$$\theta(t) = \frac{\mu - r}{\sigma(t, S_t)}.$$

Variance Swaps

Variance swaps are forward contracts on future realized stock variance, the square of the future volatility.

The easy way to trade variance is to use variance swaps, sometimes called realized variance forward contracts (see *Carr and Madan (1998), Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999)*).

Variance Swaps (cntd)

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$N(\sigma_R^2(S) - K_{var}),$$

where $\sigma_R^2(S)$ is the realized stock variance (quoted in annual terms) over the life of the contract,

$$\sigma_R^2(S) := \frac{1}{T}\int_0^T \sigma^2(s)ds,$$

Variance Swaps (cntd)

K_{var} is the delivery price for variance, and N is the notional amount of the swap in dollars per annualized volatility point squared.

The holder of variance swap at expiration receives N dollars for every point by which the stock's realized variance $\sigma_R^2(S)$ has exceeded the variance delivery price K_{var}.

Variance Swaps (cntd)

The value of a forward contract $\mathcal{P}$ on future realized variance with strike price K_{var} is the expected present value of the future payoff in the risk-neutral world:

$$\mathcal{P} = E^*\{e^{-rT}(\sigma_R^2(S) - K_{var})\},$$

where r is the risk-free discount rate corresponding to the expiration date T, and E^* denotes the expectation under the risk-neutral measure P^*.

Variance Swaps (cntd)

In tis way, *a variance swap for stochastic volatility with delay* is a forward contract on annualized variance $\sigma_R^2(t, S_t)$. Its payoff at expiration equals to

$$N(\sigma_R^2(S) - K_{var}),$$

where $\sigma_R^2(S)$ is the realized stock variance(quoted in annual terms) over the life of the contract,

$$\sigma_R^2(S) := \frac{1}{T}\int_0^T \sigma^2(u, S(u-\tau))du, \quad \tau > 0.$$

Pricing of Variance Swaps (cntd)

Let us take the expectations under risk-neutral measure $\mathbb{P}^*$ on the both sides of the equation for variance $\sigma^2(t,S_t)$

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sigma(s,S_s)dW^*(s) + \int_{t-\tau}^{t}\sigma(u,S_u)d\tilde{N}(u) - (\mu - r)\tau\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t).$$

Pricing of Variance Swaps (cntd)

Denoting $v(t) = \mathbb{E}^*[\sigma^2(t, S_t)]$, we obtain the following deterministic delay differential equation:

$$\frac{dv(t)}{dt} = \gamma V + \alpha\tau(\mu - r)^2 + \frac{\alpha(1+\lambda)}{\tau}\int_{t-\tau}^{t} v(s)ds - (\alpha+\gamma)v(t).$$

Pricing of Variance Swaps (cntd)

Notice that last equation has a stationary solution ($\gamma > \alpha\lambda$)

$$v(t) \equiv X = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}.$$

Pricing of Variance Swaps (cntd)

Hence, the expectation of the realized variance, or say the fair delivery price K_{var} of *a variance swap for stochastic volatility with delay in stationary regime under risk-neutral measure* $\mathbb{P}^*$ equals to

$$K_{var} = \mathbb{E}^*[v] = \frac{1}{T}\int_0^T v(t)dt$$
$$= \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}.$$

Pricing of Variance Swaps (cntd)

The price P of a variance swap at time t given delivery price K in this case should be:

$$\mathcal{P} = e^{-r(T-t)}[\frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda} - K].$$

Pricing of Variance Swaps (cntd)

In general case, there is no way to write a solution of the above equation for $v(t)$ in explicit form for arbitrarily given initial data. But we can write an approximate solution for $v(t)$ ($\gamma > \alpha\lambda$):

$$v(t) \approx X + Ce^{-\gamma t} = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda} + Ce^{-(\gamma - \alpha\lambda)t}.$$

where

$$C = v(0) - X = \sigma_0^2 - \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}.$$

Pricing of Variance Swaps (cntd)

We note, that the characteristic equation for the equation

$$\frac{dv(t)}{dt} = \gamma V + \alpha\tau(\mu - r)^2 + \frac{\alpha(1+\lambda)}{\tau}\int_{t-\tau}^{t} v(s)ds - (\alpha+\gamma)v(t).$$

in this case has the following look

$$\rho^2 + \rho(\gamma - \alpha\lambda) = 0$$

and the solution of the equation is

$$\rho = (\alpha\lambda - \gamma).$$

Pricing of Variance Swaps (cntd)

Hence, the expectation of the realized variance, or say the fair delivery price K_{var} of *variance swap for stochastic volatility with delay in general case under risk-neutral measure* $\mathbb{P}^*$ equals to

$$\begin{aligned} K_{var} &= \mathbb{E}^*[v] = \frac{1}{T}\int_0^T v(t)dt \\ &\approx \frac{1}{T}\int_0^T [V + \alpha\tau(\mu - r)^2/\gamma + (\sigma_0^2 - V - \alpha\tau(\mu - r)^2/\gamma)e^{(\alpha\lambda-\gamma)t}]dt \\ &= \frac{\gamma V + \alpha\tau(\mu-r)^2}{\gamma-\alpha\lambda} + (\sigma_0^2 - \frac{\gamma V + \alpha\tau(\mu-r)^2}{\gamma-\alpha\lambda})\frac{e^{(\alpha\lambda-\gamma)T}-1}{T(\alpha\lambda-\gamma)}. \end{aligned}$$

Pricing of Variance Swaps (cntd)

The price $\mathcal{P}$ of a variance swap at time t given delivery price K in this case should be:

$$\mathcal{P} \approx e^{-r(T-t)}[\frac{\gamma V+\alpha\tau(\mu-r)^2}{\gamma-\alpha\lambda} + (\sigma_0^2 - \frac{\gamma V+\alpha\tau(\mu-r)^2}{\gamma-\alpha\lambda})\frac{e^{-(\gamma-\alpha\lambda)(T-t)}-1}{(T-t)(\alpha\lambda-\gamma)} - K].$$

Numerical Example: $S\&P60$ Canada Index

Statistics on Log Returns $S\&P60$ Canada Index	
Series:	Log Returns $S\&P60$ Canada Index
Sample:	1 1300
Observations:	1300
Mean	0.000235
Median	0.000593
Maximum	0.051983
Minimum	-0.101108
Std. Dev.	0.013567
Skewness	-0.665741
Kurtosis	7.787327

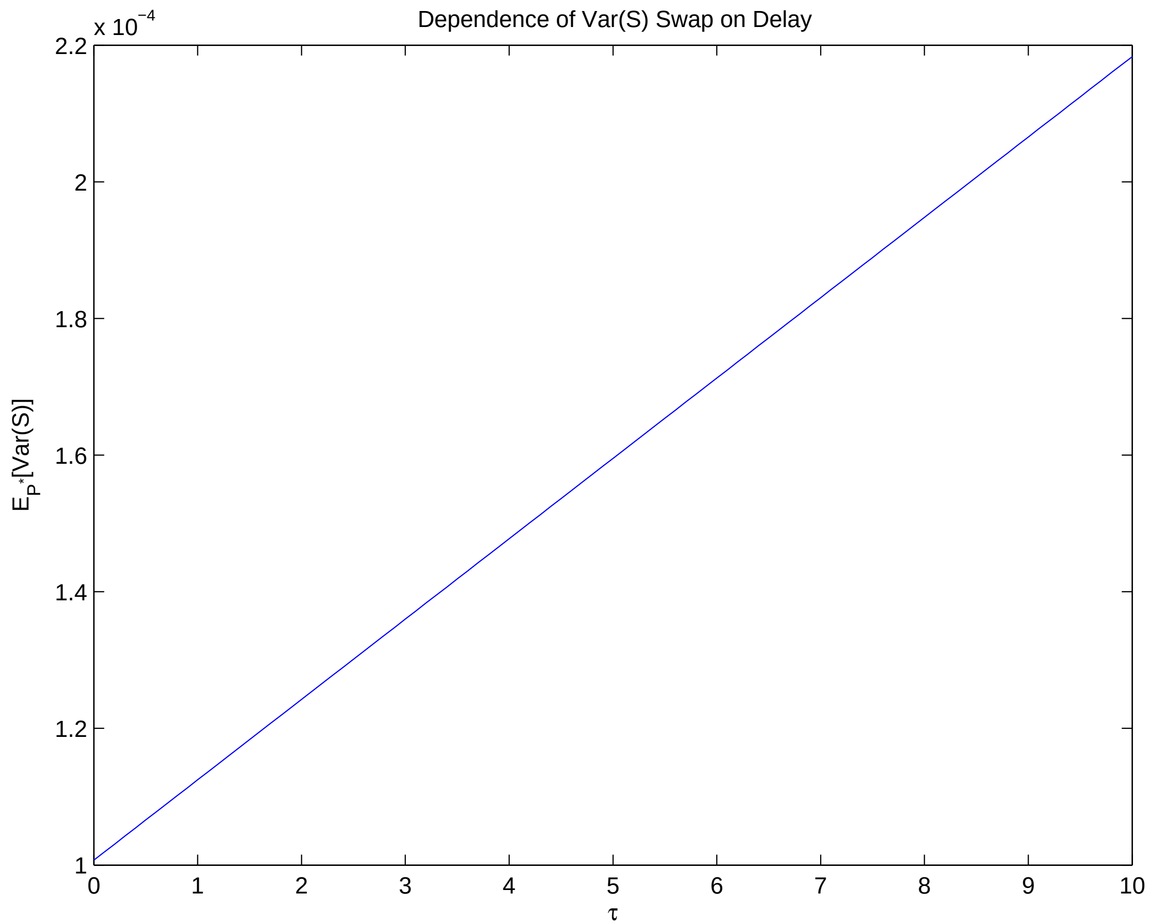

Fig. 1. Dependence of Variance Swap with Delay on Delay ($S\&P60$ Canada Index).

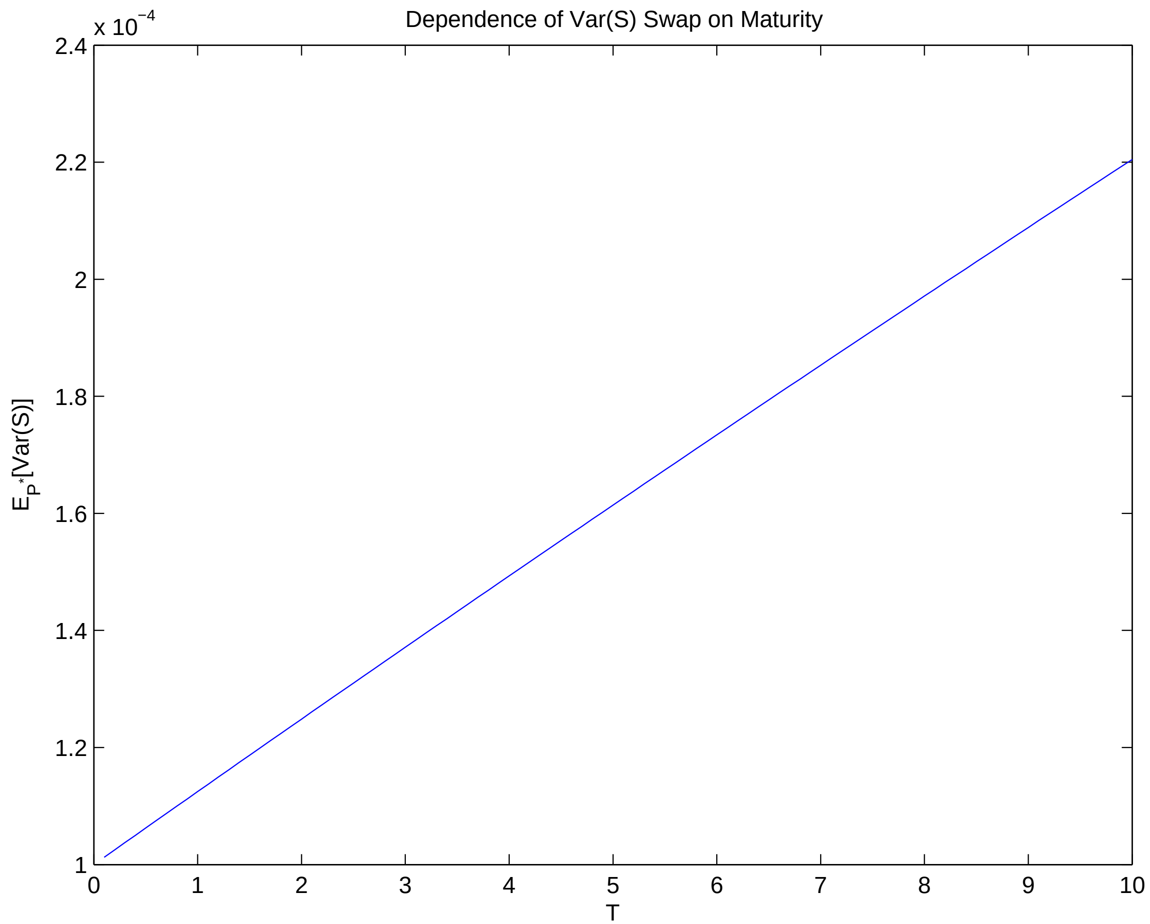

Fig. 2. Dependence of Variance Swap with Delay on Maturity ($S\&P60$ Canada Index).

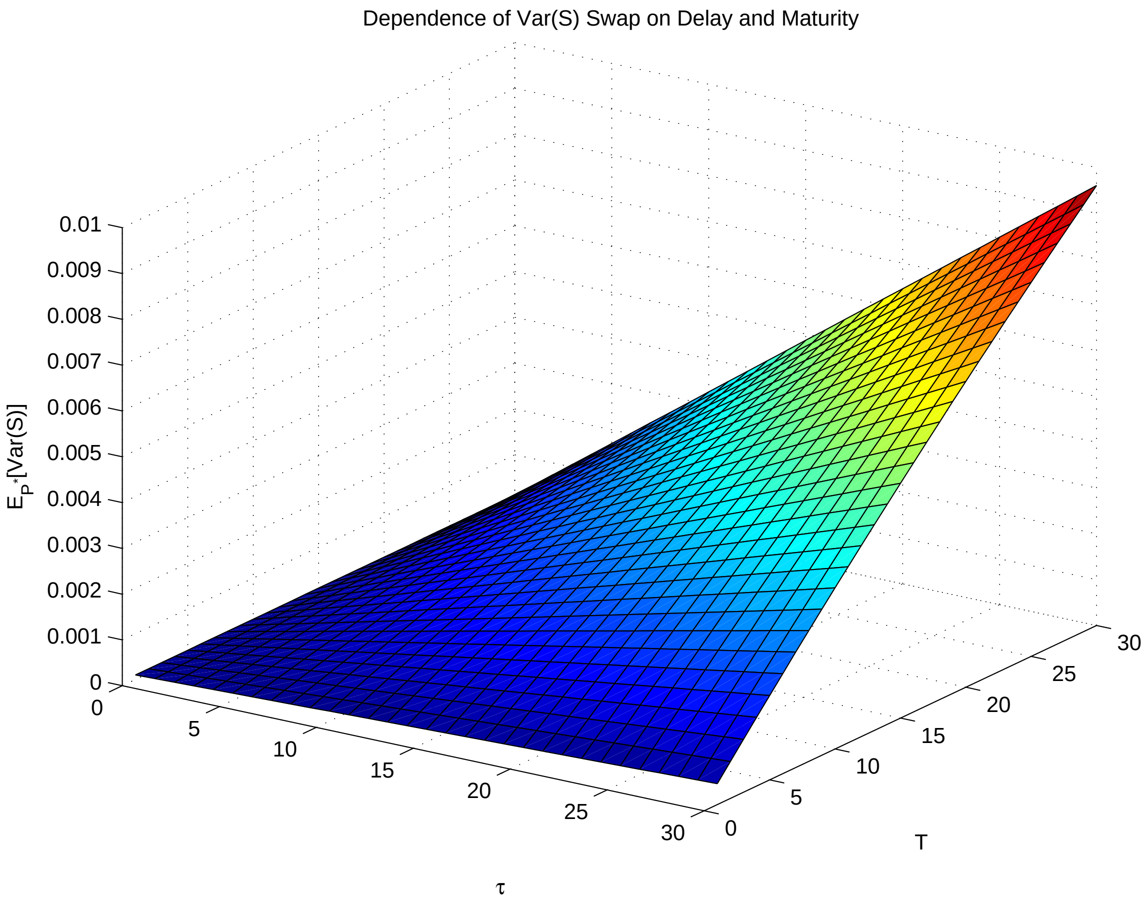

Fig. 3. Variance Swap with Delay for $S\&P60$ Canada Index.

Numerical Example 2: $S\&P500$ Index

Statistics on Log Returns $S\&P500$ Index	
Series:	Log returns $S\&P500$ Index
Sample:	1 1006
Observations:	1006
Mean	0.000263014
Median	8.84424E-05
Maximum	0.034025839
Minimum	-0.045371484
Std. Dev.	0.00796645
Variance	6.34643E-05
Skewness	-0.178481359
Kurtosis	3.296144083

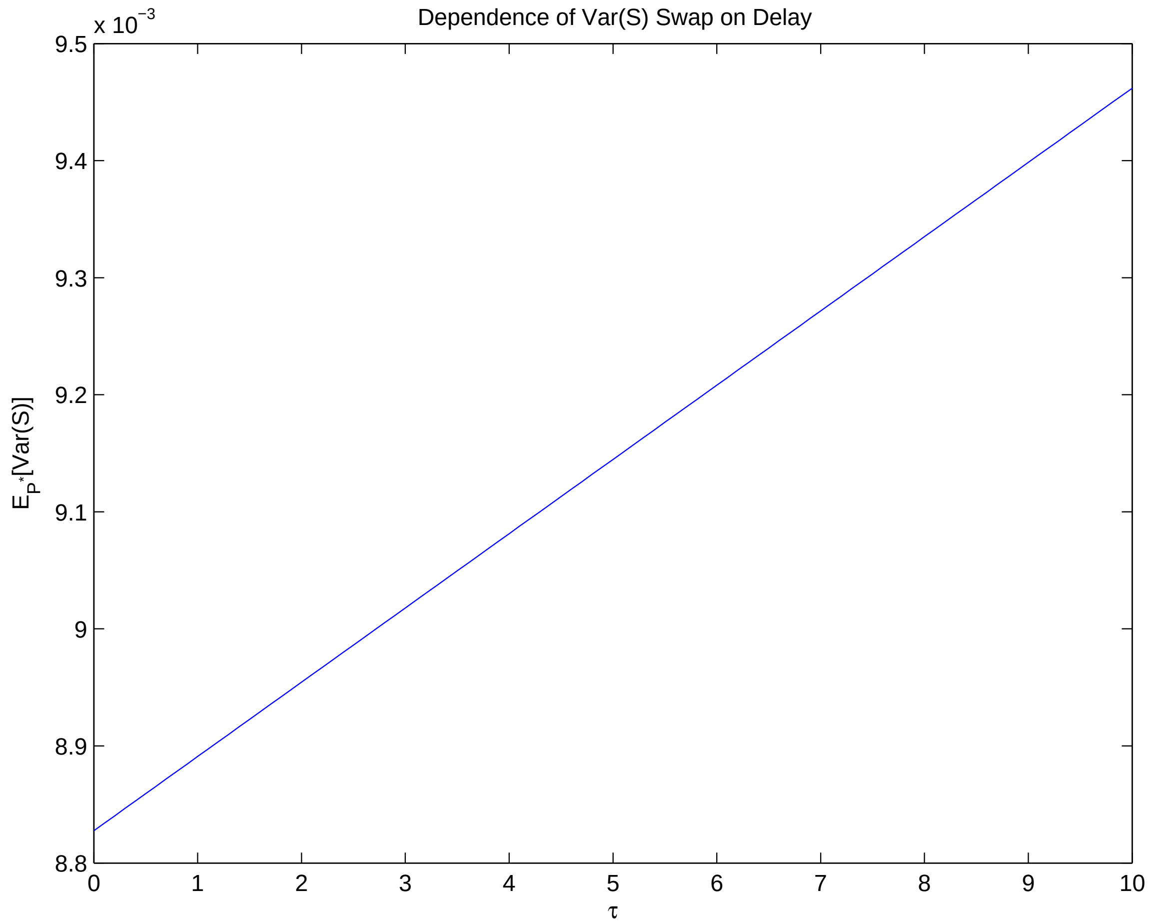

Fig. 4. Dependence of Variance Swap with Delay on Delay ($S\&P500$ Index).

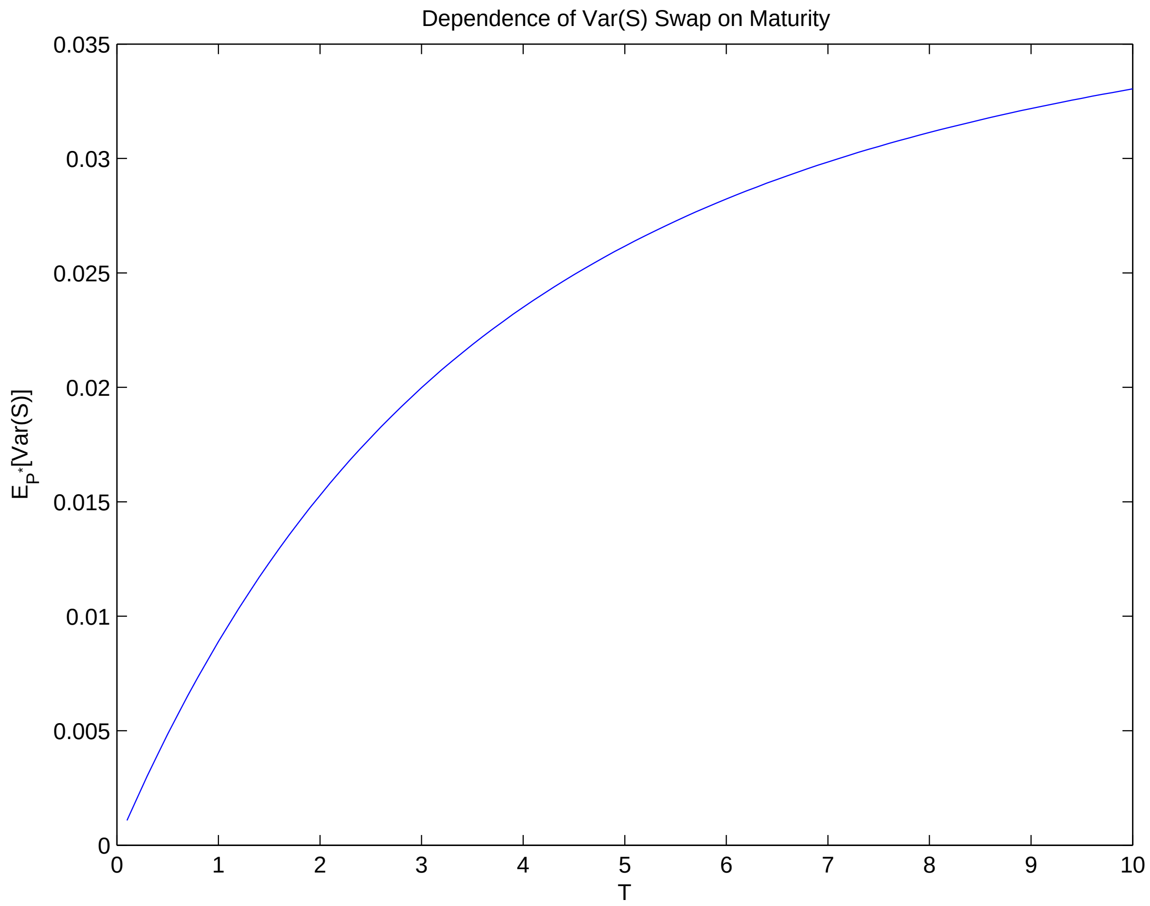

Fig. 5. Dependence of Variance Swap with Delay on Maturity ($S\&P500$ Index).

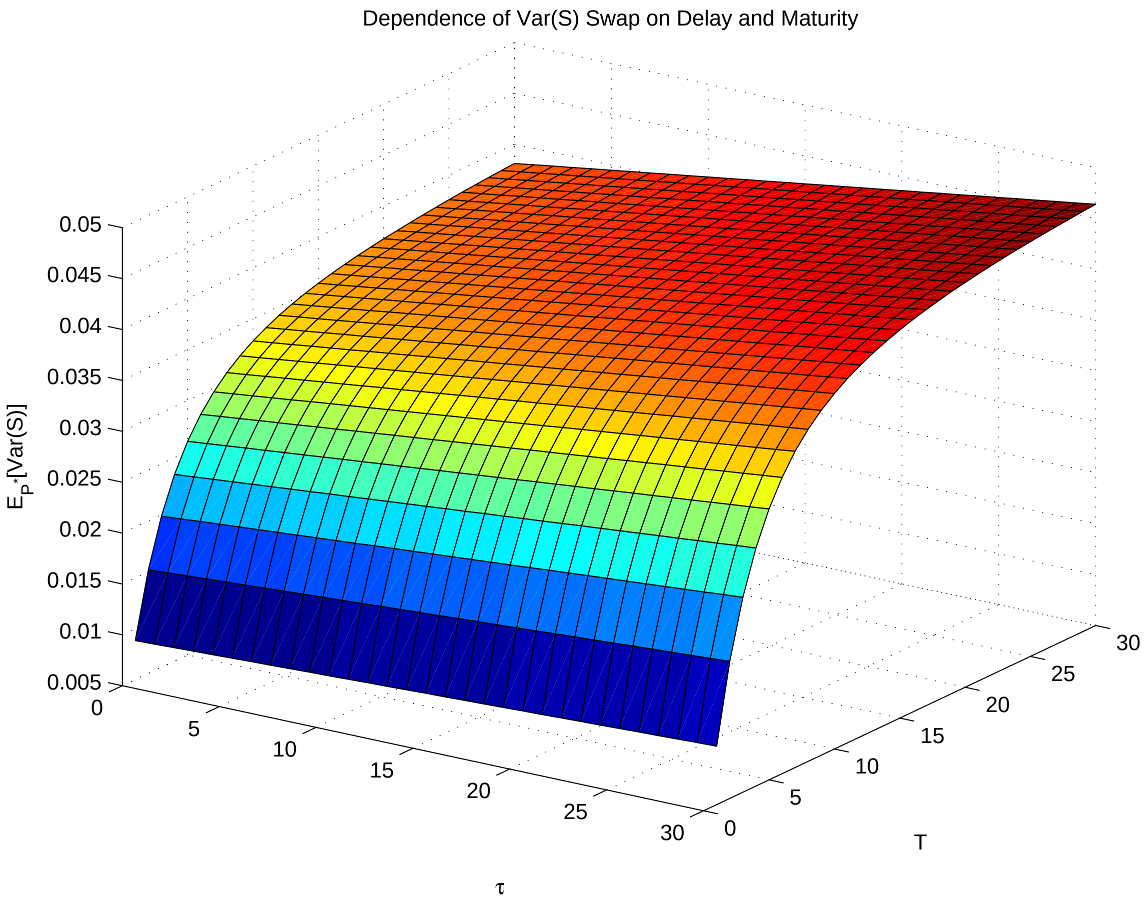

Fig. 6. Variance Swap with Delay for $S\&P500$ Index.

The End

Thank You for Your Time and Attention!