Lévy-Based Interest Rate Derivatives: Change of Time Method and PIDEs*

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Outline of Presentation

1. Literature Review and Bond Pricing

2. Stochastic Interest Rates (SIRs): Gaussian and Lévy-based

3. Change of Time Method and Pricing of Zero-Coupon Bonds

4. Interest Rate Derivatives (IRD)(Bond Options, Swaps, Caps, Floors, Swaptions, Captions and Floortions) and Pricing of IRD: PIDEs

5. Conclusion

Literature Review

Introduction to Interest Rate Models: Rebonato (1996), Bingham and Kiesel (1998), Björk (1998), James and Webber (2000), Pelsser (2000), Brigo and Mercurio (2001), Filipovic (2001)

Two-Factor Interest Rate Models: Schwartz (1979), Courtadon (1982), Schaefer and Schwartz (1984)

Multifactor Affine-Yield Models: Dai and Singleton (2000)

One-Dimensional CIR Equation with time-varying parameters: Maghsoodi (1996)

Literature Review (cntd)

Common Short-Rate Models: Vasićek (1977), Cox-Ingersoll-Ross (1985), Ho and Lee (1986), Heath, Jarrow and Morton (1987), Black, Derman and Toy (1990), Black-Karasinski (1991), Longstaff and Schwartz (2001)

Empirical comparison of various Short Rate Models: Chan, Karoly, Longstaff and Sanders (1992)

Use of a Log-Normal Simple Interest Rate to Price Caps and Floors: Miltersen, Sandman and Sondermann (1997) LIBOR Term-Structure Model: Brace, Gatarek and Musiela (1997)

Swaps Interest Rate Market: Jamshidian (1990)

Literature Review (cntd)

Term-Structure Models with Jumps: Björk, Kabanov and Rungaldier (1997), Das (1999), Das and Foresi (1996), Glasserman and Kou (2003), Glasserman and Merener (2003), Shirakawa (1991)

Levy-Based Interest Rate Models: Eberlein and Raible (1999), Raible (2000)

Change of Time Method (CTM): *Gaussian Models*: Ikeda & Watanabe (1981); *Levy-Based Models*: Rosinski & Woyczynski (1986)

Applications of CTM in Finance: Madan & Seneta (1990) (VG), Carr, Geman, Madan & Yor (2002) (CGMY), Swishchuk (2004, 2007) (pricing options and swaps)

Bond Pricing

A *bond* is a contract, paid for up-front, that yields a known amount on a known date in the future, $t = T$. If $P(t,T)$ is the value of zero-coupon bond at time t, then

$$P(t,T) = \exp(-\int_t^T r(s)ds),$$

where $r(t)$ is the instanteneous interest rate.

If there is no coupon the bond is known as a *zero-coupon bond*.

Bond Pricing (cntd)

Bond Pricing with Known Interest Rates:

Let $V(t, T)$ be the value of the bond contract, $t < T$. If the interest rate $r(t)$ and coupon payment $K(t)$ are known functions of time, the bond price is also a function of time only: $V = V(t, T)$. If this bond pays the owner Z at time $t = T$ then we know that $V(T, T) = Z$. We now derive an equation for the value of the bond at a time before maturity, $t < T$.

Suppose we hold one bond. Then arbitrage considerations lead us to the following equation:

$$\frac{dV}{dt} + K(t) = r(t)V.$$

(K(t)-coupon payment, V(t,T)-value of bond conract)

Bond Pricing (cntd)

Yield Curve:

$$Y(t,T) = -\frac{\log(V(t,T)/V(T,T))}{T-t},$$

where t is the current time.

The Short Rate:

$$r(t) := Y(t,t) = \lim_{T\to t} Y(t,T)$$

Bond Pricing (cntd)

The *short rate* is the rate on instantaneous borrowing and lending. Historically, it was the short rate which was modelled as the basic process. In practice, this rate is stochastic and can fluctuate over time. Note that the short rate is actually a theoretical entity which does not exist in real life and can not be directly observed.

A sum of 1 invested in the short rate at time zero and continuously rolled over, i.e., instantaneously reinvested, is called the money-market account. Its value $S_0(t) = \exp[\int_0^t r(s)ds]$. If r is deterministic and constant, $S_0(t)$ reduces to the classical *bank account*: $S_0(t) = B(t) = \exp[rt]$.

Stochastic Interest Rates (SIRs)

Basic Model: Cox-Ingersoll-Ross (CIR) (1985). It is based on the CIR process:

$$dr(t) = k(\theta - r(t))dt + \gamma\sqrt{r}dW(t).$$

$P(t,u)$-*price of zero-coupon bond*

$$P(t,u) = E^*[\exp(-\int_t^u r(s)ds)|\mathcal{F}_t]$$

Stochastic Interest Rates (SIRs)

"Riskless" asset

$$S_t^0 = \exp\left(\int_0^t r(s)ds\right)$$

Equation for $P(t,u)$

$$\frac{dP(t,u)}{P(t,u)} = (r(t) - \sigma(t,u)q(t))dt + \sigma(t,u)dW(t),$$

Risk-Neutral World:

$$\frac{dP(t,u)}{P(t,u)} = r(t)dt + \sigma(t,u)dW^*(t).$$

One-Factor and Multi-Factor Gaussian Interest Rate Models

One-Factor Gaussian SIRMs

1. *The Geometric Brownian Motion Model* (Rendleman and Bartter (1980)). $dr(t) = \mu r(t)dt + \sigma r(t)dW(t)$.

2. *The Ornstein-Uhlenbeck* (1930) Model. $dr(t) = -\mu r(t)dt + \sigma dW(t)$,

3. *The Vasićek* (1977) Model. $dr(t) = \mu(b - r(t))dt + \sigma dW(t)$.

4. *The Continuous-Time GARCH Model*. $dr(t) = \mu(b - r(t))dt + \sigma r(t)dW(t)$.

One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

One-Factor Gaussian SIRMs

5. *The Cox-Ingersoll-Ross* (1985) Model. $dr(t) = k(\theta - r(t))dt + \gamma\sqrt{r}dW(t)$.

6. *The Ho and Lee* (1986) Model. $dr(t) = \theta(t)dt + \sigma dW(t)$.

7. *The Hull and White* (1990) Model. $dr(t) = (a(t) - b(t)r(t))dt + \sigma(t)dW(t)$

One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

One-Factor Gaussian SIRMs

8. *The Heath, Jarrow and Morton* (1987) Model. Define the forward interest rate $f(t,s)$, for $t \leq s$, characterized by the following equality $P(t,u) = \exp[-\int_t^u f(t,s)ds]$ for any maturity u. $f(t,s)$ represents the instanteneous interest rate at time s as 'anticipated' by the market at time t. It is natural to set $f(t,t) = r(t)$. The process $f(t,u)_{0 \leq t \leq u}$ satisfies an equation $f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dW(v)$, where the processes a and b are continuous. We note, that the lasr SDE may be written in the following form: $df(t,u) = b(f(t,u))(\int_t^u b(f(t,s)))ds + b(f(t,u))d\widehat{W}(t)$, where $\widehat{W}(t) = W(t) - \int_0^t q(s)ds$ and
$q(t) = \int_t^u b(f(t,s))ds - \frac{a(t,u)}{b(f(t,u))}$.

One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Multi-Factor Gaussian SIRMs

Multi-factor models driven by Brownian motions can be obtained using various combinations of above-mentioned processes. We give one example of two-factor continuous-time GARCH SIRM:

$$\begin{cases} dr(t) &= \mu(b(t) - r(t))dt + \sigma r(t)dW^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW^2(t), \end{cases}$$

where W^1, W^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Change of Time Method for SDE driven by Brownian motion

Definition 1. A *time change* is a right-continuous increasing $[0, +\infty]$-valued process $(T_t)_{t \in R_+}$ such that T_t is a stopping time for any $t \in R_+$. By $\hat{\mathcal{F}}_t := \mathcal{F}_{T_t}$ we define the time-changed filtration $(\hat{\mathcal{F}}_t)_{t \in R_+}$. The *inverse time change* $(\hat{T}_t)_{t \in R_+}$ is defined as $\hat{T}_t :=$ inf$\{s \in R_+ : T_s > t\}$. (See Ikeda and Watanabe (1983)).

We consider the following SDE driven by a *Brownian motion*:

$$dX(t) = a(t, X(t))dW(t),$$

where $W(t)$ is a Brownian motion and $a(t, X)$ is a continuous and measurable by t and X function on $[0, +\infty) \times R$.

Change of Time Method for SDE driven by Brownian motion (cntd)

Theorem. (*Ikeda and Watanabe* (1981), Chapter IV, Theorem 4.3) Let $\widehat{W}(t)$ be an one-dimensional $\mathcal{F}_t$-Wiener process with $\widehat{W}(0) = 0$, given on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and let $X(0)$ be an $\mathcal{F}_0$-adopted random variable.

Change of Time Method for SDE driven by Brownian motion (cntd)

Define a continuous process $V = V(t)$ by the equality

$$V(t) = X(0) + \widehat{W}(t).$$

Let T_t be the change of time process:

$$T_t = \int_0^t a^{-2}(T_s, X(0) + \widehat{W}(s))ds.$$

If

$$X(t) := V(\hat{T}_t) = X(0) + \widehat{W}(\hat{T}_t)$$

and $\widehat{\mathcal{F}}_t := \mathcal{F}_{\hat{T}_t}$, then there exists $\widehat{\mathcal{F}}_t$-adopted Wiener process $W = W(t)$ such that $(X(t), W(t))$ is a solution of $dX(t) = a(t, X(t))dW(t)$ on probability space $(\Omega, \mathcal{F}, \widehat{\mathcal{F}}_t, P)$, where $\hat{T}_t$ is the inverse to T_t time change.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models

Solution of One-Factor Gaussian SIRMs Using CTM

We use change of time method (see Ikeda and Watanabe (1981)) to get the solutions to the following below equations (see Swishchuk (2007)). $W(t)$ below is an standard Brownian motion, and $\hat{W}$ is a $(\hat{T}_t)_{t\in R_+}$-adapted standard Brownian motion on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t\in R_+}, P)$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Solution of One-Factor Gaussian SIRMs Using CTM

1. *Geometric Brownian Motion*. $dr(t) = \mu r(t)dt + \sigma r(t)dW(t)$. Solution $r(t) = e^{\mu t}[r(0)+\widehat{W}(\widehat{T}_t)]$, where $\widehat{T}_t = \sigma^2 \int_0^t [r(0)+\widehat{W}(\widehat{T}_s)]^2 ds$.

2. *Ornstein-Uhlenbeck Process*. $dr(t) = -\mu r(t)dt + \sigma dW(t)$, Solution $r(t) = e^{-\mu t}[r(0) + \widehat{W}(\widehat{T}_t)]$, where $\widehat{T}_t = \sigma^2 \int_0^t (e^{\mu s}[r(0) + \widehat{W}(\widehat{T}_s)])^2 ds$.

3. *Vasićek Process*. $dr(t) = \mu(b - r(t))dt + \sigma dW(t)$, solution $r(t) = e^{-\mu t}[r(0) - b + \widehat{W}(\widehat{T}_t)]$, where $\widehat{T}_t = \sigma^2 \int_0^t (e^{\mu s}[r(0) - b + \widehat{W}(\widehat{T}_s)] + b)^2 ds$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models

Solution of One-Factor Gaussian SIRMs Using CTM (cntd)

4. *Continuous-Time GARCH Process*. $dr(t) = \mu(b - r(t))dt + \sigma r(t)dW(t)$. Solution $r(t) = e^{-\mu t}(r(0) - b + \widehat{W}(\widehat{T}_t)) + b$, where $\widehat{T}_t = \sigma^2 \int_0^t [r(0) - b + \widehat{W}(\widehat{T}_s) + e^{\mu s} b)^2 ds$.

5. *Cox-Ingersoll-Ross Process*. $dr^2(t) = k(\theta - r^2(t))dt + \gamma r(t)dW(t)$, solution $r^2(t) = e^{-kt}[r_0^2 - \theta^2 + \widehat{W}(\widehat{T}_t)] + \theta^2$, where $T_t = \gamma^{-2} \int_0^t [e^{kT_s}(r_0^2 - \theta^2 + \widehat{W}(s)) + \theta^2 e^{2kT_s}]^{-1} ds$.

6. *Ho and Lee Process*. $dr(t) = \theta(t)dt + \sigma dW(t)$. Solution $r(t) = r(0) + \widehat{W}(\sigma^2 t) + \int_0^t \theta(s)ds$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models

Solution of One-Factor Gaussian SIRMs Using CTM (cntd)

7. *Hull and White*. $dr(t) = (a(t) - b(t)r(t))dt + \sigma(t)dW(t)$.
Solution $r(t) = \exp[-\int_0^t b(s)ds][r(0) - \frac{a(s)}{b(s)} + \widehat{W}(\hat{T}_t)],$
where $\hat{T}_t = \int_0^t \sigma^2(s)[r(0) - \frac{a(s)}{b(s)} + \widehat{W}(\hat{T}_s) + \exp[\int_0^s b(u)du]\frac{a(s)}{b(s)}]^2 ds.$

8. *Heath, Jarrow and Morton*. $f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dW(v)$. Solution $f(t,u) = f(0,u) + \widehat{W}(\hat{T}_t) + \int_0^t a(v,u)dv,$
where $\hat{T}_t = \int_0^t b^2(f(0,u) + \widehat{W}(\hat{T}_s) + \int_0^s a(v,u)dv)ds.$

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Solution of Multi-Factor Gaussian SIRMs Using CTM

Solution of *multi-factor models driven by Brownian motions* can be obtained using various combinations of solutions of the above-mentioned processes, see subsection 5.1, and CTM. We give one example of two-factor Continuous-Time GARCH model driven by Brownian motions:

$$\begin{cases} dr(t) &= \mu(b(t)-r(t))dt + \sigma r(t)dW^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW^2(t), \end{cases}$$

where W^1, W^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Solution of Multi-Factor Gaussian SIRMs Using CTM

Solution, using CTM for the first and the second equations:

$$r(t) = e^{-\mu t}[r(0) - e^{\xi t}(b(0) + \hat{W}^2(\hat{T}_t^2)) + \hat{W}^1(\hat{T}_t^1)] + e^{\xi t}[b(0) + \hat{W}^2(\hat{T}_t^2)],$$

where $\hat{T}^i$ are defined in 4. ($i = 1$) and 1. ($i = 2$), respectevely. Here, $W^1(t)$ and $W^2(t)$ are independent.

Lévy-based Stochastic Interest Rate Models (SIRMs)

Lévy Processes

Definition 2. By *Lévy process* we define a stochastically continuous process with *stationary and independent increments*, Sato (1999), Applebaum (2003), Schoutens (2003).

Examples of Lévy Processes:

- linear function $L(t) = \gamma t$
- Brownian motion with drift
- Poisson process
- compound Poisson process

Lévy-based Stochastic Interest Rate Models (SIRMs)

Lévy-Khintchine Formula for Lévy Processes

$$\begin{aligned} E(e^{i(u,X(t))}) &= \exp\{t[i(u,\gamma) - \tfrac{1}{2}(u,Au) \\ &+ \int_{R^d-\{0\}}[e^{i(u,y)} - 1 - i(u,y)\mathbf{1}_{B_1(0)}]\nu(dy)]\} \end{aligned}$$

(γ, A, ν)-*Lévy-Khintchine triplet*

Lévy-based Stochastic Interest Rate Models (SIRMs)

Interpretation of Lévy-Khintchine triplet

- γ stands for linear function, drift
- Diffusion matrix A stands for Brownian motion
- Lévy measure ν stands for jumps

Lévy-based Stochastic Interest Rate Models (SIRMs)

Lévy-Itô Decomposition

If X is a Lévy process, then rhere exists $\gamma \in R^d$, a Brownian motion B_A with covariance matrix A and an independent Poisson random measure N on $R^+ \times (R^d - \{0\})$ such that, for each $t \geq 0$,

$$X(t) = \gamma t + B_A(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx).$$

Lévy-based Stochastic Interest Rate Models (SIRMs)

Lévy Processes in Finance

- Brownian motion with drift (only continuous Lévy process)
- Merton model=Brownian motion+drift+Gaussian jumps
- Kou model=Brownian motion+drift+exponential jumps
- VG, IG, NIG, Generalized hyperbolic processes
- *α-stable Lévy processes*

α-stable Lévy Processes

Definiton 3. Let $\alpha \in (0,2]$. An *α-stable Lévy process* L such that L_1 (or equivalently any L_t) has a strictly α-stable distribution (i.e., $L_1 \equiv S_\alpha(\sigma,\beta,\nu)$) for some $\alpha \in (0,2] \setminus \{1\}$, $\sigma \in R_+$, $\beta \in [-1,1]$, $\nu = 0$ or $\alpha = 1$, $\sigma \in R_+$, $\beta = 0$, $\nu \in R$). (See Sato (1999)). We call L a *symmetric α-stable Lévy process* if the distribution of L_1 is even symmetric α-stable (i.e., $L_1 \equiv S_\alpha(\sigma,0,0)$ for some $\alpha \in (0,2]$, $\sigma \in R_+$.) A process L is called $(T_t)_{t\in R_+}$-adapted if L is constant on $[T_{t-}, T_t]$ for any $t \in R_+$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

α-stable Lévy Processes (cntd)

- the only self-similar Lévy processes

- either Brownian motion or pure jump

- characteristic exponent, Lévy-Khintchine triplet known in closed form

- 4 parameters

- infinite variance (except for Brownian motion)

Lévy-based Stochastic Interest Rate Models (SIRMs)

α-stable Lévy Processes (cntd)

• α-stable Lévy Processes are semimartingales ($\int_0^t f_s dL_s$ can be defined)

• α-stable Lévy Processes are pure discontinuous Markov processes with generator

$$Af(x) = \int_{R-\{0\}} [f(x+y) - f(x) - yf'(y)\mathbf{1}_{|y|<1}(y)] \frac{K_\alpha}{|y|^{1+\alpha}} dy$$

• Lévy-Khintchine triplet is $(0, 0, \nu)$, where $\nu(dx) = \frac{\alpha dx}{\Gamma(1-\alpha)|x|^{1+\alpha}}$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy SIRMs

$L(t)$ below is a symmetric α-stable Lévy process. We define below various processes via SDE driven by α-stable Lévy process.

1. *Geometric α-stable Lévy Motion*. $dr(t) = \mu r(t-)dt + \sigma r(t-)dL(t)$.

2. *Ornstein-Uhlenbeck Process Driven by α-stable Lévy Motion*. $dr(t) = -\mu r(t-)dt + \sigma dL(t)$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy SIRMs

3. *Vasićek Process Driven by α-stable Lévy Motion*.
$dr(t) = \mu(b - r(t-))dt + \sigma dL(t)$.

4. *Continuous-Time GARCH Process Driven by α-stable Lévy process*. $dr(t) = \mu(b - r(t-))dt + \sigma r(t-)dL(t)$.

5. *Cox-Ingersoll-Ross Process Driven by α-stable Lévy Motion*.
$dr(t) = k(\theta - r(t-))dt + \gamma\sqrt{r(t-)}dL(t)$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy SIRMs

6. *Ho and Lee Process Driven by α-stable Lévy Motion*.
$dr(t) = \theta(t-)dt + \sigma dL(t).$

7. *Hull and White Process Driven by α-stable Lévy Motion*.
$dr(t) = (a(t-) - b(t-)r(t-))dt + \sigma(t)dL(t)$

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy SIRMs

8. *Heath, Jarrow and Morton Process Driven by α-stable Lévy Motion*. Define the forward interest rate $f(t,s)$, for $t \leq s$, characterized by the following equality $P(t,u) = \exp[-\int_t^u f(t,s)ds]$ for any maturity u. $f(t,s)$ represents the instanteneous interest rate at time s as 'anticipated' by the market at time t. It is natural to set $f(t,t) = r(t)$. The process $f(t,u)_{0 \leq t \leq u}$ satisfies an equation $f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dL(v)$, where the processes a and b are continuous.

Lévy-based Stochastic Interest Rate Models (SIRMs)

Multi-Factor Lévy SIRMs

Multi-factor models driven by α-stable Lévy motions can be obtained using various combinations of above-mentioned processes, see subsection 3.3. We give one example of two-factor continuous-time GARCH model driven by α-stable Lévy motions.

$$\begin{cases} dr(t) &= \mu(b(t-) - r(t-))dt + \sigma r(t-)dL^1(t) \\ db(t) &= \xi b(t-)dt + \eta b(t-)dL^2(t), \end{cases}$$

where L^1, L^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

Multi-Factor Lévy SIRMs

Also, we can consider various combinations of models, presented above, i.e., mixed models containing Brownian and Lévy motions. For example,

$$\begin{cases} dr(t) &= \mu(b(t-) - r(t-))dt + \sigma r(t-)dL(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW(t), \end{cases}$$

where Brownian motion $W(t)$ and Lévy process $L(t)$ may be correlated.

Change of Time Method for SDE Driven by Levy Motion

We denote by $L^{\alpha}_{a.s.}$ the family of all real measurable $\mathcal{F}_t$-adapted processes a on $\Omega \times [0, +\infty)$ such that for every $T > 0$, $\int_0^T |a(t, \omega)|^{\alpha} dt < +\infty$ a.s. We consider the following *SDE driven by a Lévy motion*:

$$dX(t) = a(t, X(t-))dL(t).$$

Change of Time Method for SDE Driven by Levy Motion

Theorem. *(Rosinski and Woyczynski (1986)*, Theorem 3.1., p.277). Let $a \in L^{\alpha}_{a.s.}$ be such that $T(u) := \int_0^u |a|^{\alpha} dt \to +\infty$ a.s. as $u \to +\infty$. If $\hat{T}(t) := \inf\{u : T(u) > t\}$ and $\hat{\mathcal{F}}_t = \mathcal{F}_{\hat{T}(t)}$, then the time-changed ctochastic integral $\hat{L}(t) = \int_0^{\hat{T}(t)} a dL(t)$ is an $\hat{\mathcal{F}}_t - \alpha$-stable Lévy process, where $L(t)$ is $\mathcal{F}_t$-adapted and $\mathcal{F}_t$-α-stable Lévy process. Consequently, a.s. for each $t > 0$ $\int_0^t a dL = \hat{L}(T(t))$, i.e., the stochastic integral with respect to a α-stable Lévy process is nothing but another α-stable Lévy process with randomly changed time scale.

Solutions of One-Factor Lévy-based SIRMs using CTM

Below we give the solutions to some one-factor Lévy IRM decribed by SDE driven by α-stable Lévy process. $L(t)$ below is a symmetric α-stable Lévy process, and $\hat{L}$ is a $(\hat{T}_t)_{t\in R_+}$-adapted symmetric α-stable Levy process on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t\in R_+}, P))$.

1. *Geometric α-stable Lévy Motion*. $dr(t) = \mu r(t-)dt + \sigma r(t-)dL(t)$. *Solution* $r(t) = e^{\mu t}[r(0) + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t [r(0) + \hat{L}(\hat{T}_s)]^\alpha ds$.

2. *Ornstein-Uhlenbeck Process Driven by α-stable Lévy Motion*. $dr(t) = -\mu r(t-)dt + \sigma dL(t)$. *Solution* $r(t) = e^{-\mu t}[r(0) + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t (e^{\mu s}[r(0) + \hat{L}(\hat{T}_s)])^\alpha ds$.

Change of Time Method for SDE Driven by Levy Motion

3. *Vasićek Process Driven by α-stable Lévy Motion*. $dr(t) = \mu(b - r(t-))dt + \sigma dL(t)$. Solution $r(t) = e^{-\mu t}[r(0) - b + \hat{L}(\hat{T}_t)],$ where $\hat{T}_t = \sigma^\alpha \int_0^t (e^{\mu s}[r(0) - b + \hat{L}(\hat{T}_s)] + b)^\alpha ds$.

4. *Continuous-Time GARCH Process Driven by α-stable Lévy process*. $dr(t) = \mu(b - r(t-))dt + \sigma r(t-)dL(t)$. Solution $r(t) = e^{-\mu t}(r(0) - b + \hat{L}(\hat{T}_t)) + b,$ where $\hat{T}_t = \sigma^\alpha \int_0^t [r(0) - b + \hat{L}(\hat{T}_s) + e^{\mu s} b]^\alpha ds$.

5. *Cox-Ingersoll-Ross Process Driven by α-stable Lévy Motion*. $dr(t) = k(\theta^2 - r(t-))dt + \gamma\sqrt{r(t-)}dL(t)$. Solution $r^2(t) = e^{-kt}[r_0^2 - \theta^2 + \hat{L}(\hat{T}_t)] + \theta^2,$ where $T_t = \gamma^{-\alpha} \int_0^t [e^{kT_s}(r_0^2 - \theta^2 + \hat{W}(s)) + \theta^2 e^{2kT_s}]^{-\alpha/2} ds$.

Change of Time Method for SDE Driven by Levy Motion

6. *Ho and Lee Process Driven by α-stable Lévy Motion*. $dr(t) = \theta(t-)dt + \sigma dL(t)$. *Solution* $r(t) = r(0) + \hat{L}(\sigma^2 t) + \int_0^t \theta(s)ds.$

7. *Hull and White Process Driven by α-stable Lévy Motion*. $dr(t) = (a(t-) - b(t-)r(t-))dt + \sigma(t-)dL(t)$.
Solution $r(t) = \exp[-\int_0^t b(s)ds][r(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T}_t)]$,
where $\hat{T}_t = \int_0^t \sigma^\alpha(s)[r(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T}_s) + \exp[\int_0^s b(u)du]\frac{a(s)}{b(s)}]^\alpha ds.$

8. *Heath, Jarrow and Morton Process Driven by α-stable Lévy Motion*. $f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dL(v)$. *Solution* $f(t,u) = f(0,u) + \hat{L}(\hat{T}_t) + \int_0^t a(v,u)dv$, where $\hat{T}_t = \int_0^t b^\alpha(f(0,u) + \hat{L}(\hat{T}_s) + \int_0^s a(v,u)dv)ds.$

Change of Time Method for SDE Driven by Levy Motion

Solution of Multi-Factor Lévy SIRMs Using CTM

Solution of multi-factor models driven by α-stable Lévy motions can be obtained using various combinations of solutions of the above-mentioned processes and CTM. We give one example of two-factor continuous-time GARCH model driven by α-stable Lévy motions:

$$\begin{cases} dr(t) &= \mu(b(t-) - r(t-))dt + \sigma r(t-)dL^1(t) \\ db(t) &= \xi b(t-)dt + \eta b(t-)dL^2(t), \end{cases}$$

where L^1, L^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Change of Time Method for SDE Driven by Levy Motion

Solution of Multi-Factor Lévy SIRMs Using CTM

Solution, using CTM for the first and the second equations:

$$r(t) = e^{-\mu t}[r(0) - e^{\xi t}(b(0) + \hat{L}^2(\hat{T}_t^2)) + \hat{L}^1(\hat{T}_t^1)] + e^{\xi t}[b(0) + \hat{L}^2(\hat{T}_t^2)],$$

where $\hat{T}^i$ are defined in 4. $(i = 1)$ and 1. $(i = 2)$, respectevely.

Bond Pricing

European Call Option

To calculate the value of a *European options* with maturity θ on the zero-coupon bond with maturity equal to T we could proceed as follows. If it is a call with strike price K, the value of the option at time θ is $\max(P(\theta, T) - K), 0)$ and it seems reasonable to hedge this call with a portfolio of riskless asset S_t^0 and zero-coupon bond $P(\theta, T)$ with maturity T. Using the classical arguments (and using the money-market account as numeraire), the price at time 0 of a European call option on a bond is given by

$$V(\theta, T, r) = E^*[\frac{1}{S_0(t)} \max(P(\theta, T) - K, 0)].$$

Bond Pricing (cntd)

Gaussian Bond Pricing for One-Factor SIRMs via CTM

The solution of the SDE

$$\frac{dP(t,u)}{P(t,u)} = r(t)dt + \sigma(t,u)dW^*(t).$$

can be written in the form

$$P(t,T) = P(0,T)\exp[\int_0^T r(s)ds]\frac{\exp[\int_0^t \sigma(s,T)dW^*(s)]}{E\{\exp[\int_0^t \sigma(s,T)dW^*(s)]\}}$$

$$= P(0,T)\exp[\int_0^T r(s)ds]\exp[\int_0^t \sigma(s,T)dW^*(s) - \frac{1}{2}\int_0^t \sigma(s,T)^2 ds].$$

We see that the log returns under risk-neutral measure approximately follow a Normally distributed random variable.

Bond Pricing (cntd)

Gaussian Bond Pricing for One-Factor SIRMs via PDE

Consider $V(t, T, r)$-bond price at time t, where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dW(t).$$

For example, for GBM, $a = \mu r$ and $b = \sigma r$, for OU process, $a = -\mu r$ and $b = \sigma$, and so on.

Bond Pricing (cntd)

Gaussian Bond Pricing for One-Factor SIRMs via PDE

The zero-coupon bond pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial r^2} + (a - \lambda b) \frac{\partial V}{\partial r} - rV = 0,$$

where the function λ is often called the market price of risk. The final condition is

$$V(T, T, r) = Z.$$

This PDE may be solved approximately by standard numerical methods, see, for example, Wilmott, Howison and Dewynne (1995).

Bond Pricing (cntd): Drawback of Gaussian Models

Remark. Empirical studies (see, for example, Eberlein and Raible (1999), Raible (2000)) show that the normality assumption does not reflect reality. Empirically observed log returns of bonds turn out to have leptokurtic distribution.

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform

The zero-coupon bond price is modeled with the following process (see Eberlein and Raible (1999), Raible (2000)):

$$P(t,T) = P(0,T)\,\mathrm{exp}[\int_0^T r(s)ds]\frac{\mathrm{exp}[\int_0^t \sigma(s,T)dL(s)]}{E\{\mathrm{exp}[\int_0^t \sigma(s,T)dL(s)]\}},$$

where $r(t)$ has one of the representations mentioed above.

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

Eberlein and Raible (1999) derived the bond price process in the form

$$P(t,T) = P(0,T)\exp[\int_0^T r(s)ds]\frac{\exp[\int_0^t \sigma(s,T)dL(s)]}{\exp[\int_0^t \theta(\sigma(s,T))dL(s)]},$$

where $\theta(u) := \log(E[\exp(uL(1))])$ denotes the logarithm of the moment-generating function of the Lévy process at time 1. For example, in the classical Gaussian model we choose $\theta(u) = u^2/2$ and $L(s) = W(s)$. We note, that we know the expressions for $r(t)$ in the above formula for many SIRMs.

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

Except when $L(t)$ is a Poisson or a Brownian motion, our *Lévy market model is an incomplete model*. It means that there are many different equivalent martingales measures to choose. In general this leads to many different possible prices for European options or bond options, etc.

One of the way to price bond is to use for the P^* the *Esscher transform* equivalent martingale measure. Following Gerber and Shiu (1994), we can by using the so-called Esscher transform find in some cases at least one equaivalent martingale measure P^*.

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

Let $f(t,x)$ be the density of our model's (real world, i.e. under P) distribution of $L(t)$. For some real number $\theta \in \{\theta \in R | \int_{-\infty}^{+\infty} \exp(\theta y) f(t,y) dy < +\infty\}$ we can define a new density

$$f^{(\theta)}(t,x) = \frac{\exp(\theta x) f(t,x)}{\int_{-\infty}^{+\infty} \exp(\theta y) f(t,y) dy}.$$

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

In order to assume finiteness of the expectation in the denominator above in the case of general Lévy processes, we assume that

$$\int_{\{|x|>1\}} \exp[vx]\nu(dx) < \infty, \quad for \quad |v| < (1+\epsilon)M,$$

where $\epsilon > 0$ and M is such that $0 \leq \sigma(s,T) \leq M$ (a.s.) for $0 \leq s \leq T$ and $\nu(dx)$ is the Lévy measure of L_1 (see Schoutens (2003)).

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

Another way to price bond is to consider *characteristic function (or Fourier transform)*, if it is known, of the risk-neutral log returns (see Carr and Madan (1998)). We note, that if we know the explicit expression for $r(t)$, then we can find the characteristic function of the risk-neutral bond price.

Partial Differential Equation (PDE) and Feynman-Kac Formula (Diffusion Case)

$$dr(t) = \gamma(r,t)dt + \sigma(r,t)dW(t)$$

$$V(t,x) = E_{t,x}[e^{-r(T-t)}v(r(T))] = E[e^{-r(T-t)}v(r(T))|r(t) = x].$$

Then $V(t,x)$ satisfies the PDE

$$V_t(t,x) + \gamma(r,t)V_x(r,t) + \frac{1}{2}\sigma^2(r,t)V_{xx}(t,x) = rV(t,x)$$

with the terminal condition $V(T,x) = v(x)$.

Generator of Diffusion Process with Drift γ and Diffusion σ ("duplet" $[\gamma, \sigma]$)

$$Af(x) = \gamma \partial f(r) + \frac{1}{2} \sigma^2 \partial^2 f(x)$$

Generator of Lévy Process with Triplet $[\gamma, \sigma, \nu(dy)]$

$$\begin{aligned}(Af)(x) &= \gamma\partial f(x) + \tfrac{1}{2}\sigma^2\partial^2 f(x) \\ &+ \int_{R^d-\{0\}}[f(x+y) - f(x) - y\partial f(x)\mathbf{1}_{\hat{B}}(y)]\nu(dy).\end{aligned}$$

Lévy Bond Pricing for One-Factor SIRMs via PIDE (cntd)

Consider $V(T, t, r)$-bond price at time t, where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dL(t),$$

$L(t)$ is a Lévy process.

For example, for GBM, $a = \mu r$ and $b = \sigma r$, for OU process, $a = -\mu r$ and $b = \sigma$, and so on.

Partial Integro-Differential Equation (PIDE) for the Price Process (Lévy Case with triplet $[\gamma, \sigma, \nu(dy)]$)

The price process $V(T, t, r(t))$ satisfies the following PIDE:

$$\begin{aligned} \frac{\partial V}{\partial t} \;&+\; \tfrac{1}{2} b^2 \sigma^2 \frac{\partial^2 V}{\partial r^2} + (a + \gamma b) \frac{\partial V}{\partial r} \\ &+\; \int_{-\infty}^{+\infty} [V(t, r + by) - V(t, r) - by \frac{\partial V(t,r)}{\partial r}] \nu(dy) - rV = 0, \end{aligned}$$

with final condition

$$V(T, T, r) = v(r(T)).$$

with deterministic function $v(r)$.

This PDIE is the analogue of the famous Black-Scholes PDE and follows from the Feynman-Kac formula for Lévy processes:

$$V(T, t, r) = E[\exp(-\int_t^T r(s) ds) v(r(T)) | r(t) = r].$$

Lévy Bond Pricing for One-Factor SIRMs via PIDE

One more way to price bond is to consider the solution of a partial differential integral equation with boundary condition, all in terms of the triplet of Lévy characteristics $[\gamma, \sigma, \nu(dy)]$ of the Lévy process under the risk-neutral measure P^*.

Lévy Bond Pricing for One-Factor SIRMs via PIDE (cntd)

The zero-coupon bond pricing equation is

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V}{\partial r^2} + (a + \gamma b - \lambda b\sigma)\frac{\partial V}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[V(t, r + by) - V(t, r) - by\frac{\partial V(t,r)}{\partial r}]\nu(dy) - rV = 0, \end{aligned}$$

where λ is the *market price of risk*. The final condition is

$$V(T, T, r) = Z.$$

Numerical methods for solving above PIDE can be found, for example, in Matache, Schwab and Wihler (2004), in Matache, Ptersdorff and Schwab (2005) and in Cont and Tankov (2003, Chapter 12).

Interest Rate Derivatives

- *Bond Option* - option on bond

- *Swaps*-exchange the interest rate payments on a certain amount for a certain length of time

- *Cap*-loan at the floating interest rate (not to exceed a specified value, the cap)

- *Floor*-similar to cap but does not go below a specified value, floor

- *Swaption*-option on swap

- *Caption*-option on cap

- *Floortion*-option on floor

Pricing of Gaussian and Lévy Bond Options

Pricing of Gaussian Bond Options

Let interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r,t)dt + b(r,t)dW(t),$$

where $W(t)$ is a standard Wiener process.

Consider the European call bond option, with exercise price K and expity date T, on a zero-coupon bond with maturity date $T_B \geq T$.

Pricing of Gaussian and Lévy Bond Options (cntd)

Pricing of Gaussian Bond Options

To find the value of the call option on bond (to buy a bond) we proceed with the following steps:

1. To find the value of the bond: $V_B(r,t;T_B)$, that satisfies the following PDE:

$$\frac{\partial V_B}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V_B}{\partial r^2} + (a - \lambda b)\frac{\partial V_B}{\partial r} - rV_B = 0.$$

with the final condition

$$V_B(r,T_B;T_B) = Z.$$

Pricing of Gaussian and Lévy Bond Options (cntd)

Pricing of Gaussian Bond Options

2. Let $C_B(r,t)$ be the value of the call option on this bond. Since C_B also depends on $r(t)$, it must satisfy previous equation too:

$$\frac{\partial C_B}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 C_B}{\partial r^2} + (a - \lambda b)\frac{\partial C_B}{\partial r} - rC_B = 0.$$

with the final condition

$$C_B(r,T) = \max(V_B(r,T;T_B) - K, 0).$$

Remark. These PDEs can be solved numerically using standard methods, see Wilmott, Howison and Dewynne (1995).

Pricing of Gaussian and Lévy Bond Options (cntd)

Pricing of Lévy Bond Options

Let interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r,t)dt + b(r,t)dL(t),$$

where $L(t)$ is a Lévy process.

Pricing of Gaussian and Lévy Bond Options (cntd)

Pricing of Lévy Bond Options

Consider the European Call Bond Option, with exercise price K and expity date T, on a zero-coupon bond with maturity date $T_B \geq T$.

To find the value of the call option on bond (to buy a bond) we proceed with the following steps:

Pricing of Gaussian and Lévy Bond Options (cntd)

Pricing of Lévy Bond Options

1. Find the value of the bond: $V_B(r, t; T_B)$, that satisfies the following PDE:

$$\begin{aligned}\frac{\partial V_B}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_B}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_B}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[V_B(t, r + by) - V_B(t, r) - by\frac{\partial V_B(t,r)}{\partial r}]\nu(dy) - rV_B = 0.\end{aligned}$$

with the final condition

$$V_B(r, T_B; T_B) = Z.$$

Pricing of Gaussian and Lévy Bond Options (cntd)

Pricing of Lévy Bond Options

2. Let $C_B(r,t)$ be the value of the call option on this bond. Since C_B also depends on $r(t)$, it must satisfy equation (27) too:

$$\begin{aligned}\frac{\partial C_B}{\partial t} \; &+ \; \tfrac{1}{2}b^2\sigma^2\frac{\partial^2 C_B}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_B}{\partial r} \\ &+ \; \textstyle\int_{-\infty}^{+\infty}[C_B(t, r+by) - C_B(t,r) - by\frac{\partial C_B(t,r)}{\partial r}]\nu(dy) - rC_B = 0.\end{aligned}$$

with the final condition

$$C_B(r,T) = \max(V_B(r,T;T_B) - K, 0).$$

Pricing of Gaussian and Lévy Bond Options (cntd)

Pricing of Lévy Bond Options

Remark. One of the approach to solve this PDIE could be numerical using different finite difference methods, see *Duffy (2005)*. Numerical methods for solving above PIDE can be also found, for example, in *Matache, Schwab and Wihler (2004), in Matache, Ptersdorff and Schwab (2005)* and in *Cont and Tankov (2003, Chapter 12)*.

Pricing of Swaps, Caps and Floors

Pricing of Swaps, Caps and Floors for Gaussian IRMs

Let interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r,t)dt + b(r,t)dW(t),$$

where $W(t)$ is a standard Wiener process.

Pricing of Swaps, Caps and Floors (cntd)

Pricing of Swaps for Gaussian IRMs

We consider to value such swaps in general. Suppose that A pays the interest on an amount Z to B at a fixed rate r^* and B pays interest to A at the floating rate r. These payments continue until time T_S. Denote the value of this swap to A by $ZV_S(r,t)$. We note, that in a time-step dt A receives $(r - r^*)Zdt$.

Pricing of Swaps, Caps and Floors (cntd)

Pricing of Swaps for Gaussian IRMs

If we think of this payment as being similar to a coupon payment on a simple bond then we find that:

$$\frac{\partial V_S}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V_S}{\partial r^2} + (a - \lambda b) \frac{\partial V_S}{\partial r} - r V_S + (r - r^*) = 0.$$

with the final condition

$$V_S(r, T_S) = 0.$$

We note, that r can be greater or less than r^* and so $V_S(r,t)$ need not be positive.

Pricing of Swaps, Caps and Floors (cntd)

Pricing of Caps for Gaussian IRMs

The loan of Z is to be paid back at time T_C. The value of the capped loan, $ZV_C(r,t)$ satisfies the following PDE:

$$\frac{\partial V_C}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V_C}{\partial r^2} + (a - \lambda b)\frac{\partial V_C}{\partial r} - rV_C + \min(r, r^*) = 0.$$

with the final condition

$$V_C(r, T_C) = 1.$$

Pricing of Swaps, Caps and Floors (cntd)

Pricing of Floors for Gaussian IRMs

The value of the floored loan, $ZV_F(r,t)$, satisfies the following PDE:

$$\frac{\partial V_F}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V_F}{\partial r^2} + (a - \lambda b) \frac{\partial V_F}{\partial r} - r V_F + \max(r, r^*) = 0.$$

with the final condition

$$V_F(r, T_F) = 1,$$

where T_F is an expiry time for floor.

Remark. These PDE can be solved numerically using standard methods, see Wilmott, Howison and Dewynne (1995).

Pricing of Swaps, Caps and Floors for Lévy IRMs

Consider $V(r,t)$-bond price at time t, where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r,t)dt + b(r,t)dL(t),$$

$L(t)$ is a Lévy process.

Pricing of Swaps, Caps and Floors for Lévy IRMs

Pricing of Swaps

We consider to value such swaps in general. Suppose that A pays the interest on an amount Z to B at a fixed rate r^* and B pays interest to A at the floating rate r. These payments continue until time T_S. Denote the value of this swap to A by $ZV_S(r,t)$. We note, that in a time-step dt A receives $(r - r^*)Zdt$.

Pricing of Swaps, Caps and Floors for Lévy IRMs

Pricing of Swaps

If we think of this payment as being similar to a coupon payment on a simple bond then we find that:

$$
\begin{aligned}
\frac{\partial V_S}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_S}{\partial r} \\
&+ \int_{-\infty}^{+\infty}[V_S(t, r + by) - V_S(t, r) - by\frac{\partial V_S(t,r)}{\partial r}]\nu(dy) \\
&- rV_S + (r - r^*) = 0.
\end{aligned}
$$

with the final condition

$$V_S(r, T_S) = 0.$$

We note, that r can be greater or less than r^* and so $V_S(r, t)$ need not be positive.

Pricing of Swaps, Caps and Floors for Lévy IRMs

Pricing of Caps

The loan of Z is to be paid back at time T_C. The value of the capped loan, $ZV_C(r,t)$ satisfies the following PDE:

$$\begin{array}{rl} \frac{\partial V_C}{\partial t} & + \ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_C}{\partial r} \\ & + \ \int_{-\infty}^{+\infty}[V_C(t, r+by) - V_C(t,r) - by\frac{\partial V_C(t,r)}{\partial r}]\nu(dy) \\ & - \ rV_C + \min(r, r^*) = 0. \end{array}$$

with the final condition

$$V_C(r, T_C) = 1.$$

Pricing of Swaps, Caps and Floors for Lévy IRMs

Pricing of Floors

The value of the floored loan, $ZV_F(r,t)$, satisfies the following PDE:

$$\begin{aligned}\frac{\partial V_F}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_F}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_F}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[V_F(t, r + by) - V_F(t,r) - by\frac{\partial V_F(t,r)}{\partial r}]\nu(dy) \\ &- rV_F + \max(r, r^*) = 0.\end{aligned}$$

with the final condition

$$V_F(r, T_F) = 1,$$

where T_F is an expiry time for floor.

Pricing of Swaptions, Captions and Floortions

Pricing of Swaptions, Captions and Floortions for Gaussian IRMs

Let interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r,t)dt + b(r,t)dW(t),$$

where $W(t)$ is a standard Wiener process.

Pricing of Swaptions, Captions and Floortions

Pricing of Swaptions for Gaussian IRMs

Consider European Swap Call Option, option to buy this swap (a call swaption) for an amount K at time $T < T_S$, where T_S is an expiry time for swap with value $V_S(r,t)$, $t \le T_S$.

1. Thus, this value V_S satisfies the following PDE (see (32)):

$$\frac{\partial V_S}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V_S}{\partial r^2} + (a - \lambda b) \frac{\partial V_S}{\partial r} - r V_S + (r - r^*) = 0.$$

with the final condition

$$V_S(r, T_S) = 0.$$

Pricing of Swaptions, Captions and Floortions

Pricing of Swaptions for Gaussian IRMs

2. Then the value $C_S(r,t)$ of this call swap option (call swaption) satisfies the following PDE:

$$\frac{\partial C_S}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 C_S}{\partial r^2} + (a - \lambda b) \frac{\partial C_S}{\partial r} - r C_S = 0.$$

with the final condition

$$C_S(r,T) = \max(V_S(r,T) - K, 0).$$

We solve for the value of the swap first and then use this value as the final data for the value of the swaption.

Pricing of Swaptions, Captions and Floortions

Pricing of Captions for Gaussian IRMs

Consider European Cap Call Option, option to buy this cap (a call caption) for an amount K at time $T < T_C$, where T_C is an expiry time for cap with value $V_C(r,t)$, $t \leq T_C$.

1. Thus, this value V_C satisfies the following PDE:

$$\frac{\partial V_C}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V_C}{\partial r^2} + (a - \lambda b)\frac{\partial V_C}{\partial r} - rV_C + \min(r, r^*) = 0.$$

with the final condition

$$V_C(r, T_C) = 1.$$

Pricing of Swaptions, Captions and Floortions

Pricing of Captions for Gaussian IRMs

2. Then the value $C_C(r,t)$ of this call cap option (call caption) satisfies the following PDE:

$$\frac{\partial C_C}{\partial t} + \frac{1}{2} b^2 \sigma^2 \frac{\partial^2 C_C}{\partial r^2} + (a - \lambda b) \frac{\partial C_C}{\partial r} - r C_C = 0.$$

with the final condition

$$C(r,T) = \max(V_C(r,T) - K, 0).$$

We solve for the value of the cap first and then use this value as the final data for the value of the caption.

Pricing of Swaptions, Captions and Floortions

Pricing of Floortions for Gaussian IRMs

Consider European Floor Call Option, option to buy this floor (a call floortion) for an amount K at time $T < T_F$, where T_F is an expiry time for floor with value $V_F(r,t)$, $t \le T_F$.

1. Thus, this value V_F satisfies the following PDE:

$$\frac{\partial V_F}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 V_F}{\partial r^2} + (a - \lambda b) \frac{\partial V_F}{\partial r} - r V_F + \min(r, r^*) = 0.$$

with the final condition

$$V_C(r, T_F) = 1.$$

Pricing of Swaptions, Captions and Floortions

Pricing of Floortions for Gaussian IRMs

2. Then the value $C_F(r,t)$ of this call floor option (call floortion) satisfies the following PDE:

$$\frac{\partial C_F}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 C_F}{\partial r^2} + (a - \lambda b) \frac{\partial C_F}{\partial r} - r C_F = 0.$$

with the final condition

$$C_F(r,T) = \max(V_F(r,T) - K, 0).$$

We solve for the value of the floor first and then use this value as the final data for the value of the floortion.

Remark. These PDE can be solved numerically using standard methods, see Wilmott, Howison and Dewynne (1995).

Pricing of Swaptions, Captions and Floortions for Lévy IRMs

Consider $V(r,t)$-bond price at time t, where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r,t)dt + b(r,t)dL(t),$$

$L(t)$ is a Lévy process.

Pricing of Swaptions, Captions and Floortions for Lévy IRMs

Pricing of Swaptions

Consider European Swap Call Option, option to buy this swap (a call swaption) for an amount K at time $T < T_S$, where T_S is an expiry time for swap with value $V_S(r,t)$, $t \leq T_S$.
1. Thus, this value V_S satisfies the following PDE:

$$\begin{aligned}\frac{\partial V_S}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_S}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[V_S(t, r + by) - V_S(t,r) - by\frac{\partial V_S(t,r)}{\partial r}]\nu(dy) \\ &- rV_S + (r - r^*) = 0.\end{aligned}$$

with the final condition

$$V_S(r, T_S) = 0.$$

Pricing of Swaptions, Captions and Floortions for Lévy IRMs

Pricing of Swaptions

2. Then the value $C_S(r,t)$ of this call swap option (call swaption) satisfies the following PDE:

$$\begin{aligned}\frac{\partial C_S}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_S}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_S}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[C_S(t,r+by) - C_S(t,r) - by\frac{\partial C_S(t,r)}{\partial r}]\nu(dy) - rC_S = 0.\end{aligned}$$

with the final condition

$$C_S(r,T) = \max(V_S(r,T) - K, 0).$$

We solve for the value of the swap first and then use this value as the final data for the value of the swaption.

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Pricing of Captions

Consider European Cap Call Option, option to buy this cap (a call caption) for an amount K at time $T < T_C$, where T_C is an expiry time for cap with value $V_C(r,t)$, $t \le T_C$.
1. Thus, this value V_C satisfies the following PDE:

$$\begin{aligned}\frac{\partial V_C}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_C}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[V_C(t, r + by) - V_C(t,r) - by\frac{\partial V_C(t,r)}{\partial r}]\nu(dy) \\ &- rV_C + \min(r, r^*) = 0.\end{aligned}$$

with the final condition

$$V_C(r, T_C) = 1.$$

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Pricing of Captions

2. Then the value $C_C(r,t)$ of this call cap option (call caption) satisfies the following PDE:

$$\begin{aligned}\frac{\partial C_C}{\partial t} &+ \tfrac{1}{2}b^2\sigma^2\frac{\partial^2 C_C}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_C}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[C_C(t,r+by) - C_C(t,r) - by\frac{\partial C_C(t,r)}{\partial r}]\nu(dy) - rC_C = 0.\end{aligned}$$

with the final condition

$$C(r,T) = \max(V_C(r,T) - K, 0).$$

We solve for the value of the cap first and then use this value as the final data for the value of the caption.

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Pricing of Floortions

Consider European Floor Call Option, option to buy this floor (a call floortion) for an amount K at time $T < T_F$, where T_F is an expiry time for floor with value $V_F(r,t)$, $t \le T_F$.
1. Thus, this value V_F satisfies the following PDE:

$$\begin{aligned}\frac{\partial V_F}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 V_F}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial V_F}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[V_F(t, r + by) - V_F(t, r) - by\frac{\partial V_F(t,r)}{\partial r}]\nu(dy) \\ &- rV_F + \min(r, r^*) = 0.\end{aligned}$$

with the final condition

$$V_C(r, T_F) = 1.$$

Pricing of Swaptions, Captions and Floortions for Lévy IRMs

Pricing of Floortions

2. Then the value $C_F(r,t)$ of this call floor option (call floortion) satisfies the following PDE:

$$\begin{aligned} \frac{\partial C_F}{\partial t} &+ \frac{1}{2}b^2\sigma^2\frac{\partial^2 C_F}{\partial r^2} + (a + b\gamma - \lambda b\sigma)\frac{\partial C_F}{\partial r} \\ &+ \int_{-\infty}^{+\infty}[C_F(t,r+by) - C_F(t,r) - y\frac{\partial C_F(t,r)}{\partial r}]\nu(dy) - rC_F = 0. \end{aligned}$$

with the final condition

$$C_F(r,T) = \max(V_F(r,T) - K, 0).$$

We solve for the value of the floor first and then use this value as the final data for the value of the floortion.

Conclusion

- Levy-Based Interest Rate Models
- Pricing of Bond (CTM and PIDE)
- Pricing of Bond Options
- Pricing Swaps, Caps and Floors
- Pricing Swaptions, Captions and Floortions

Paper

This presentation is based on the following paper

A. Swishchuk 'Interest Rate Derivatives and Change of Time Method'

accepted by *2008 Stochastic Modeling Symposium*, Montréal, December 1-2

The End

Thank You for Your Time and Attention!