Stochastic Modeling and Pricing of Energy Markets' Contracts with Local Stochastic Delayed and Jumped Volatilities *

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Outline of Presentation

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Introduction: Motivation

Since the early 1990s, the markets for electricity and related products have been liberalized worldwide. It all started off with the Nordic market NordPool and the England&Wales market at around 1992, and over the last two decades trade in electricity and related products on all continents in the world has liberalized.

Introduction: Motivation

In 1999 the Chicago Mercantile Exchange organized a market for temperature derivatives that has gained momentum in recent years.

Exchange-based markets for gas have emerged and are now actively traded at the New York Mercantile Exchange and the Intercontinental Exchange in London.

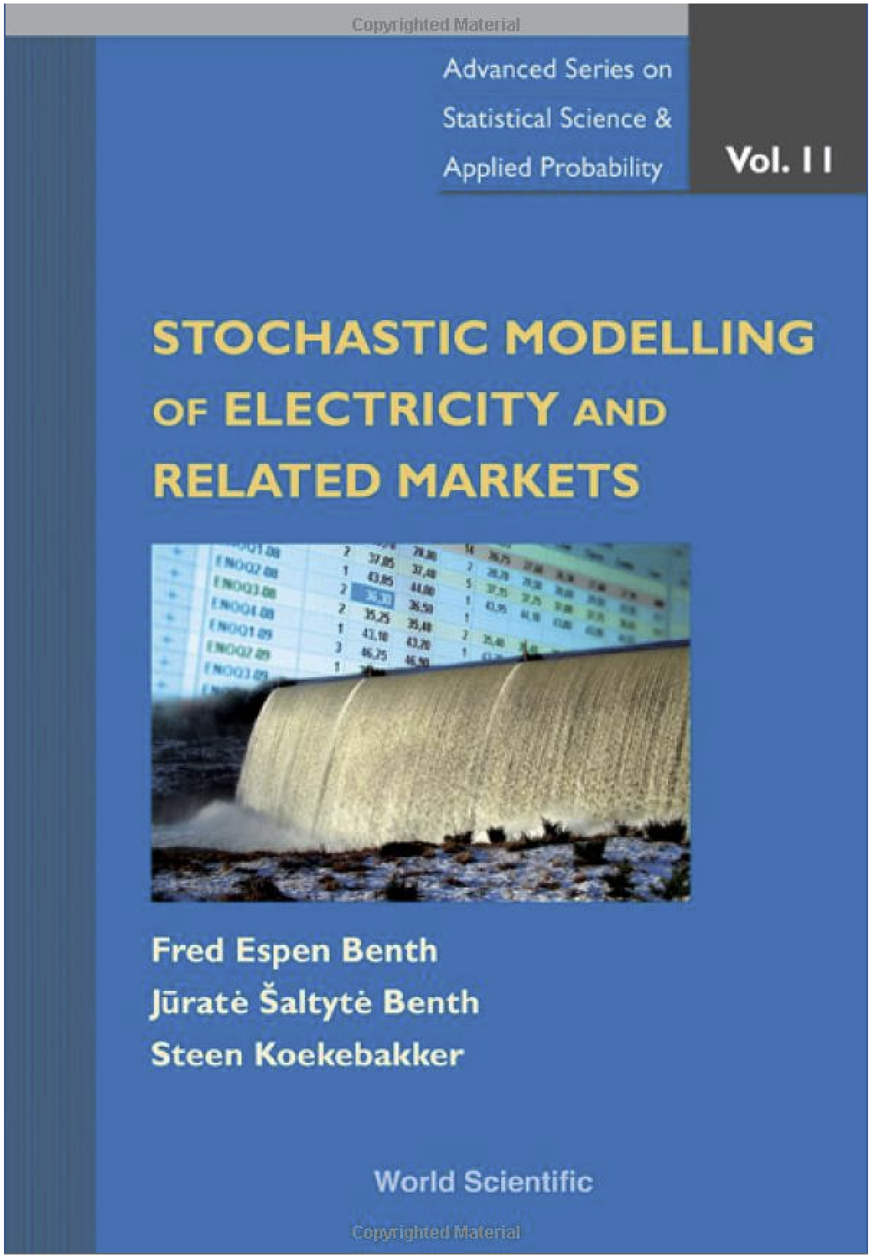
Introduction: Motivation

The basic products in the energy markets, including electricity, gas and temperature markets, are spot, futures and forward contracts and options on these. With organized markets comes the need to have consistent *stochastic models* describing the price evolution of the products.

Introduction: Motivation

Energy-related spot prices have several typical characteristics, with the most prominent being *mean reversion* towards a *seasonally* varying mean level, and frequently occuring *spikes* resulting from an imbalance between supply and demand.

Further, since the energy commodities are driven by the balance between demand and production, the prices tend to mean-revert. A natural class of stochastic models to describe such dynamics is the Ornstein-Uhlenbeck processes. We use these mean-reverting stochastic processes as our modelling tool.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Book Cover

Benth *et al.* **Book (2008, World Sci. Publ. Co)**

We reviewed this book at our 'Lunch at the Lab' finance seminar.

This is a nice book from mathematical point of view: first application of IIPs.

This is also nice book from practical point of view: contains many applications to electricity, gas and temperature markets.

This book was also one of the motives to develop present research.

Stochastic Analysis for Independent Increments Processes (IIP): Definition and Examples

$(\Omega, \mathcal{F}, \mathcal{F}_t, P)$-complete filtered probability space

$X : \Omega \to R^d$-random variable, if it's $\mathcal{F}$-measurable

$X(t) \equiv X(t, \omega)$-stochastic process, a family of random variables parametrized over the time t

$X(t)$ is $\mathcal{F}_t$-adapted, if every $X(t)$ is measurable wrt $\mathcal{F}_t$

$X(t)$ is RCLL, if its paths $t \to X(t, \omega)$ are right-continuous and has left-limits a.s.

Stochastic Analysis for IIP: Definitions (cntd)

An adapted RCLL stochastic process $I(t)$ starting at zero is an *II process* (*Independent Increment process*) if it satisfies the following two conditions:

1) The increments $I(t_0), I(t_1) - I(t_0), ..., I(t_n) - I(t_{n-1})$ are independent r.v. for any partition $0 \leq t_0 < t_1 < ... < t_n$, and $n \geq 1$.

2) It is continuous in probability, that is, for every $t \geq 0$ and $\epsilon > 0$, it

$$\lim_{s \longrightarrow t} P(|I(s) - I(t)| > \epsilon) = 0.$$

Stochastic Analysis for IIP: Definitions (cntd)

If we add the condition that increments are stationary, then $I(t)$ is called a *Lévy process.*

If the increments of Lévy process are normally distributed then we have a *Brownian motion.*

Lévy processes which are increasing, that is, having only positive jumps, are often called *subordinators*.

Sometimes (Sato (1999)) the II processes are called *additive processes*.

Stochastic Analysis for IIP: Definitions (cntd)

The characteristic function of the II process $I(t)$ is

$$E[\exp(i\theta(I(s) - I(t)))] = \exp(\psi(s,t;\theta)), \quad 0 \leq s < t, \theta \in R$$

and

$$\begin{aligned} \psi(s,t;\theta) &= i\theta(\gamma(t) - \gamma(s)) - \tfrac{1}{2}\theta^2(C(t) - C(s)) \\ &+ \int_s^t \int_R (e^{i\theta z} - 1 - i\theta z \mathbf{1}_{|z|\leq 1}) l(dz, du). \end{aligned}$$

Stochastic Analysis for IIP: Definitions (cntd)

The function $\psi(s,t;\theta)$ is called *the cumulant function* of the process $I(t)$, and the generating triplet of the II process is $(\gamma(t), C(t), l)$, with the properties

1) $\gamma : R \to R$ is a continuous function with $\gamma(0) = 0$,

2) $C : R \to R$ is non-decreasing and continuous, with $C(0) = 0$,

3) l is a σ-finite measure on the Boreal σ-algebra of $[0, +\infty) \times R$, with the property

$$l(A \times \{0\}) = 0, \quad l(\{t\} \times R) = 0, \quad t \geq 0, A \in \mathcal{B}(R_+)$$

and

$$\int_0^t \int_R \min(1, z^2) l(ds, dz) < +\infty.$$

Stochastic Analysis for IIP: Definitions (cntd)

If

$$l(ds, dz) = ds\tilde{l}(dz), \quad \gamma(t) = \gamma t, \quad C(t) = ct$$

for constants γ and $c \geq 0$, we have $\psi(t, s, ; \theta) = (t - s)\tilde{\psi}(\theta)$ with

$$\tilde{\psi}(\theta) = i\theta\gamma - \frac{1}{2}\theta^2 c + \int_R (e^{i\theta z - 1 - i\theta z \mathbf{1}_{|z| \leq 1}})\tilde{l}(dz)$$

The function $\tilde{\psi}$ is called *the cumulant of a Lévy process* L.

Stochastic Analysis for IIP: The Lévy-Khintchine Decomposition and Semimartingales

The *Lévy-Khintchine decomposition* for II process $I(t)$:

$$I(t) = \gamma(t) + M(t) + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(ds, dz),$$

where $M(t)$ is a local square integrable continuous martingale.

An adapted RCLL stochastic process $S(t)$ is a *semimartingale* if it has the representation

$$\begin{aligned} S(t) &= S(0) + A(t) + M(t) + \textstyle\int_0^t \int_{R\setminus\{0\}} X_1(s,z)\tilde{N}(ds,dz) \\ &+ \textstyle\int_0^t \int_{R\setminus\{0\}} X_2(s,z) N(ds,dz), \end{aligned}$$

where $A(t)$ is an adapted continuous stochastic process.

Stochastic Analysis for IIP: The Lévy-Khintchine Decomposition and Semimartingales (cntd)

In general, an II process is not a semimartingale. An II process $I(t)$ may be represented as the sum of a deterministic RCLL function and a semimartingale.

Examples of II Processes

- Brownian motion
- Compound Poisson process
- Time Inhomogeneous Compound Poisson process
- Normal inverse Gaussian (NIG)
- CGMY Lévy process

Stochastic Volatilities with Delay and Jumps

Introduction: Stock Price with Delayed Volatility (Kazmerchuk, Swishchuk & Wu (2005))

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dB(t), \quad t > 0,$$

where $\mu \in R$ is the mean rate of return, the volatility term $\sigma > 0$ is a continuous and bounded function and $B(t)$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$. We denote $S_t = S(t + \theta), \quad \theta \in [-\tau, 0]$ and the initial data of $S(t)$ is defined by $S(t) = \varphi(t)$, where $\varphi(t)$ is a deterministic function with $t \in [-\tau, 0], \quad \tau > 0$.

Introduction: Stochastic Volatility with Delay and Jumps (Swishchuk (2009))

$$\begin{aligned}\frac{d\sigma^2(t,S_t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sigma(u,S_u)dB(u) + \int_{t-\tau}^{t}\sigma(u,S_u)d\tilde{N}(u)\right]^2 \\ &- (\alpha+\gamma)\sigma^2(t,S_t)\end{aligned}$$

where $N(t)$ is a Poisson process independent of $B(t)$ with intensity $\lambda > 0$ and $\tilde{N}(t) := N(t) - \lambda t$.

Introduction: Stochastic Volatility with Delay and Jumps (cntd)

Our model of stochastic volatility exhibits jumps and also past-dependence: the behavior of a stock price right after a given time t not only depends on the situation at t, but also on the whole past (history) of the process $S(t)$ up to time t. This draws some similarities with fractional Brownian motion models (see Mandelbrot (1997)) due to a long-range dependence property. Another advantage of this model is mean-reversion.

Introduction: Stochastic Volatility with Delay and Jumps (cntd)

This model is also a continuous-time version of GARCH(1,1) model (see Bollerslev (1986)) with jumps (one of the motivations for delay). Another advantage of our stochastic volatility model with delay and jumps is mean-reversion: the volatility is allowed to mean revert. Such models have shown some success in modeling interest rate (i.e., Aït-Sahalia (1996)).

Motivation: Why Jumps?

There is currently fairly compelling evidence for jumps in the level of financial prices. The most convincing evidence comes from recent nonparametric work using high-frequency data as in Barndorff-Nielsen and Shephard (2007) and Aït-Sahalia and Jacod (2008) among others. Also, paper by Todorov and Tauchen (2008) conducts a non-parametric analysis of the market volatility dynamics using high-frequency data on the VIX index compiled by the CBOE and the $S\&P500$ index.

Introduction: Our Model

In this talk we concentrate on stochastic modelling and pricing of energy markets' contracts for stochastic volatilites with delay and jumps. The basic products in these markets are spot, futures and forward contracts and options written on these. We shall study forward and swaps.

Stochastic Models for the Energy Spot Dynamics with Local Stochastic Delayed and Jumped Volatilities

We study different types of stochastic processes for modelling energy spot dynamics. The classical stochastic process for the spot dynamics of commodity prices is given by the so-called Schwartz' model (see Schwartz (1997)).

Stochastic Models for the Energy Spot Dynamics with Local Stochastic Delayed and Jumped Volatilities

It is defined as the exponential of an OU process, and has become the standard model for energy prices possessing mean-reverting features. We present in this talk spot price models based on a sum of non-Gaussian OU processes with stochastic delayed and jumped volatility. We will consider geometric models of the kind proposed by Schwartz, but also arithmetic models which may be more analytically tractable in markets where the forward contracts have a delivery period.

Stochastic Models for the Energy Spot Price Dynamics

The Schwartz model

The starting point for the models is the Schwartz one-factor model from his 1997 paper 'The stochastic behavior of commodity prices: implications for valuation and hedging', Journal of Finance, Vol.52(3), 923–973.

$$S(t) = S(0)\,\mathrm{exp}[X(t)]$$

with

$$dX(t) = \kappa\Big(\alpha - X(t)\Big)dt + \sigma dW(t).$$

Stochastic Models for the Energy Spot Price Dynamics: Spot price modelling with OU processes

$I(t)$ is an II process with a Lévy-Kintchine representation

$$\begin{aligned}\psi(t,s;\theta) &= i\theta\Big(\gamma(s)-\gamma(t)\Big)-\tfrac{1}{2}\theta^2\Big(C(s)-C(t)\Big)\\ &+ \textstyle\int_t^s\int_R\Big\{e^{iz\theta}-1-iz\theta\mathbf{1}_{|z|<1}\Big\}\,l(dz,du),\end{aligned}$$

where γ is of finite variation.

An RCLL process $X(s)$ $(t \le s \le T)$ is an OU process if it is the unique strong solution to

$$dX(s) = \Big(\mu(s)-\alpha(s)X(s)\Big)ds + \sigma(s)dI(s), \quad X(t) = x.$$

Stochastic Models for the Energy Spot Price Dynamics

The unique solution can be written

$$X(s) = x\, e^{-\int_t^s \alpha(v)dv} + \int_t^s \mu(u) e^{-\int_u^s \alpha(v)dv} du + \int_t^s \sigma(u) e^{-\int_u^s \alpha(v)dv} dI(u).$$

We can calculate expected value and other moments, because we know characteristics function of $X(t)$.

Our Stochastic Models for Pricing Energy Derivatives with Local Stochastic Delayed and Jumped Volatilities

Geometric Models with Local Stochastic Dealyed and Jumped Volatility

We introduce n pure jump semimartingale II processes $I_j(t), j = 1, ..., n$, where I_j and I_k are independent of each other for all $j \neq k$. We can repreent each process via its associated random jump measure $N_j(dt, dz)$ by the Lévy-Itô representation

$$I_j(t) = \gamma_j(t) + \int_0^t \int_{|z|<1} z\tilde{N}_j(du, dz) + \int_0^t \int_{|z|\geq 1} z\tilde{N}_j(du, dz). \quad (1)$$

Here, $\gamma_j(t)$ has bounded variation and the compensator measure for $N_j(dt, dz)$ is denoted by $l_j(dt, dz)$. Assume, that B is Brownian motion independent of $N_j(dt, dz)$.

Geometric Models with Local Stochastic Dealyed and Jumped Volatility

Let the stochastic process $S(t)$ be denoted as

$$\ln S(t) = \ln \Lambda(t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t),$$

where for $i = 1, ..., m$

$$dX_i(t) = (\mu_i(t) - \alpha_i(t)X_i(t))dt + \sigma_i(t, X_i(t+\theta))dB(t),$$

and for $j = 1, ..., n$

$$dY_j(t) = (\delta_j(t) - \beta_j(t)Y_j(t))dt + \eta_j(t, Y_j(t+\theta))dI_j(t).$$

Geometric Models with Local Stochastic Dealyed and Jumped Volatility

Here, $\theta \in [-\tau, 0], \tau > 0$, is the delay, and on the interval $[-\tau, 0]$, $X_i(t) = \phi_i(t)$ and $Y_j(t) = \psi_j(t)$, where $\phi_i(t)$ and $\psi_j(t)$ are deterministic functions, $i = 1, ..., m$ and $j = 1, ..., n$. We remark that two factors $X_i(t), i = 1, ..., m$, and $Y_j(t), j = 1, ..., n$, represent the long- and short-term fluctuations of the spot dynamics which may be correlated. We suppose that jumps components I_j are independent, which is an obvious restriction of generality.

Geometric Models with Local Stochastic Dealyed and Jumped Volatility

The deterministic seasonal price level is modelled by the function $\Lambda(t)$, (seasonal function) which is assumed to be continuously differentuable. The coefficients $\mu_i, \alpha_i, \delta_j \beta_j$ are all continuous functions. We suppose that volatilities $\sigma_{ik}(t)$ and $\eta_j(t)$ are stochastic volatilities with delay and jumps. We consider two cases in this situation:

Geometric Model: Stochastic Volatilities with Delay and with Jumps

$$\begin{aligned}\frac{d\sigma_i^2(t,X_i(t+\theta))}{dt} &= \gamma_i^1 V_i + \frac{\alpha}{\tau}[\int_{t-\tau}^{t} \sigma_i(u, X_i(u+\theta) dB(u) \\ &+ \int_{t-\tau}^{t} \sigma_i(u, X_i(u+\theta) d\tilde{N}_1(t)]^2 \\ &- (a_i + b_i)\sigma_i^2(t, X_i(t+\theta))\end{aligned}$$

and

$$\begin{aligned}\frac{d\eta_j^2(t,Y_j(t+\theta))}{dt} &= \gamma_j^2 W_i + \frac{\alpha}{\tau}[\int_{t-\tau}^{t} \eta_j(u, X_j(u+\theta) dB_1(u) \\ &+ \int_{t-\tau}^{t} \sigma_i(u, X_i(u+\theta) d\tilde{N}_2(t)]^2 \\ &- (c_j + d_j)\eta_j^2(t, X_i(t+\theta)\end{aligned}$$

Geometric Model: Stochastic Volatilities with Delay and with Jumps

Here, $B(t)$ and $B_1(t)$ are two independent Brownian motions and $\tilde{N}_1(t)$ and $\tilde{N}_2(t)$ are two independent compensated Poisson processes with intensities λ_1 and λ_2, independent of $B(t)$ and $B_1(t)$.

We note, that Benth *et al.* (2008) considered only deterministic $\sigma_i(t)$ and $\eta_j(t)$.

We study the pricing of forwards, futures, swaps and options for the above-mentioned model with delayed and jumped volatilities.

Arithmetic Models with Local Stochastic Dealyed and Jumped Volatility

Let the stochastic process $S(t)$ be defined as

$$S(t) = \Lambda(t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t), \tag{9}$$

where $X_i(t), i = 1, ..., m$, and $Y_j(t), j = 1, ..., n$, are defined for the geometric models above and the seasonality function $\Lambda(t)$ is the same.

We suppose that for this model the volatilites $\sigma_i^2(t, X_i(t+\theta))$ and $\eta_j^2(t, Y_j(t+\theta))$ satisfied the same equations as for the case of geometric models.

We study the pricing of forwards and swaps for the above-mentioned model with delayed and jumped volatilites.

Risk-Neutral Forward and Swap Price Modelling: Forward Pricing

When entering the forward contract, one agrees on a future deliverty time and the price to be paid for receiving the underlying. Suppose that the delivery time is τ, with $0 \leq t \leq T < +\infty$, and that the agreed price to pay upon delivery is $f(t, T)$. At time T, we will effectively receive a (possibly negative) payment

$$S(T) - f(t, T).$$

It is costless to enter such contracts, which gives us a relation where we can extract the forward price:

$$e^{-r(T-t)} E_Q[S(T) - f(t, T) | \mathcal{F}_t] = 0.$$

Risk-Neutral Forward and Swap Price Modelling: Forward Pricing

$$f(t,T) = E_Q[S(T)|\mathcal{F}_t] -$$

fundamental pricing relation between the spot and forward price. Since the energy markets are incomplete, the choice of martingale measure Q is open.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

Let us consider swaps, using the electricity market as the typical example. The buyer of an electricity futures receives power during a setlement period (physically or financially), against paying a fixed price per MWh. The time t value of the payoff from the continuous flow electricity is given as

$$\int_{\tau_1}^{\tau_2} e^{-r(u-t)}(S(u) - F(t,\tau_1,\tau_2))du,$$

where $F(t,\tau_1,\tau_2)$ is the electricity futures price at time t for the delivery period $[\tau_1,\tau_2]$ with $\tau_1 \leq \tau_2$.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

Since it is costless to enter an electricity futures contract, the risk-neutral price is defined by the equation

$$e^{-rt}E_Q[\int_{\tau_1}^{\tau_2} e^{-r(u-t)}(S(u) - F(t,\tau_1,\tau_2))du|\mathcal{F}_t] = 0.$$

As long as F is adapted we have

$$F(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} \frac{re^{-ru}}{e^{-r\tau_1} - e^{-r\tau_2}} S(u)du|\mathcal{F}_t].$$

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

One may have that the settlement takes place finacially at the end of the delivery period τ_2. The payoff from the contract at time τ_2 is then

$$e^{-r\tau_2} E_Q[\int_{\tau_1}^{\tau_2} (S(u) - F(t, \tau_1, \tau_2))du|\mathcal{F}_t] = 0,$$

which yields an electricity futures price

$$F(t, \tau_1, \tau_2) = E_Q[\int_{\tau_1}^{\tau_2} \frac{1}{\tau_2 - \tau_1} S(u)du|\mathcal{F}_t].$$

The same considerations could be done for gas futures contracts, and in the following we refer to $F(t, \tau_1, \tau_2)$ simply as the *swap price*.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

Let us introduce a weight function $\widehat{w}(u)$, being equal to one if the swap is settled at the end of the delivery period, or $\widehat{w}(u) = \exp(-ru)$ if the contract is settled continuously over the delivery period. Define the function

$$w(u,s,t) = \frac{\widehat{w}(u)}{\int_s^t \widehat{w}(v)dv},$$

where $0 \leq u \leq s \leq t$. Observe that $w = 1/(t-s)$, when $\widehat{w} = 1$, while we have

$$w(u,s,t) = \frac{re^{-ru}}{e^{-rs} - e^{-rt}},$$

for the case when $\widehat{w} = \exp(-ru)$. We note that $\int_s^t w du = 1$.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

In general, we can write the link between a swap contract and the underlying spot as

$$F(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2)S(u)du|\mathcal{F}_t].$$

Proposition 4.1. (Benth *et al. (2008)***)** Suppose $E_Q[\int_{\tau_1}^{\tau_2} |w(u,\tau_1,\tau_2)S(u)|du] < +\infty$. It holds that

$$F(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2)f(t,u)du].$$

This means that holding a swap contract can be considered as holding a (weighted) continuous stream of forwards.

Risk-Neutral Probabilities and the Esscher Transform

The Esscher transform is a generalization of the Girsanov transform of Brownian motion to jump processes.

The Esscher transform is preserving the distributional properties of the jump process in the sense of transforming the cumulant function by a linear change of the argument.

Effectively, the Esscher transform yields an explicit change of measure, where we have access to the characteristics of the jump processes I_j also under the new risk-neutral measure.

Risk-Neutral Probabilities and the Esscher Transform

It was intorduced by Esscher (1932) to study risk theory and used by Gerber and Siu (1994) for derivatives pricing. Suppose we have f probability density and θ is a real number. Then, as long as $\int_R e^{\theta y} f(y)dy < +\infty$, we can define a new probability density

$$f(x;\theta) = \frac{e^{\theta x} f(x)}{\int_R e^{\theta y} f(y)dy}.$$

Risk-Neutral Probabilities and the Esscher Transform

Let now $\theta(t)$ be a $(p+n)$-dimensional vector of real-valued continuous functions on $[0, T]$

$$\theta(t) = (\widehat{\theta}_1(t), ..., \widehat{\theta}_p(t), \tilde{\theta}_1(t), ..., \tilde{\theta}_n(t)).$$

Define for $0 \leq t \leq \tau$ the stochastic exponential

$$Z^\theta(t) = \prod_{k=1}^{p} \widehat{Z}_k^\theta(t) \times \prod_{j=1}^{n} \tilde{Z}_j^\theta(t),$$

where

Risk-Neutral Probabilities and the Esscher Transform

$$\widehat{Z}_k^{\theta}(t) = \exp(\int_0^t \widehat{\theta}_k(s)dB_k(s) - \frac{1}{2}\int_0^t \widehat{\theta}_k^2(s)ds), \quad k = 1, 2, ..., p,$$

and

$$\tilde{Z}_j^{\theta}(t) = \exp(\int_0^t \tilde{\theta}_j(s)dI_j(s) - \phi_j(0, t; \tilde{\theta}_j(\cdot))), \quad j = 1, 2, ..., n.$$

We note, that $\widehat{Z}_k^{\theta}(t)$ and $\tilde{Z}_j^{\theta}(t)$ are a positive local martingales with expectation equals to one. Hence, we can define an equivalent probability measure Q^{θ} such that Z^{theta} is the density process of the Radon-Nikodym derivative dQ^{θ}/dP, that is,

$$\frac{dQ^{\theta}}{dP}|_{\mathcal{F}_t} = Z^{\theta}(t).$$

Risk-Neutral Probabilities and the Esscher Transform

The expectation operator wrt the probability Q^θ is denoted by $E_\theta[\cdot]$. We observe that the Radon-Nikodym derivative dQ^θ/dP can be factorized as

$$\frac{dQ^\theta}{dP}|_{\mathcal{F}_t} = \prod_{k=1}^{p} \hat{Z}_k^\theta(t) \times \prod_{j=1}^{n} \tilde{Z}_j^\theta(t).$$

Hence, we associate a price of risk to each random source given by the Brownian motion B_k and the jump factors I_j, $k = 1, 2, ..., p, j = 1, 2, ..., n$ in the model of spot price.

Risk-Neutral Probabilities and the Esscher Transform

Let us study how the characteristics of B and I are changing when we apply the Esscher transform.

Proposition 4.4. (Benth *et al. (2008)***)** Under measure Q^θ the processes

$$B_k^\theta(t) = B_k(t) - \int_0^t \widehat{\theta}_k(s)ds$$

are Brownian motions, $k = 1, 2, ..., p$. Furthermore, for each $j = 1, 2, ..., n$, $I_j(t)$ ia an II process with drift

$$\gamma_j(t) + \int_0^t \int_{|z|<1} z(e^{\tilde{\theta}_j(u)z} - 1)l_j(dz, du),$$

and compensatior measure $e^{\tilde{\theta}_j(t)z}l_j(dz, dt)$. We denote the new random jump measure by N_j^θ and its compensator by $\tilde{N}_j^\theta$.

Risk-Neutral Probabilities and the Esscher Transform

We note, that $N_j^\theta = N_j$. However, $\tilde{N}_j^\theta$ is not coinciding with $\tilde{N}_j$:

$$
\begin{aligned}
\tilde{N}_j^\theta &= N_j^\theta - e^{\tilde{\theta}_j(t)z} l_j(dz, dt) \\
&= N_j - l_j - (e^{\tilde{\theta}_j(t)z} - 1) l_j \\
&= \tilde{N} - (e^{\tilde{\theta}_j(t)z} - 1) l_j.
\end{aligned}
$$

Hence, $\tilde{N}_j$ translates to $\tilde{N}_j^\theta$ by sybtraction of a drift, exactly as the Girsanov transform of B_k to B_k^θ.

Esscher Transform for Some Specific Models: Time Inhomogeneous Compound Poisson Process

Compensator $l(dz, dt) = \lambda(t) F_X(dz) dt$, where F_X is the distribution of the jump size r.v. X and $\lambda(t)$ is the time-dependent jump intensity. The compensator measure onder Q^θ is (Prop. 4.4.):

$$l^\theta(dt, dz) = \lambda(t) e^{\tilde{\theta} z} F_X(dz) dt.$$

A common choice of jump size is the exponential distribution with exapectation μ_J. Then, we find the compensator measure under Q^θ to be

$$l^\theta(dt, dz) = \frac{\lambda(t)}{\mu_J} \exp(-(\frac{1}{\mu_J} - \tilde{\theta}) z) dz dt.$$

Hence, with $\tilde{\theta} < 1/\mu_J, I(t)$ will remain a compound Poisson process under Q^θ, with expectation $1/(1/\mu_J - \tilde{\theta})$ and intensity $\lambda(t)/(1 - \mu_J \tilde{\theta})$.

Pricing Forwards and Swaps under SV with Delay and Jumps

Pricing Forwards: The Geometric Case

Let us assume a geometric spot price model:

$$S(t) = \Lambda(t) \exp(\sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t)) \quad (geometric \quad case)$$

The forward price at time $t \geq 0$ for contracts with settlement at $T \geq t$ is explicitly given in the following Proposition.

Pricing Forwards: The Geometric Case

Proposition 1. Suppose that

$$\sup_{0\leq t\leq T} |\eta_j(t, Y_j(t+\theta))e^{-\int_t^{\tau} \beta_j(v)dv} + \tilde{\theta}_j(t)| \leq c_j, \quad a.s., \quad \theta \in [-\tau, 0].$$

The dynamics of $t \to f(t,T)$ wrt Q^θ is

$$\begin{aligned} \frac{df(t,T)}{f(t,T)} &= \{\textstyle\sum_{i=1}^m \sigma_i(t, X_i(t+\theta)) \exp(-\int_t^T \alpha_i(u)du)\} dB^\theta(t) \\ &+ \textstyle\sum_{j=1}^n \{\int_R \exp(z\eta_j(t, Y_j(t+\theta))e^{-\int_t^{\tau} \beta_j(u)du}) - 1\} \tilde{N}_j^\theta(dt, dz). \end{aligned}$$

Pricing Forwards: The Geometric Case (Samuelson Effect)

We see that the forward price dynamics becomes a geometric model, and in the case when we do not have any jumps terms Y_i, we are back to a geometric Brownian motion with time-dependent local stochastic volatility with delay and jumps

$$\frac{df(t,T)}{f(t,T)} = \{\sum_{i=1}^{m} \sigma_i(t, X_i(t+\theta)) \exp(-\int_t^T \alpha_i(u)du)\} dB^\theta(t).$$

Pricing Forwards: The Geometric Case (Samuelson Effect)

Hence, we find that the volatilites of the forward contract are decreasing with time to delivery, being smaller than the spot volatility.

When time to delivery approaches zero, however, the forward volatility converges to the volatilities of the underlying spot $\sigma_i(t, \cdot)$.

This is known as the *Samuelson effect (1965)* and is a direct result of the mean-reverting spot price dynamics.

In our case, we have *delayed Samuelson effect*.

Pricing Forwards: The Geometric Case (Samuelson Effect)

We observe a similar Samuelson effect when including jumps in the spot dynamics, where so-called 'jump volatility' is expressed through the integrands

$$\exp(z\eta_j(t, Y_j(t+\theta))e^{-\int_t^T \beta_j(u)du}) - 1.$$

Suppose that $\eta_j(tY_j(t+\theta)) > 0$. When $t \to T$, the integrands are converging to $\exp(z\eta_j(t)) - 1$, which is identical to the corresponding terms of the spot price dynamics.

Pricing Forwards: The Geometric Case (Samuelson Effect)

However, when $t < T$, we find for $z \geq 0$,

$$0 \leq \{\exp(z\eta_j(t, Y_j(t+\theta))e^{-\int_t^T \beta_j(u)du}) - 1\} \leq \exp(z\eta_j(t, Y_j(t+\theta))) - 1,$$

meaning that the positive jumps in the spot price dynamics are scaled down in the forward price dynamics, and the downscaling is exponential wrt the mean reversion.

For the negative jumps (z<0) we find

$$0 \geq \{\exp(z\eta_j(t, Y_j(t+\theta))e^{-\int_t^T \beta_j(u)du}) - 1\} \geq \exp(z\eta_j(t, Y_j(t+\theta))) - 1.$$

Thus, also the negative jumps in the spot are scaled down in the forward.

Pricing Forwards: The Geometric Case (Samuelson Effect)

All in all, the 'jump volatility' of the forward dynamics is a downscaling of the jump volatility of the spot, in line with the observations we made for the Brownian motion terms.

The downscaling of the jump volatility is dependent on time to maturity. The farther away from maturity, the less influence the jump volatility gets from the spot. The influence is 'discounted' by the speed of mean reversion. The stronger the speed of mean reversion, the faster jumps in the spot price are whipped out along the term structure.

Pricing of Swaps (The Geometric Case)

It is in general not possible to state an explicit formula for the swap price $F(t,\tau_1,\tau_2)$ of contracts with settlement over the period $[\tau_1,\tau_2]$ when we choose to work with a geometric model of the spot price dynamics. We now elaborate on the approximation of the swap price.

Recall the relation between forward and swaps

$$F(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2) f(t,u)du.$$

Pricing of Swaps (The Geometric Case)

There exist a few cases where we can derive explicit forward prices for contracts with delivery over a period in the geometric case.

Let's restrict our attention to a non-stationary dynamics of the spot price represented by setting the mean reversion coefficients α_i and β_j equal to zero. Hence, the spot price becomes a geometric Brownian motion in the case of no jumps.

Pricing of Swaps (The Geometric Case)

$$S(t) = \Lambda(t)\,\mathrm{exp}(\sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t)) \quad (geometric \quad case)$$

We find the following risk-neutral dynamics of $F(t, \tau_1, \tau_2)$.

Pricing of Swaps (The Geometric Case)

Proposition 2. Suppose that the mean reversion coefficients α_i and β_j are set equal to zero for $j = 1, 2, ..., n$ and $i = 1, 2, ..., m$. The the risk-neutral dynamics of the forward price $F(t, \tau_1, \tau_2)$ is given by

$$\begin{aligned} \frac{dF(t,\tau_1,\tau_2)}{F(t-,\tau_1,\tau_2)} &= \sum_{i=1}^m \sigma_i(t, X_i(t+\theta))dB^\theta(t) \\ &+ \sum_{j=1}^n \int_R (e^{\eta_j(t,Y_j(t+\theta))z} - 1)\tilde{N}_j^\theta(dz, dt). \end{aligned}$$

However, having zero speed of mean reversion creates a market which does not have the right properties for modelling the riskness of forward, and henceforth swaps.

Pricing of Swaps (The Arithmetic Case)

Suppose we have the spot price process that is modelled as the arithmetic dynamics:

$$S(t) = \Lambda(t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t) \quad (arithmetic \quad case)$$

Pricing of Swaps (The Arithmetic Case): The Asymptotics of the Swap Prices

Letting $\tau_1 \to +\infty$ and the length of the delivery period $\tau_1 - \tau_2$ be fixed, we obtain

$$F(t,\tau_1,\tau_2) - \int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2)\Lambda(u)du - \Theta(t,\tau_1,\tau_2;\theta(\cdot)) \to_{\tau_1 \to +\infty} 0.$$

Hence, the swap price behaves asymptotically as the weighted average seasonal function $\Lambda(u)$ and risk-adjustment function Θ. This is in line with the asymptotics of forwards.

Pricing of Swaps (The Arithmetic Case)

Proposition 3. The risk-neutral dynamics of the stochastic process $F(t,\tau_1,\tau_2)$ is given by

$$\begin{aligned} dF(t,\tau_1,\tau_2) &= \sum_{i=1}^{m} \sigma_i(t, X_i(t+\theta)) \int_{\tau_1)}^{\tau_2} w(u,\tau_1,\tau_2) e^{-\int_v^u \alpha_i(s)ds} du dB^{\theta}(t) \\ &+ \sum_{j=1}^{n} \int_R z\eta_j(t, Y_j(t+\theta)) \\ &\times \int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2) e^{-\int_v^u \beta_j(s)ds} du \tilde{N}_j^{\theta}(dt,dz). \end{aligned}$$

Pricing of Swaps (The Arithmetic Case)

The volatility in the dynamics of $F(t,\tau_1,\tau_2)$ has an average delayed Samuelson effect. For constant α_i and β_j we have:

$$\begin{aligned}\sigma_i(t,X_i(t+\theta))\int_{\tau_1)}^{\tau_2} w(u,\tau_1,\tau_2)e^{-\int_v^u \alpha_i(s)ds}du &= \frac{\sigma_i(t,X_i(t+\theta))}{\alpha_i(\tau_2-\tau_1)} \\ &\times \left(e^{-\alpha_i(\tau_1-t)} - e^{-\alpha_i(\tau_2-t)}\right).\end{aligned}$$

The same argument holds true for the jump volatility.

Conclusion

1. *Independent Increments Processes*
2. *Stochastic Volatilites with Delay and Jumps*
3. *Stochastic Modelling of Energy Markets' Contracts*: Geometric and Arithmetic Models
4. *Some Results*: Pricing of Forwards and Swaps Based on the Spot Price

The End

Thanks the Audience for Coming, for Your Time and Attention!