Delayed Heston Model: Improvement of Vol Surface and Hedging of Vol Swaps
*

Anatoliy Swishchuk
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada
ASMDA 2013
Mataró, Barcelona, Spain
June 25-29, 2013

*This research is supported by NSERC

Outline of Presentation

1. Delayed Heston Model: Motivation & Definition

2. Delayed Heston Model: Improvement of Volatility Surface

3. Delayed Heston Model: Variance and Volatility Swaps

4. Delayed Heston Model: Hedging Volatility Swaps

5. Conclusion

Classical Heston Model

The Heston model is one of the most popular stochastic volatility models in the industry, as semi-closed formulas for vanilla option prices are available, few (five) parameters need to be calibrated, and it accounts for the mean-reverting feature of the volatility:

$$\begin{cases} dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^Q \\ dV_t &= [\gamma(\theta^2 - V_t)]dt + \delta\sqrt{V_t} dW_t^Q, \end{cases}$$

where S_t is a stock price, V_t is the stochastic variance, W_t^Q is the Wiener process with respect to the the risk-neutral probability Q and r is the interest rate.

Delayed Heston Model

• Motivation: to include past history (a.k.a. delay) of the variance (over some delayed time interval $[t-\tau, t]$)

• Advantage: Improvement of the Volatility Surface Fitting (44% reduction of the calibration error) compare with Classical Heston model

• Goal: to price and hedge volatility swaps

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

We'd like to take into account not only its current state (as it is the case in the Heston model) but also its past history over some interval $[t-\tau, t]$, where τ is a positive constant and is called the delay. Namely, at each time t, the immediate future volatility at time $t+\epsilon$ will not only depend on its value at time t but also on all its history over $[t-\tau, t]$.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

Namely, at each time t, the immediate future volatility at time $t+\epsilon$ will not only depend on its value at time t but also on all its history over $[t-\tau, t]$. Starting from the well-known discrete-time GARCH(1,1) model, a continuous-time GARCH variance diffusion incorporating delay (let's refer to it as 'delayed vol') was introduced in a paper Sw. (2005). Unfortunately, the latter model doesn't lead to (semi-)closed formulas for the vanilla options, making it difficult to use for practitioners willing to calibrate on vanilla market prices (which can be a serious drawback).

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

Nevertheless, one can notice that the Heston model and 'delayed vol' are very similar in the sense that the expected values of the variances are the same - when we make the delay tends to 0 in 'delayed vol'. As mentioned before, the Heston framework is very convenient for practitioners, and therefore it is naturally tempting to adjust the Heston dynamics in order to incorporate - in some way - the delay introduced in 'delayed vol'.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

We considered in a first approach adjusting the Heston drift by a deterministic function of time so that the expected value of the variance under our new delayed Heston model is equal to the one under 'delayed vol'. Our approach can therefore be seen as a variance 1st moment correction of the Heston model, in order to account for the delay. It is important to note that our model is a generalization of the classical Heston model (the latter corresponding to the zero delay case $\tau = 0$ of our model).

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

We performed numerical tests to validate our approach. With recent market data (Sept. 30th 2011, underlying EURUSD), we performed the model calibration on the whole market vanilla option price surface (14 maturities from 1M to 10Y, 5 strikes ATM, 25 Delta Call/Put, 10 Delta Call/Put). The results show a significant (44%) reduction of the average absolute calibration error compared to the Heston model (i.e. average of the absolute differences between market and model prices).

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

Further, we considered variance and volatility swaps hedging and pricing in our delayed Heston framework. These contracts are widely used in the financial industry and therefore it is relevant to know their price processes (how much they worth at each time t) and how we can hedge a position on them, i.e. theoretically cancel the risk inherent to holding one unit of them.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

Using the fact that every continuous local martingale can be represented as a time-changed Brownian motion, as well as the Brockhaus & Long approximation (that allows to approximate the expected value of the square-root of an almost surely non negative random variable using a 2nd order Taylor expansion approach), we were able to derive closed formulas for variance and volatility swaps price processes.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

In addition, as variance swaps are relatively liquid instruments in the market (i.e. they can be easily bought and sold), we considered the question of hedging a position on a volatility swap using variance swaps in our framework.

We were able to derive a closed formula for the dynamic hedge ratio, i.e. the number of units of variance swaps to hold at each time in order to hedge a position on a volatility swap.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

- Motivation: past history of the varinace in its diffusion (over some delayed time interval $[t-\tau, t]$)

- Non-Markov continuous-time GARCH model (Sw. (2005))

$$\frac{dV_t}{dt} = \gamma(\theta^2 - V_t) + \alpha\left[\frac{1}{\tau}(\int_{t-\tau}^{t} \sqrt{V_s}dZ_s^Q - (\mu - r)\tau)^2 - V_t\right]$$

- $$\begin{cases} dV_t & = & [\gamma(\theta^2 - V_t) + \epsilon_\tau(t)]dt + \delta\sqrt{V_t}dW_t^Q \\ \epsilon_\tau(t) & := & \alpha\left[\tau(\mu - r)^2 + \frac{1}{\tau}\int_{t-\tau}^{t} E^Q(V_s)ds - E^Q(V_t)\right]. \end{cases}$$

We note, that $\lim_{\tau\to 0}\sup_{t\in R_+}|\epsilon_\tau(t)| = 0$.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Calibration Results

- Semi-closed formulas available for call options

- September 30th 2011 for underlying EURUSD on the whole volatility surface (14 maturities from 1M to 10Y, 5 strikes: ATM, 25D call/put, 10D call/put)

- 44% reduction of the average absolute calibration error: 46bp for delayed Heston, 81bp for Heston

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Variance & Volatility Swaps Pricing

- Realized variance: $V_R := \frac{1}{T}\int_0^T V_s ds$
- $K_{var} = E^Q[V_R], \quad K_{vol} = E^Q[\sqrt{V_R}]$
- Brockhaus & Long approximation: $E[\sqrt{Z}] \approx \sqrt{E[Z]} - \frac{Var[Z]}{8E[Z]^{3/2}}$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Variance & Volatility Swaps Pricing

- Using time-changed Brownian motion representation for continuous local martingales, we get closed formula for VarSwap and VolSwap fair strikes

- $x_t := -(V_0 - \theta_\tau^2)e^{\gamma - \gamma_\tau t} + e^{\gamma t}(V_t - \theta_\tau^2)$

- $dx_t = f(t, x_t)dW_t^Q, \quad x_t = \widehat{W}_{T_t}, \quad T_t = < x >_t = \int_0^t f^2(s, x_s)ds$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Variance & Volatility Swaps Pricing

- $\theta_\tau^2 := \theta^2 + \frac{\alpha\tau(\mu - r)^2}{\gamma}, \quad \gamma_\tau := \alpha + \gamma + \frac{\alpha}{\gamma_\tau \tau}(1 - e^{\gamma_\tau \tau})$

- $V_t = \theta_\tau^2 + (V_0 - \theta_\tau^2)e^{-\gamma_\tau t} + e^{-\gamma t}\widehat{W}_{T_t} = E^Q[V_t] + e^{-\gamma t}\widehat{W}_{T_t}$

The parameter θ_τ^2 can be interpreted as the delayed-adjusted long-range variance. We note, that $\theta_\tau^2 \to \theta^2$ as $\tau \to 0$.

The parameter γ_τ can be interpreted as the delayed-adjusted mean-reverting speed. We note, that $\gamma_\tau \to \gamma$ as $\tau \to 0$.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Volatility Swap Hedging

- Price Processes:

- VarSwap: $X_t(T) := E_t^Q[V_R]$,

- VolSwap: $Y_t(T) := E_t^Q[\sqrt{V_R}]$,

- $V_R := \frac{1}{T}\int_0^T V_s ds$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

- Portfolio containing 1 VolSwap and β_t VarSwaps:

$$\Pi_t = e^{-r(T-t)}[Y_t(T) - K_{vol} + \beta_t(X_t(T) - K_{var})]$$

- If $I_t := \int_0^t V_s ds$ is the accumulated varinace at time t, then:

$$\begin{aligned} X_t(T) &= E_t^Q[\frac{I_t}{T} + \frac{1}{T}\int_t^T V_s ds] := g(t, I_t, V_t) \\ Y_t(T) &= E_t^Q[\sqrt{\frac{I_t}{T} + \frac{1}{T}\int_t^T V_s ds}] := h(t, I_t, V_t) \end{aligned}$$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Volatility Swap Hedging

- We compute the infinitesimal variations (using the fact that $X_t(T)$ and $Y_t(T)$ are martingales):

$$\begin{aligned} dX_t(T) &= \frac{\partial g}{\partial V_t}\delta\sqrt{V_t}dW_t^Q \\ dY_t(T) &= \frac{\partial h}{\partial V_t}\delta\sqrt{V_t}dW_t^Q \\ d\Pi_t &= r\Pi_t dt + e^{-r(T-t)}[\frac{\partial h}{\partial V_t} + \beta_t\frac{\partial g}{\partial V_t}]\delta\sqrt{V_t}dW_t^Q \end{aligned}$$

$\Rightarrow$

$$\beta_t = -\frac{\frac{\partial h}{\partial V_t}}{\frac{\partial g}{\partial V_t}} = -\frac{\frac{\partial Y_t(T)}{\partial V_t}}{\frac{\partial X_t(T)}{\partial V_t}}$$

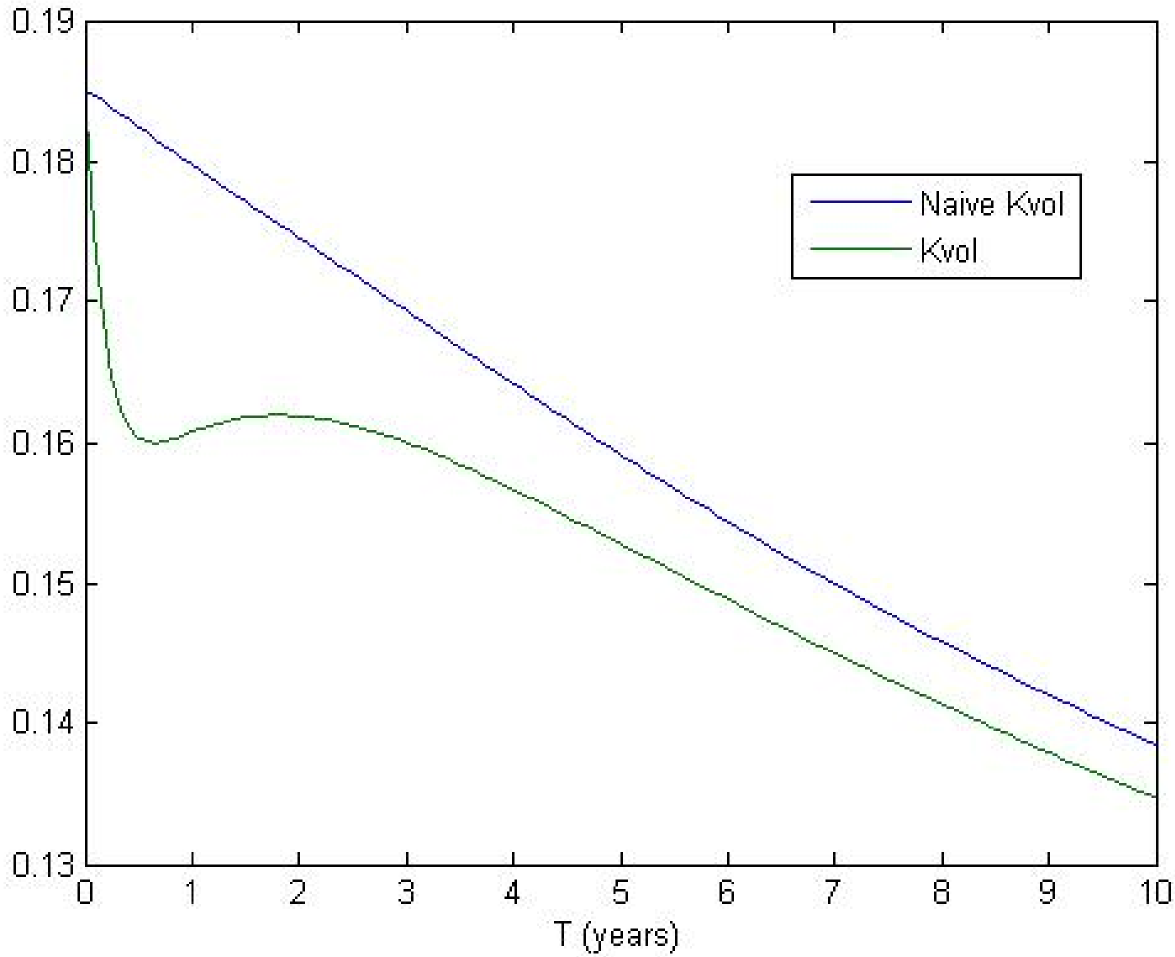
-hedge ratio

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Numerical Results)

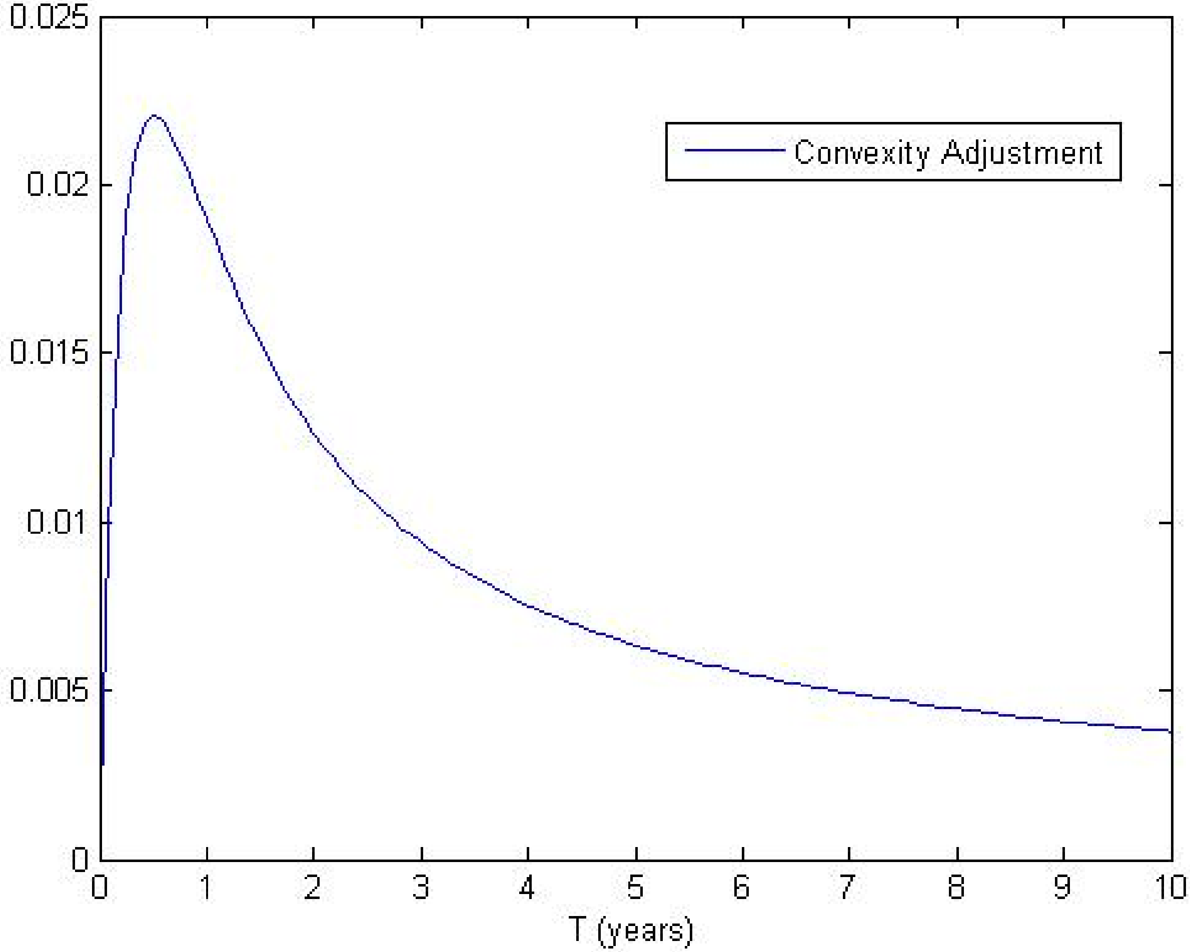
We take the parameters that have been calibrated above (vanilla options on September 30^{th} 2011 for underlying EURUSD, maturities from 1M to 10Y, strikes ATM, 25D Put/Call, 10D Put/Call), namely

$$(v_0, \gamma, \theta^2, \delta, c, \alpha, \tau) = (0.0343, 3.9037, 10^{-8}, 0.808, -0.5057, 71.35, 0.7821).$$

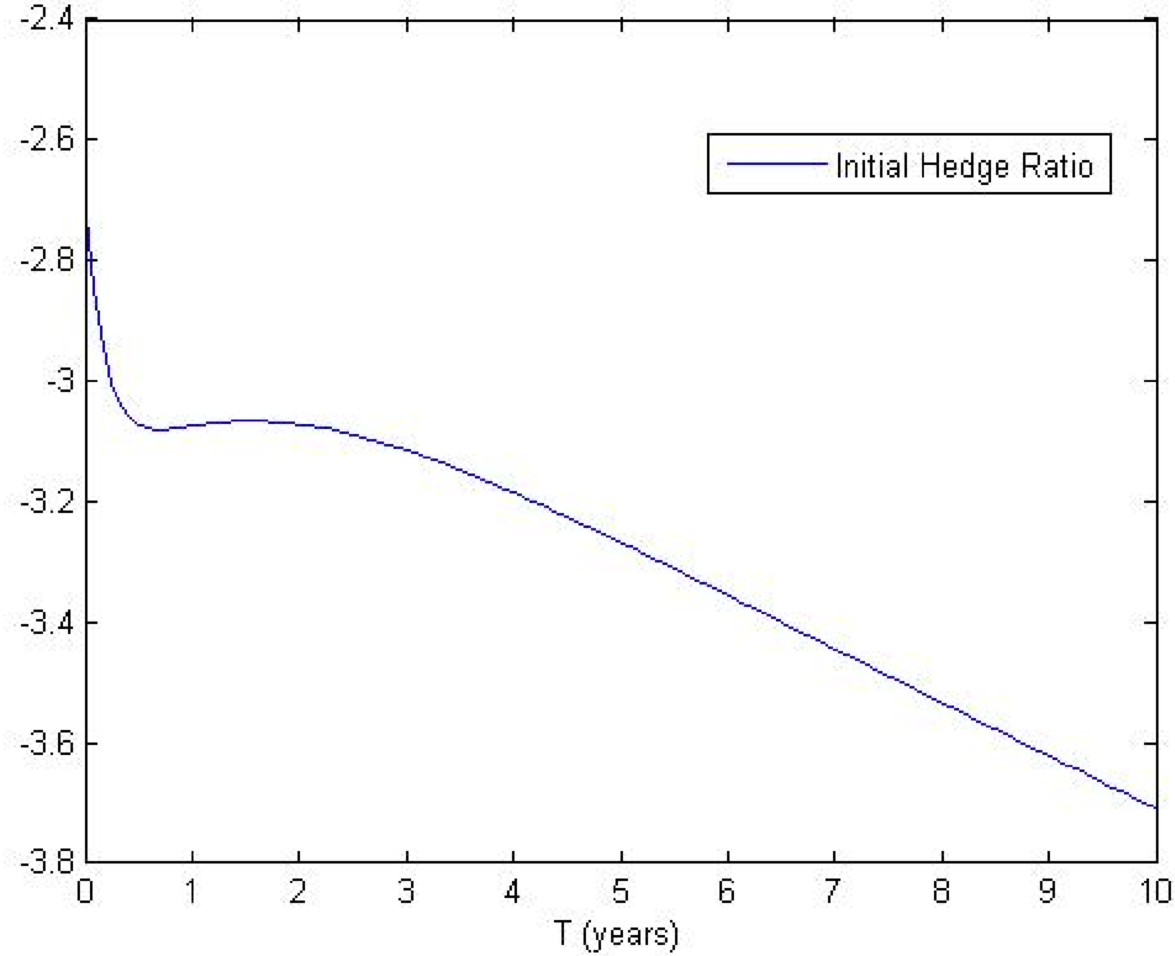
We plot the naive Volatility Swap strike $\sqrt{K_{var}}$ and the adjusted Volatility Swap strike $\sqrt{K_{var}} - \frac{Var^Q(V_R)}{8K_{var}^{\frac{3}{2}}}$ along the maturity dimension, as well as the convexity adjustment $\frac{Var^Q(V_R)}{8K_{var}^{\frac{3}{2}}}$:



Naive VolSwap vs. Adjusted VolSwap Strikes



Convexity Adjustment



Initial Hedge ratio $\beta_0(T)$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps

These results had been obtained together with my PhD student Nelson Vadori and have been submitted to Wilmott J. as two papers:

1. 'Delayed Heston Model: Improvement of the Volatility Surface Fitting'

2. 'Pricing and Hedging of Volatility Swap in the Delayed Heston Model: Part 2'

Conclusion

1. Delayed Heston Model: Motivation & Definition

2. Delayed Heston Model: Improvement of Volatility Surface

3. Delayed Heston Model: Variance and Volatility Swaps

4. Delayed Heston Model: Hedging Volatility Swaps

The End

Thank You for Your Time and Attention!

E-mail: aswish@ucalgary.ca

UNIVERSITY OF
CALGARY