General Compound Hawkes Processes in Limit Order Books *

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Outline of Presentation

- *Introduction*

- *Hawkes Process* (HP) and *General Compound Hawkes Process* (GCHP)

- *Applications of GCHP:* Finance/Limit Order Books-Stock Mid-Price

- *Functional Central Limit Theorems* (FCLT) and *Law of Large Numbers* (LLN) for GCHP

- *Some Numerical Examples* for GCHP

- *Monday, February 5, 2018*-Another 'Black Monday'?!

Structure of the Talk

$\boxed{\text{Introduction}}$ $\Downarrow$

$\boxed{\text{Hawkes Processes: Definition and Examples}}$ $\Downarrow$

$\boxed{\text{General Compound Hawkes Processes (GCHP)}}$ $\Downarrow$

$\boxed{\text{Application of GCHP: Mid-Price in Limit Order Books (LOB)}}$ $\Downarrow$

$\boxed{\text{Various LLN and FCLT for GCHP}}$ $\Downarrow$

$\boxed{\text{Numerical Example: CISCO Data (5 Days, 3-7 Nov 2014)}}$

Introduction

The Hawkes process is a self-exciting simple point process first introduced by A. Hawkes in 1971. The future evolution of a self-exciting point process is influenced by the timing of past events. The process is non-Markovian except for some very special cases. Thus, the Hawkes process depends on the entire past history and has a long memory. The Hawkes process has wide applications in neuroscience, seismology, genome analysis, finance, insurance, and many other fields. The present talk is devoted to the introduction to the Hawkes process and their applications in finance and insurance.

Introduction I

The Hawkes process (Hawkes (1971)) is a simple point process that has self-exciting property, clustering effect and long memory.

It has been widely applied in seismology, neuroscience, DNA modelling and many other fields, including finance (Embrechts et al. (2011)) and insurance (Stabile et al. (2010)).

Introduction II

The most recent application of HP is financial analysis, in particular, limit order books. In this talk, we further study various new Hawkes processes, namely, so-called general (and regime-switching general) compound Hawkes processes to model the mid-price processes in the limit order books. We prove Law of Large Numbers and Functional Central Limit Theorems (FCLT) for these processes.

Introduction III

The latter two FCLTs are applied to limit order books where we use these asymptotic methods to study the link between price volatility and order flow in our models by studying the diffusion limits of these mid-price processes.

The volatilities of price changes are expressed in terms of parameters describing the arrival rates and price changes. We also present some numerical examples.

Introduction VI

The general compound Hawkes process was first introduced in Sw. (2017) (IME 2017, Vienna) to model the risk process in insurance, and studied in detailed in Sw. 2018 (Wilmott Magazine, v. 2018, Issue 94, pp. 50-57).

In the paper Sw. *et al.* (2018) (Handbook of Applied Econometrics, Routledge Taylor and Francis, 2018) we obtained functional CLTs and LLNs for so-called compound Hawkes process with dependent orders and regime-switching compound Hawkes process.

We note, that Stabile & Torrisi (2010) were the first who replaced Poisson process by a simple Hawkes process in studying the classical problem of the probability of ruin.

Hawkes Process: Counting Process I

Definition 1 (Counting Process). A counting process is a stochastic process $N(t), t \geq 0$, taking positive integer values and satisfying: $N(0) = 0$. It is almost surely finite, and is a right-continuous step function with increments of size $+1$.

Denote by $\mathcal{F}^N(t), t \geq 0$, the history of the arrivals up to time t, that is, $\{\mathcal{F}^N(t), t \geq 0\}$, is a filtration, (an increasing sequence of σ-algebras).

Hawkes Process: Counting Process II

A counting process $N(t)$ can be interpreted as a cumulative count of the number of arrivals into a system up to the current time t.

The counting process can also be characterized by the sequence of random arrival times $(T_1, T_2, ...)$ at which the counting process $N(t)$ has jumped. The process defined by these arrival times is called a point process.

Hawkes Process: Point Process

Definition 2 (Point Process). If a sequence of random variables $(T_1, T_2, ...)$, taking values in $[0, +\infty)$, has $P(0 \leq T_1 \leq T_2 \leq ...) = 1$, and the number of points in a bounded region is almost surely finite, then, $(T_1, T_2, ...)$ is called a point process.

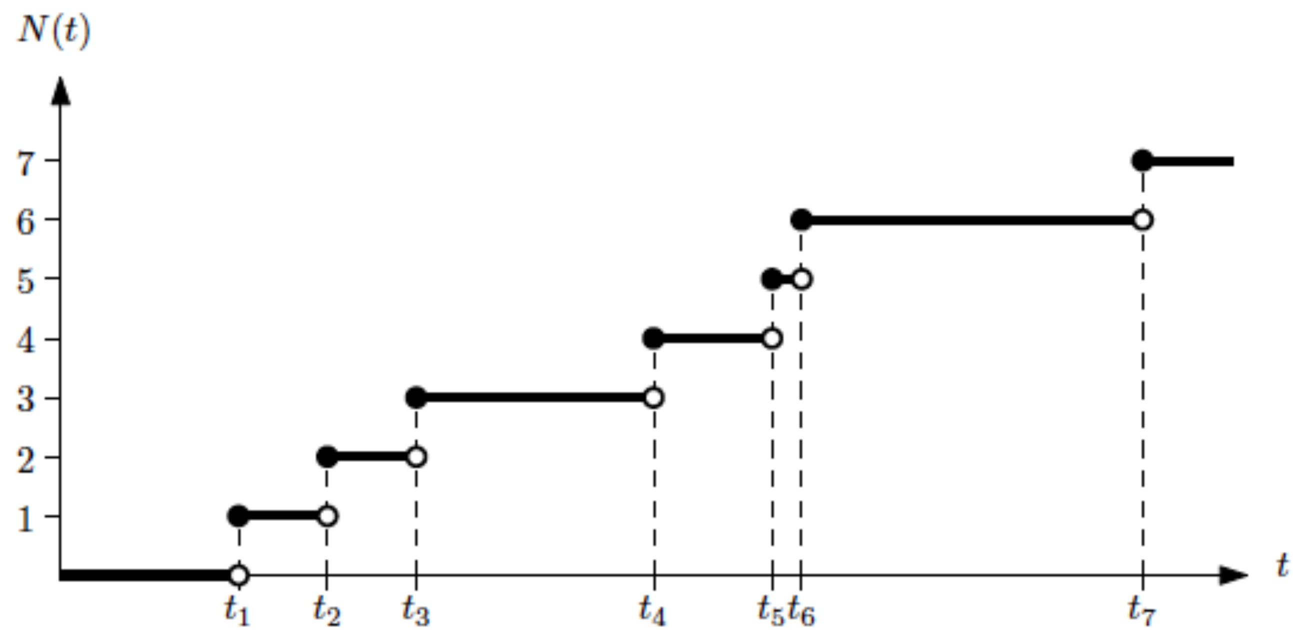


Fig. 1: An example point process realisation $\{t_1, t_2, \ldots\}$ and corresponding counting process $N(t)$.

Hawkes Process: Conditional Intensity Function

Definition 3 (Conditional Intensity Function). Consider a counting process $N(t)$ with associated histories $\mathcal{F}^N(t), t \geq 0$. If a non-negative function $\lambda(t)$ exists such that

$$\lambda(t) = \lim_{h \to 0} \frac{E[N(t+h) - N(t) | \mathcal{F}^N(t)]}{h}, \quad (1)$$

then it is called the conditional intensity function of $N(t)$. We note, that sometimes this function is called the hazard function.

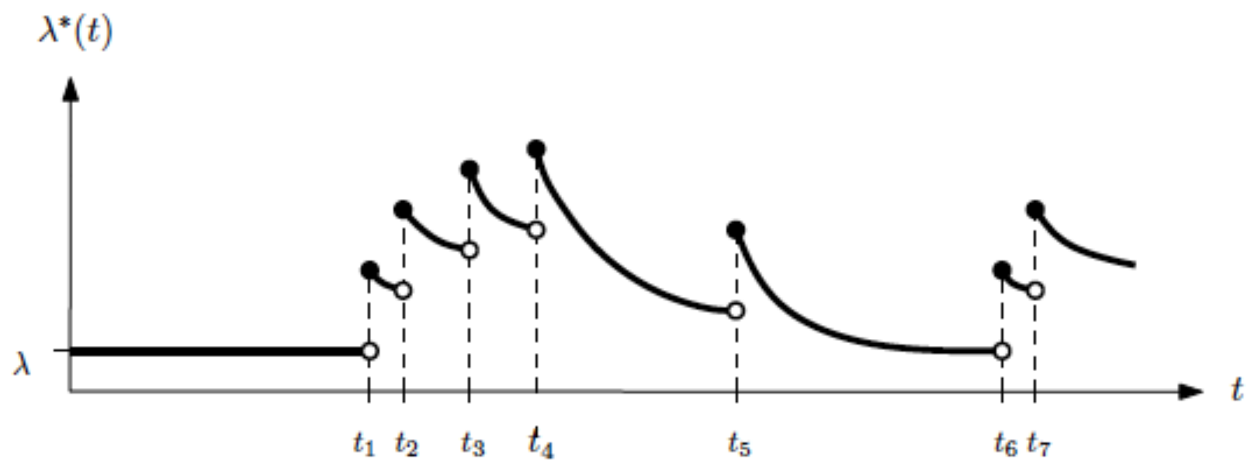


Fig. 2: An example conditional intensity function for a self-exciting process.

Hawkes Process: Definition I

Definition 4 (One-dimensional Hawkes Process). The one-dimensional Hawkes process is a point process $N(t)$ which is characterized by its intensity $\lambda(t)$ with respect to its natural filtration:

$$\lambda(t) = \lambda + \int_0^t \mu(t-s)dN(s), \tag{2}$$

where $\lambda > 0$, and the response function $\mu(t)$ is a positive function and satisfies $\int_0^{+\infty} \mu(s)ds < 1$.

Hawkes Process: Definition II

The constant λ is called the background intensity and the function $\mu(t)$ is sometimes also called the excitation function. We suppose that $\mu(t) \neq 0$ to avoid the trivial case, which is, a homogeneous Poisson process. Thus, the Hawkes process is a non-Markovian extension of the Poisson process.

Hawkes Process: Definition III

The interpretation of equation (2) is that the events occur according to an intensity with a background intensity λ which increases by $\mu(0)$ at each new event then decays back to the background intensity value according to the function $\mu(t)$. Choosing $\mu(0) > 0$ leads to a jolt in the intensity at each new event, and this feature is often called a self-exciting feature, in other words, because an arrival causes the conditional intensity function $\lambda(t)$ in (1)-(2) to increase then the process is said to be self-exciting.

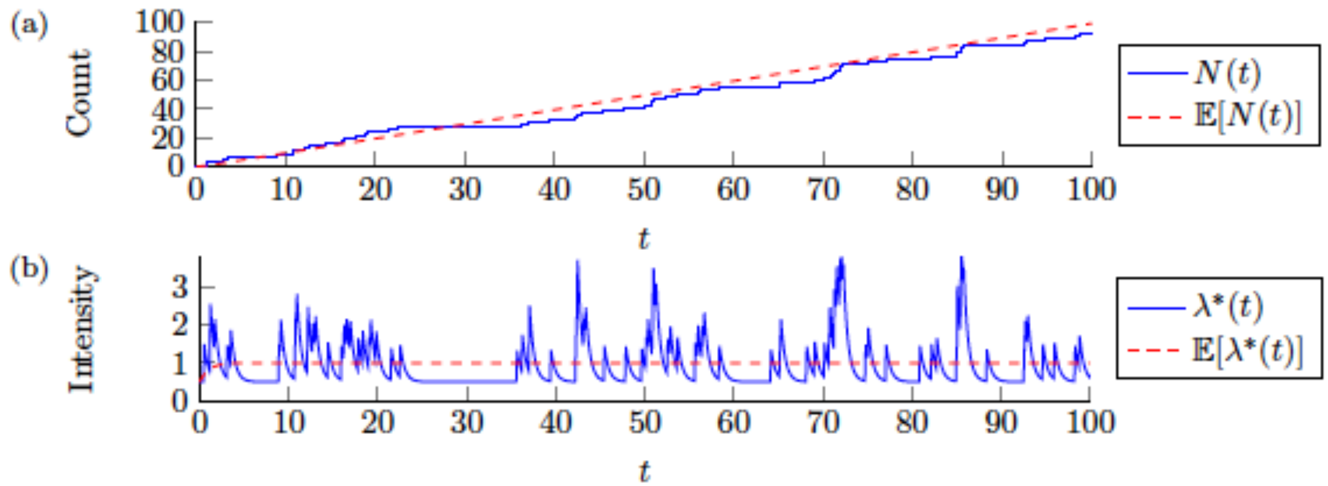


Fig. 3: (a) A typical Hawkes process realisation $N(t)$, and its associated $\lambda^*(t)$ in (b), both plotted against their expected values.

Hawkes Process: Definition IV

With respect to definitions of $\lambda(t)$ in (1) and $N(t)$ (2), it follows that

$$P(N(t+h)-N(t)=m|\mathcal{F}^N(t))=\begin{cases} \lambda(t)h+o(h), & m=1 \\ o(h), & m>1 \\ 1-\lambda(t)h+o(h), & m=0. \end{cases}$$

Hawkes Process: Definition V

We should mention that the conditional intensity function $\lambda(t)$ in (1)-(2) can be associated with the compensator $\Lambda(t)$ of the counting process $N(t)$, that is:

$$\Lambda(t) = \int_0^t \lambda(s)ds. \tag{3}$$

Thus, $\Lambda(t)$ is the unique $\mathcal{F}^N(t), t \geq 0$, predictable function, with $\Lambda(0) = 0$, and is non-decreasing, such that

$$N(t) = M(t) + \Lambda(t) \quad a.s.,$$

where $M(t)$ is an $\mathcal{F}^N(t), t \geq 0$, local martingale (This is the Doob-Meyer decomposition of N.)

Hawkes Process: Definition VI

A common choice for the function $\mu(t)$ in (2) is one of exponential decay:

$$\mu(t) = \alpha e^{-\beta t}, \tag{4}$$

with parameters $\alpha, \beta > 0$. In this case the Hawkes process is called the Hawkes process with exponentially decaying intensity.

Thus, the equation (2) becomes

$$\lambda(t) = \lambda + \int_0^t \alpha e^{-\beta(t-s)} dN(s), \tag{5}$$

We note, that in the case of (4), the process $(N(t), \lambda(t))$ is a continuous-time Markov process, which is not the case for the choice (2).

Hawkes Process: Definition VII

With some initial condition $\lambda(0) = \lambda_0$, the conditional density $\lambda(t)$ in (5) with the exponential decay in (4) satisfies the SDE

$$d\lambda(t) = \beta(\lambda - \lambda(t))dt + \alpha dN(t), \quad t \geq 0,$$

which can be solved (using stochastic calculus) as

$$\lambda(t) = e^{-\beta t}(\lambda_0 - \lambda) + \lambda + \int_0^t \alpha e^{-\beta(t-s)} dN(s),$$

which is an extension of (5).

Hawkes Process: Definition VIII

Another choice for $\mu(t)$ is a power law function:

$$\lambda(t) = \lambda + \int_0^t \frac{k}{(c + (t - s))^p} dN(s) \tag{6}$$

for some positive parameters c, k, p.

This power law form for $\lambda(t)$ in (6) was applied in the geological model called Omori's law, and used to predict the rate of aftershocks caused by an earthquake.

Hawkes Process: Immigration-birth Representation I

Stability properties of the HP are often simpler to divine if it is viewed as a branching process.

Imagine counting the population in a country where people arrive either via immigration or by birth. Say that the stream of immigrants to the country form a homogeneous Poisson process at rate λ.

Each individual then produces zero or more children independently of one another, and the arrival of births form an inhomogeneous Poisson process.

Hawkes Process: Immigration-birth Representation II

An illustration of this interpretation can be seen in the next Fig. 4.

In branching theory terminology, this immigration-birth representation describes a Galton-Watson process with a modified time dimension.

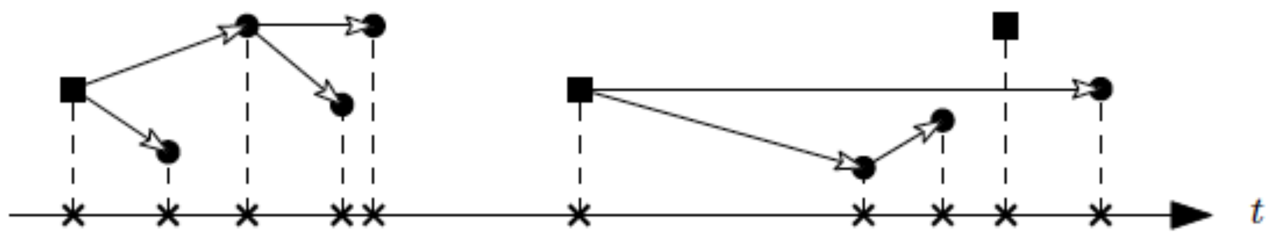


Fig. 4: Hawkes process represented as a collection of family trees (immigration–birth representation). Squares (■) indicate immigrants, circles (●) are offspring/descendants, and the crosses (×) denote the generated point process.

Hawkes Process: Law of Large Numbers (LLN) and Central Limit Theorem (CLT)

LLN for HP. Let $0 < \widehat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$. Then

$$\frac{N(t)}{t} \to_{t\to+\infty} \frac{\lambda}{1-\widehat{\mu}}.$$

CLT for HP. Under LLN and $\int_0^{+\infty} s\mu(s)ds < +\infty$ conditions

$$P\left(\frac{N(t) - \lambda t/(1-\widehat{\mu})}{\sqrt{\lambda t/(1-\widehat{\mu})^3}} < y\right) \to_{t\to+\infty} \Phi(y),$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution.

Remark. For exponential decaying intensity $\widehat{\mu} = \alpha/\beta$.

Hawkes Process: Some Generalizations

Many generalizations of Hawkes processes have been proposed.

They include, in particular, multi-dimensional Hawkes processes, non-linear Hawkes processes, mixed diffusion-Hawkes models, Hawkes models with shot noise exogenous events, Hawkes processes with generation dependent kernels.

Some Generalizations: Nonlinear Hawkes Process

Definition 7 (Nonlinear HP). Consider a counting process with conditional intensity function of the form

$$\lambda(t) = h\left(\int_{-\infty}^{t} \mu(t-s)dN(s)\right),$$

where $h : R \to [0, +\infty)$, $\mu : (0, +\infty) \to R$.

Then $N(t)$ is a nonlinear HP.

Selecting $h(x) = \lambda + x$ reduces $N(t)$ to linear HP.

Applications: Finance/
Limit Order Books (LOB)-Stock Mid-Price

New Directions in Finance: Big Data in Finance

Big data has now become a driver of model building and analysis in a number of areas, including finance.

Main problem: how to deal with big data arising in electronic markets for algorithmic and high-frequency (milliseconds) trading that contain two types of orders, limit orders and market orders.

More than half of the markets in today's highly competitive and relentlessly fast-paced financial world now use a limit order book (LOB) mechanism to facilitate trade.

Big Data in Finance-Limit Order Books/Markets

Orders to buy and sell an asset arrive at an exchange:

1. *Market buy/sell order* - specifes number of shares to be bought/sold at the best available price, right away.

2. *Limit buy/sell order* - specifes a price and a number of shares to be bought/sold at that price, when possible.

3. Order cancellation - agents who have submitted a limit order may cancel the order before it is executed.

New Directions in Finance: Limit Order Books/Markets II

- *Market orders* are executed immediately

- *Limit orders* are queued for later execution, but may cancel

- The Limit-Order Book is the collection of queued limit orders awaiting execution or cancellation

Big Data in Finance-Limit Order Books:
The Bid and Ask Prices

The *bid price* s_t^b is the highest limit buy order price in the book.
It is the *best available price* for a *market sell*.

The *ask price* s_t^a is the lowest limit sell order price in the book.
It is the *best available price* for a *market buy*.

Usually, we are interested in *mid-price*:

$$S_t := \frac{s_t^a + s_t^b}{2}.$$

Big Data in Finance-Limit Order Books: Can We Model it?

There are hundred thousands of orders for one stock just for one day: e.g., for *CISCO data* on Nov 3, 2014, we have 0.5 million price orders for that day, and that is only for *1-level order book*, meaning the limit orders sitting at the best bid and ask. And if we take hundreds of stocks on an exchange and not only 1-level orders book, then we will get an example of really big data in finance!

And the question is: *can we model this mechanism of trading* and *describe this big data in finance*? And the answer is 'Yes'.

Applications: Finance/Limit Order Books (LOB)-Markovian Stock Mid-Price

If S_t is a stock mid-price, then the standard model compound Poisson process (or Markovian model):

$$S_t = S_0 + \sum_{k=1}^{N(t)} X_k,$$

where $N(t)$ is Poisson process and X_k is i.i.d.r.v.

The model is popular in limit order books/markets (Cont & Larrard (2013), SIAM J. Fin. Math.) and also in insurance.

Applications: Finance-General Compound Hawkes Process (GCHP)

We propose a new model for the mid-price: general compound Hawkes process (GCHP).

Definition 8 (General Compound Hawkes Process (GCHP)). Let $N(t)$ be any one-dimensional Hawkes process defined above. Let also X_n be ergodic continuous-time finite (or possibly infinite but countable) state Markov chain, independent of $N(t)$, with space state X, and $a(x)$ be any bounded and continuous function on X. The general compound Hawkes process is defined as

$$S_t = S_0 + \sum_{k=1}^{N(t)} a(X_k). \tag{7}$$

General Compound Hawkes Process (GCHP): Some Examples

1. Compound Poisson Process: $S_t = S_0 + \sum_{k=1}^{N(t)} X_k$, where $N(t)$ is a Poisson process and $a(X_k) = X_k$ are i.i.d.r.v.

2. Compound Hawkes Process: $S_t = S_0 + \sum_{k=1}^{N(t)} X_k$, where $N(t)$ is a Hawkes process and $a(X_k) = X_k$ are i.i.d.r.v.

3. Compound Markov Renewal Process: $S_t = S_0 + \sum_{k=1}^{N(t)} a(X_k)$, where $N(t)$ is a renewal process and X_k is a Markov chain.

Application: General Compound Hawkes Process-Our Main Model for a Stock Mid-Price

The general compound Hawkes process (GCHP) as a model for the stock mid-price is defined by

$$S_t = S_0 + \sum_{k=1}^{N(t)} a(X_k),$$

where $N(t)$ be any one-dimensional Hawkes process defined above, X_n is ergodic continuous-time finite (or possibly infinite but countable) state Markov chain, independent of $N(t)$, with space state X, and $a(x)$ be any bounded and continuous function on X.

How to Study Stochastic Evolution of the Mid-Price S_t?

Order arrivals and cancellations are very frequent and occur at the millisecond time scale, whereas, in many applications, such as order execution, the metric of success is the volume-weighted average price (VWAP), so one is interested in the dynamics of order flow over a large time scale, typically tens of seconds or even minutes.

As long as high-frequency trading happen in milliseconds, the question is "How we can study the stochastic evolution of the mid-price S_t?"

How to Study Stochastic Evolution of the Mid-Price S_t?

One of the ways is to look over a larger time scale, e.g., $5, 10$ or 20 minutes, i.e., considering time scale tn instead of t, n can be $n = 100, 1000, ..,$ etc.

It means that we can use asymptotic methods to study the link between price volatility and order flow in our model by studying the diffusion limit of the price process.

Thus, we consider Law of Large Numbers (LLN) and Functional Central Limit Theorems (FCLT) for the time scaled stock mid-price S_t.

Structure of the Results for LLN and FCLT (from General to the Specific Ones)

$\boxed{\text{Non-linear GCHP with } n \text{ Dependent Orders-LLN \& FCLT}}$ $\Downarrow$

$\boxed{\text{Linear GCHP with } n \text{ Dependent Orders-LLN \& FCLT}}$ $\Downarrow$

$\boxed{\text{Linear GCHP with 2 Different Dependent Orders-LLN \& FCLT}}$ $\Downarrow$

$\boxed{\text{Linear GCHP with 2 Ticks Dependent Orders-LLN \& FCLT}}$ $\Downarrow$

$\boxed{\text{Numerical Example for Linear GCHP with 2 Ticks}}$ $\Downarrow$

$\boxed{\text{CISCO Data (5 Days, 3-7 Nov 2014 (see [Cartea \textit{et al.}, 2015]))}}$

Non-linear General Compound Hawkes Process
with n-state Dependent Orders in LOB
and LLN and FCLT for Them

Non-linear General Compound Hawkes Process with n-state Dependent Orders in LOB

In what follows, we consider the mid-price process S_t in the form

$$S_t = S_0 + \sum_{i=1}^{N_t} a(X_k),$$

where $X_k \in \{1, 2, ..., n\} := X$ is a continuous-time n-state Markov chain, $a(x)$ is continuous and bounded function on $X = \{1, 2, ..., n\}$, $N(t)$ is the non-linear Hawkes process defined by the intensity function in the following form (see also Definition 6 above):

$$\lambda(t) = h\Big(\lambda + \int_0^t \mu(t-s)dN(s)\Big),$$

with $h(\cdot)$ being a non-linear increasing function with support in R^+, α-Lipschitz and such that $\alpha||h||_{L^1} < 1$.

Non-linear General Compound Hawkes Process with n-state Dependent Orders in Limit Order Books

Under the latter conditions, it was proved in Brémaud, P. and Massoulié, L. (1996): Stability of nonlinear Hawkes processes. *The Annals of Probab.*, 24(3), 1563, that there exists a unique stationary and ergodic Hawkes process satisfying the above dynamics.

LLN for Non-linear General Compound Hawkes Process (NLGCHP) with n-state Dependent Orders in Limit Order Books

Lemma (LLN for NLGCHP). The process

$$S_{nt} = S_0 + \sum_{i=1}^{N_{tn}} a(X_k)$$

above satisfies the following weak convergence in the Skorokhod topology:

$$\frac{S_{nt}}{n} \to_{n\to+\infty} a^* E[N[0,1]]t,$$

where $a^* := \sum_{i\in X} \pi_i^* a(X_i)$, and $E[N[0,1]]$ is the mean of $N[0,1]$ (the number of points in the interval $[0,1]$) under the stationary and ergodic measure.

LLN for Non-linear General Compound Hawkes Process (NLGCHP) with n-state Dependent Orders in Limit Order Books

Remark. We note, that this result follows from Brémaud, P. and Massoulié, L. (1996) (Stability of nonlinear Hawkes processes. *The Annals of Probab.*, 24(3), 1563)

$$\frac{N(t)}{t} \to_{t\to+\infty} E[N[0,1]]$$

or

$$\frac{N(nt)}{n} \to_{t\to+\infty} tE[N[0,1]].$$

FCLT for Non-linear General Compound Hawkes Process (NLGCHP) with n-state Dependent Orders in LOB

Theorem (Diffusion Limit for NLGCHP). Let X_k be an ergodic Markov chain with n states $X := \{1, 2, ..., n\}$ and with ergodic probabilities $(\pi_1^*, \pi_2^*, ..., \pi_n^*)$. Let also S_t be GCHP defined above with non-linear function $h(t)$ satisfying the above conditions, and $0 < \hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1 \quad and \quad \int_0^{+\infty} \mu(s)sds < +\infty$. Then

$$\frac{S_{nt} - N(nt) \cdot a^*}{\sqrt{n}} \to_{n \to +\infty} \sigma^* \sqrt{E[N[0,1]]} W(t),$$

where $W(t)$ is a standard Wiener process, a^* and $E[N[0,1]]$ are defined above, and

FCLT for Non-linear General Compound Hawkes Process (NLGCHP) with n-state Dependent Orders in Limit Order Books II (Parameters)

Here:

$(\sigma^*)^2 := \sum_{i\in X}\pi_i^* v(i)$

$$\begin{array}{rcl} v(i) & = & b(i)^2 + \sum_{j\in X}(g(j)-g(i))^2 P(i,j) \\ & - & 2b(i)\sum_{j\in X}(g(j)-g(i))P(i,j), \\ b & = & (b(1), b(2), ..., b(n))', \\ b(i): & = & a(i) - a^*, \\ g: & = & (P + \Pi^* - I)^{-1} b, \\ a^* & := & \sum_{i\in X}\pi_i^* a(X_i), \end{array}$$

P is a transition probability matrix for X_k,, i.e., $P(i,j) = P(X_{k+1} = j|X_k = i)$, Π^* denotes the matrix of stationary distributions of P, and $g(j)$ is the jth entry of g.

Linear General Compound Hawkes Process
with n-state Dependent Orders in LOB
and LLN and FCLT for Them

Linear General Compound Hawkes Process (LGCHP) with n-state Dependent Orders in LOB

We consider here the mid-price process S_t (LGCHP):

$$S_t = S_0 + \sum_{i=1}^{N(t)} a(X_k),$$

where $X_k \in \{1, 2, ...n\} := X$ is continuous-time n-state Markov chain, $a(x)$ is continuous and bounded function on $X = \{1, 2, ..., n\}$, and $N(t)$ is the number of price changes up to moment t, described by a linear one-dimensional Hawkes process defined above, Definition 4. It means that we have the case with *non-fixed tick*, n-values price change and dependent orders.

LLN for Linear General Compound Hawkes Process (LGCHP) with n-state Dependent Orders in Limit Order Books

Lemma (LLN for LGCHP). The process S_{nt} above (LGCHP) satisfies the following weak convergence in the Skorokhod topology:

$$\frac{S_{nt}}{n} \to_{n\to+\infty} a^* \cdot \frac{\lambda}{1-\hat{\mu}} t,$$

where a^* is defined in Lemma (LLN for NLGCHP)) and $\hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$.

FCLT for Linear General Compound Hawkes Process (LGCHP) with n-state Dependent Orders in LOB

Theorem (Diffusion Limit for LGCHP). Let X_k be an ergodic Markov chain with n states $\{1, 2, ..., n\}$ and with ergodic probabilities $(\pi_1^*, \pi_2^*, ..., \pi_n^*)$. Let also S_t be LGCHP, and $0 < \hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1 \quad and \quad \int_0^{+\infty} \mu(s)sds < +\infty$. Then

$$\frac{S_{nt} - N(nt) \cdot a^*}{\sqrt{n}} \to_{n \to +\infty} \sigma^* \sqrt{\lambda/(1-\hat{\mu})} W(t),$$

where $W(t)$ is a standard Wiener process, σ^* is defined in Theorem (Diffusion Limit for NLGCHP) and here $E[N[0,1]] = \lambda/(1-\hat{\mu})$.

Linear General Compound Hawkes Process
with 2-state Dependent Orders in LOB
and LLN and FCLT for Them

Linear General Compound Hawkes Process (LGCHP) with Two-state Dependent Orders in LOB

We consider here the mid-price process S_t in the form of linear GCHP with two-state dependent order (LGCHP2SDO):

$$S_t = S_0 + \sum_{i=1}^{N(t)} a(X_k),$$

where $X_k \in \{1,2\} := X$ is continuous-time 2-state Markov chain, $a(x)$ is continuous and bounded function on $X = \{1,2\}$, and $N(t)$ is the number of price changes up to moment t, described by linear one-dimensional Hawkes process defined in Definition 4. It means that we have the case with *non-fixed tick*, two-values price change and dependent orders.

LLN for Linear General Compound Hawkes Process (LGCHP) with Two-state Dependent Orders in LOB

Lemma (LLN for LGCHP2SDO). The process S_{nt} as LGCHP2SDO satisfies the following weak convergence in the Skorokhod topology (see [**?**]):

$$\frac{S_{nt}}{n} \to_{n\to+\infty} a^* \cdot \frac{\lambda}{1-\hat{\mu}} t,$$

where $a^* := \pi_1^* a(1) + \pi_2^* a(2)$ and $\hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$.

FCLT for Linear General Compound Hawkes Process(LGCHP) with Two-state Dependent Orders in LOB

Theorem (Diffusion Limit for LGCHP2SDO). Let X_k be an ergodic Markov chain with two states $\{1, 2\}$ and with ergodic probabilities (π_1^*, π_2^*). Let also S_t is defined as LGCHP2SDO and $0 < \hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1 \quad and \quad \int_0^{+\infty} \mu(s)sds < +\infty$. Then

$$\frac{S_{nt} - N(nt) \cdot a^*}{\sqrt{n}} \to_{n \to +\infty} \sigma^* \sqrt{\lambda/(1-\hat{\mu})} W(t),$$

where $W(t)$ is a standard Wiener process, and

FCLT for Linear General Compound Hawkes Process(LGCHP) with Two-state Dependent Orders in LOB (Parameters)

$$
\begin{aligned}
(\sigma^*)^2 &:= \pi_1^* a_1^2 + \pi_2^* a_2^2 + (\pi_1^* a_1 + \pi_2^* a_2)[-2a_1\pi_1^* - 2a_2\pi_2^* \\
&+ (\pi_1^* a_1 + \pi_2^* a_2)(\pi_1^* + \pi_2^*)] \\
&+ \frac{(\pi_1^*(1-p)+\pi_2^*(1-p'))(a_1-a_2)^2}{(p+p'-2)^2} \\
&+ 2(a_2 - a_1) \cdot \left[\frac{\pi_2^* a_2(1-p')-\pi_1^* a_1(1-p)}{p+p'-2}\right. \\
&+ \left.\frac{(\pi_1^* a_1+\pi_2^* a_2)(\pi_1^*-p\pi_1^*-\pi_2^*+p'\pi_2^*)}{p+p'-2}\right], \\
a^* &:= \pi_1^* a(1) + \pi_2^* a(2),
\end{aligned}
$$

where (p, p') are transition probabilities of Markov chain X_k.

Linear General Compound Hawkes Process
with 2-ticks State Dependent Orders in LOB
and LLN and FCLT for Them

Linear General Compound Hawkes Process(LGCHP) with Two-ticks State Dependent Orders in LOB

We consider here the mid-price process S_t (LGCHP2TSDO):

$$S_t = S_0 + \sum_{i=1}^{N(t)} X_k,$$

where $X_k \in \{-\delta, +\delta\}$ is continuous-time 2-state Markov chain, δ is the fixed tick size, and $N(t)$ is the number of price changes up to moment t, described by linear one-dimensional Hawkes process defined in Definition 4.

It means that we have the case with fixed tick size δ, two-values price change, $X_k \in \{-\delta, +\delta\}$, and dependent orders, i.e.,X_k is a 2-state Markov chain.

LLN for Linear General Compound Hawkes Process(LGCHP) with Two-ticks State Dependent Orders in LOB

Lemma (LLN for LGCHP2TSDO). The process S_{nt} above satisfies the following weak convergence in the Skorokhod topology:

$$\frac{S_{nt}}{n} \to_{n\to+\infty} a^* \cdot \frac{\lambda}{1-\hat{\mu}} t,$$

where $a^* := \delta(2\pi^* - 1)$ and $\hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1$.

FCLT for Linear General Compound Hawkes Process(LGCHP) with Two-ticks State Dependent Orders in LOB

Theorem (Diffusion Limit for LGCHP2TSDO). Let X_k be an ergodic Markov chain with two states $\{-\delta, +\delta\}$ and with ergodic probabilities $(\pi^*, 1-\pi^*)$. Let also S_t is defined above and $0 < \hat{\mu} := \int_0^{+\infty} \mu(s)ds < 1 \quad and \quad \int_0^{+\infty} \mu(s)sds < +\infty$. Then

$$\frac{S_{nt} - N(nt) \cdot a^*}{\sqrt{n}} \to_{n \to +\infty} \sigma^* \sqrt{\lambda/(1-\hat{\mu})} W(t),$$

where $W(t)$ is a standard Wiener process,

$$a^* := \delta(2\pi^* - 1) \quad and \quad (\sigma^*)^2 := 4\delta^2 \Big(\frac{1 - p' + \pi^*(p' - p)}{(p + p' - 2)^2} - \pi^*(1 - \pi^*) \Big),$$

and (p, p') are transition probabilities of Markov chain X_k.

Numerical Example and Parameters Estimation

Parameters Estimation for CISCO Data (5 Days, 3-7 Nov 2014 (see [Cartea *et al.*, 2015]))

In what follows represents parameters estimation for our last model, namely, Linear General Compound Hawkes Process with 2-ticks State Dependent Orders in LOB, and FCLT for it,

$$\frac{S_{nt} - N(nt) \cdot a^*}{\sqrt{n}} \to_{n \to +\infty} \sigma^* \sqrt{\lambda/(1-\widehat{\mu})} W(t), \qquad (*)$$

using CISCO Data (5 Days, 3-7 Nov 2014 (see [Cartea *et al.*, 2015])). It contains the errors of estimation of comparison of the standard deviation of the LHS of (*) and the RHS of (*) multiplied by $\sqrt{n}$.

The last slides presents some ideas of how to implement the regime-switching case.

Parameters Estimation for CISCO Data (5 Days, 3-7 Nov 2014 (see [Cartea *et al.*, 2015]))

Convergence

$$\frac{S_{nt} - N(nt)a^*}{\sqrt{n}} \to_{n\to+\infty} \sigma^* \sqrt{\lambda/(1-\widehat{\mu})} W(t), \qquad (*)$$

in the Theorem (Diffusion Limit for LGCHP2TSDO) relates the volatility of intraday returns at lower frequencies to the high-frequency arrival rates of orders.

Parameters Estimation for CISCO Data (5 Days, 3-7 Nov 2014 (see [Cartea *et al.*, 2015]))

The typical time scale for order book events are milliseconds. This formula states that, observed over a larger time scale, e.g., 5, 10 or 20 minutes, the price has a diffusive behaviour with a diffusion coefficient given by the coefficient at $W(t)$ in (*):

$$\sigma^* \sqrt{\lambda/(1-\widehat{\mu})},$$

where all the parameters here are defined in the Theorem (Diffusion Limit for LGCHP2TSDO) above.

Numerical Examples and Parameters Estimations

We'd like to mention, that this formula for volatility contains all the initial parameters of the Hawkes process, Markov chain transition and stationary probabilities and the tick size. In this way, formula (*) links properties of the price to the properties of the order flow.

Also, the left hand side of (*) represents the variance of price, whereas the right hand side in (*) only involves the tick size and Hawkes process and Markov chain quantities.

Numerical Examples and Parameters Estimations

From here it follows that an estimator for price volatility may be computed without observing the price at all!

As we shall see below, the error of estimation of comparison of the standard deviation of the LNS and the RHS of (*) multiplied by $\sqrt{n}$ is approximately 0.08, indicating that approximation in (*) for diffusion limit for CHP in Theorem (Diffusion Limit for LGCHP2TSDO) , is pretty good.

Parameters Estimation for CISCO Data (5 Days, 3-7 Nov 2014 (see [Cartea *et al.*, 2015]))

We have the following estimated parameters for 5 days, 3-7 November 2014, from Formula (*) (we use $\mu(t) = \alpha e^{-\beta t}$):

$a^* = 0.0001040723; 0.0002371220; 0.0002965143; 0.0001263690;$
$0.0001554404;$
$\sigma^* = 1.066708e-04; 1.005524e-04; 1.165201e-04; 1.134621e-04;$
$9.954487e-05;$
$\lambda = 0.03238898; 0.02643083; 0.02590728; 0.02530517; 0.02417804;$
$\alpha = 438.2557; 401.0505; 559.1927; 418.7816; 449.8632;$
$\beta = 865.9344; 718.0325; 1132.0741; 834.2553; 878.9675;$
$\hat{\lambda} := \lambda/(1-\alpha/\beta) = 0.06560129; 0.059801686; 0.051181133;$
$0.050801432; 0.04957073.$

Volatility Coefficient $\sigma^*\sqrt{\lambda/(1-\alpha/\beta)}$

Volatility coefficients $\sigma^*\sqrt{\lambda/(1-\alpha/\beta)}$ (volatility coefficient for the Brownian Motion in the right hand-side (RHS) of (*)):

$0.04033114; 0.04098132; 0.04770726; 0.04725449; 0.04483260.$

Transition Probabilities p :

Day1:

uu	ud
0.5187097	0.4812903
du	dd
0.4914135	0.5085865

Day2:

0.4790503	0.5209497
0.5462555	0.4537445

Day3:

0.6175041	0.3824959
0.4058722	0.5941278

Transition Probabilities p :

Day4:

0.5806988	0.4193012
0.4300341	0.5699659

Day5:

0.4608844	0.5391156
0.5561404	0.4438596

We note, that stationary probabilities $\pi_i^*, i = 1, ..., 5,$ are, respectively: $0.5525; 0.6195; 0.6494; 0.5637; 0.5783$. Here, we assume that the tick δ size is $\delta = 0.01$.

Set of Parameters

The following set of parameters are related to the the following expression

$$S_{nt} - N(nt)a^* = S_0 + \sum_{k=1}^{N(nt)} (X_k - a^*),$$

-LHS of the expression in (*) multiplied by $\sqrt{n}$.

Set of Parameters for $nt = 10$ minutes

The first set of numbers are for the 10 minutes time horizon ($nt = 10$ minutes, for 5 days, the 7 sampled hours, total 35 numbers):

$nt = 10$ **minutes: St. Dev. and St. Error**

Table 1

[1]24.50981; [2]24.54490; [3]24.52375; [4]24.59209; [5]24.47209;
[6]24.57042; [7]24.61063; [8]24.76987; [9]24.68749; [10]24.81599;
[11]24.77026; [12]24.79883; [13]24.80073; [14]24.90121;
[15]24.87772; [16]24.98492; [17]25.09788; [18]25.09441;
[19]24.99085; [20]25.18195; [21]25.15721; [22]25.04236;
[23]25.18323; [24]25.15222; [25]25.20424; [26]25.14171;
[27]25.18323; [28]25.25348; [29]25.10225; [30]25.29003;
[31]25.28282; [32]25.33267; [33]25.30313; [34]25.27407;
[35]25.30438;

The Standard Deviation (SD) is: 0.2763377. The Standard Error (SE) for SD for the 10 min is: 0.01133634 (for standard error calculations see [Casella and Berger, 2002, page 257].

Set of Parameters for $nt = 5$ minutes

The second set of numbers are for the 5 minutes time horizon ($nt = 5$ minutes, for 5 days, the 7 sampled hours):

Table 2

[1]24.49896; [2]24.52906; [3]24.50417; [4]24.53417; [5]24.53500; [6]24.51458; [7]24.55479; [8]24.93026; [9]24.66931; [10]24.74263; [11]24.79358; [12]24.80310; [13]24.84500; [14]24.88405; [15]24.85729; [16]24.98907; [17]25.08085; [18]25.07500; [19]24.99322; [20]25.13381; [21]25.15144; [22]25.15197; [23]25.12475; [24]25.15449; [25]25.18475; [26]25.20348; [27]25.20500; [28]25.25348; [29]25.21251; [30]25.35376; [31]25.30407; [32]25.30469; [33]25.30469; [34]25.27500; [35]25.30469;

Set of Parameters for $nt = 5$ minutes: St. Dev. and St. Error

The Standard Deviation (SD) for those numbers is: 0.2863928.

The SE for SD for the 5 min is: 0.01233352.

Set of Parameters for $nt = 20$ minutes

The third and last set of numbers are for the 20 minutes time horizon ($nt = 20$ minutes, for 5 days, the 7 sampled hours):

Table 3

[1]24.48419; [2]24.53970; [3]24.56292; [4]24.57105; [5]24.48938; [6]24.52751; [7]24.50751; [8]24.76465; [9]24.59753; [10]24.82935; [11]24.76552; [12]24.81741; [13]24.75409; [14]24.84077; [15]24.92942; [16]24.99721; [17]25.05551; [18]25.04848; [19]25.08492; [20]25.09780; [21]25.09551; [22]24.95124; [23]25.24222; [24]25.19096; [25]25.18273; [26]25.14070; [27]25.20171; [28]25.26785; [29]25.23013; [30]25.38661; [31]25.32127; [32]25.34065; [33]25.30313; [34]25.25251; [35]25.24972;

Set of Parameters for $nt = 20$ minutes: St. Dev. and St. Error

The Standard Deviation (SD) is: 0.2912967.

The SE for SD for the 20 min is: 0.01234808.

Main Numerical Result: As we can see, the SE is approximately 0.01 for all three cases.

Error of Estimation

Here, we would like to calculate the error of estimation comparing the standard deviation for

$$S_{nt} - N(nt)a^* = S_0 + \sum_{k=1}^{N(nt)} (X_k - a^*)$$

and standard deviation in the right-hand side of (*) multiplied by $\sqrt{n}$, namely,

$$\sqrt{n}\sigma^* \sqrt{\lambda/(1 - \alpha/\beta)}.$$

Error of Estimation II

We calculate the error of estimation with respect to the following formula:

$$ERROR = (1/m) \sum_{k=1}^{m} (sd - \hat{sd})^2,$$

where $\hat{sd} = \sqrt{n} Coef$, where $Coef$ is the volatility coefficient in the right-hand side of equation (*). In this case $n = 1000$, and $Coef = 0.3276$.

We take observations of $S_{nt} - N(tn)s^*$ every 10 min and we have 36 samples per day for 5 days.

Error of Estimation III

Using the first approach with formula above we take $m = 5$ and for computing the standard deviation "sd" we take 36 samples of the first day. In that case, we have

$ERROR = 0.07617229.$

Error of Estimation IV

Using the second approach with formula above, we take $m = 36$ and for computing "sd" we take samples of 5 elements (the same time across 5 days). In that case we have

$ERROR = 0.07980041.$

As we can see, the error of estimation in both cases is approximately 0.08, indicating that approximation in (*) for diffusion limit for CHP, Theorem (Diffusion Limit for LGCHP2TSDO), is pretty good.

Monday, February 5, 2018-Another Black Monday?!

Stock Market Crash (Monday, Feb 5, 2018): Some Numbers

1,175-the number of points Dow Jones fell

100%-the amount the CBOE VIX increase

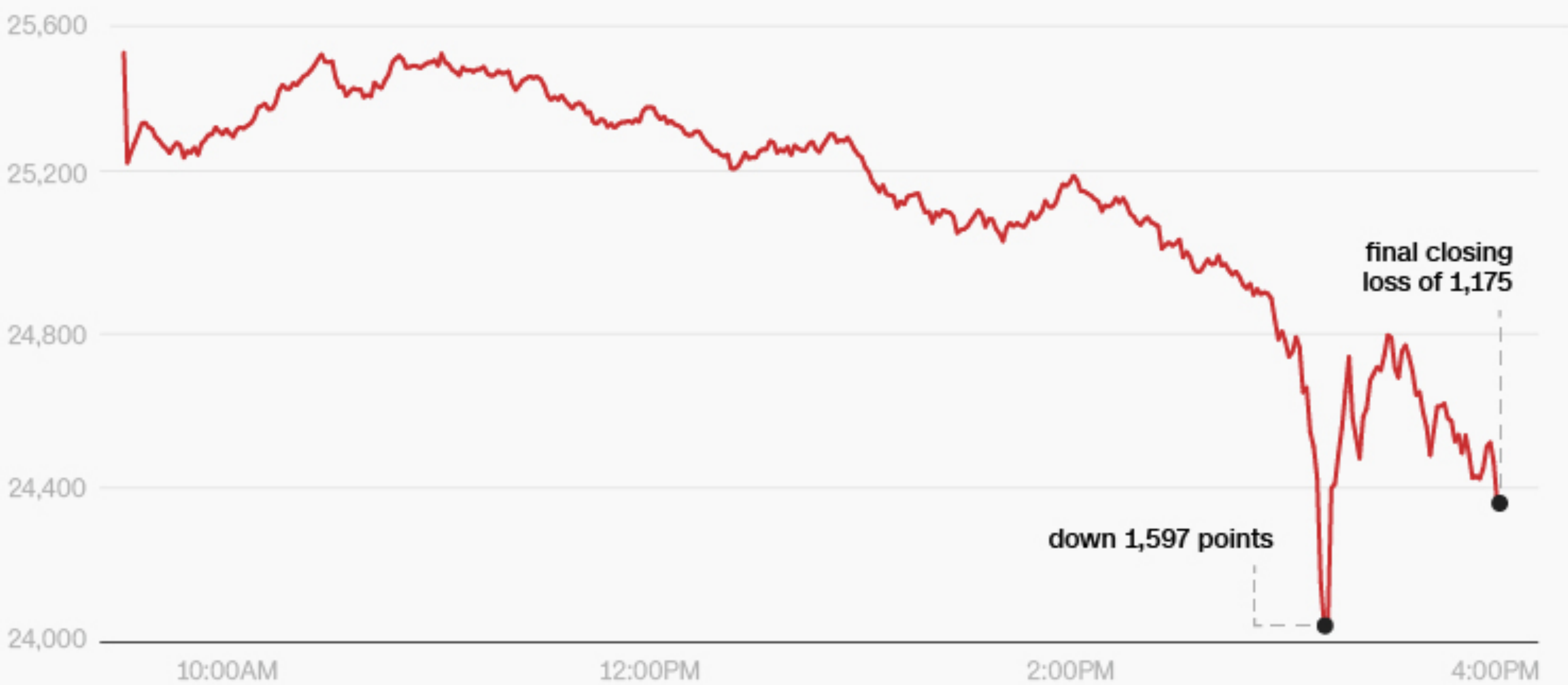
4.1%-the amount S&P 500 declined

2.71%-10-year Treasury Notes yield down

$4 Trillion - the amount Global Markets saw wiped away

$7,000-approximately what Bitcoin is worth

Dow's worst point drop ever



SOURCE: FACTSET

Stock Market Crash (Monday, Feb 5, 2018): Why it Happened?

'One of the culprits of the Flash Crash was high-frequency trading, where computers are programmed to trade a lot of stocks incredible fast.

It was a bizarre domino effect kicked off by rapid trading algorithms' (Source: money.cnn.com)

Stock Market Crash (Monday, Feb 5, 2018): Why it Happened?

"We have created a stock market that moves too darn fast for human beings", said David Weild IV, founder and chairman of CEO of Weild & Co. and a former vice chairman of Nasdaq. "And because of that," he added, "we see shocking results".

"People can make certain calls that computers can't, and explain to investors why they should or should not sell their stocks", he said. "On a day like today, traders may have told their clients to sit tight."

Computer programs sold off stocks and scared investors.

Stock Market Crash (Monday, Feb 5, 2018): Why it Happened?

"Some automated sell programs were likely triggered by the contraction in the market," explained Jonathan Corpina, a senior managing partner with Meridian Equity Partners, "those, in turn, triggered others. They start playing leapfrog with each other. At a certain point, buyers who were looking for deals also pulled back, making matters worse. That's how you get these large swings in the market".

"The sellers were really convinced at the end of the day that today was the day to sell," he said.

Corpina did not blame the volatility entirely on electronic trading.

What is driving the big global sell-off?

• Concerns that the Fed will raise rates (The Federal Reserve combats inflation by raising its interest rates)

• Rising interest rates (When interest rates rise sharply, stocks often fall)

• Worries about the bond market (bond yields hit a four-year high Friday, Feb 2; stocks are a higher-risk investment than bonds; If bond yields start to rise, investors will want to take some of their money out of stocks and put it into safer bonds)

• Too far, too fast (Stocks have been rising pretty much in a straight line since November 2016, and that's not exactly healthy. A cooling-off period would be a good thing.)

Working Paper

The results of this talk are based on the paper
'General Compound Hawkes Processes in Limit Order Books'
available on
arXiv: https://arxiv.org/abs/1706.07459

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SSRN: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2987943

Conclusion

• *Hawkes Process* (HP) and *General Compound Hawkes Process* (GCHP)

• *Applications of GCHP:* Finance/Limit Order Books-Stock Mid-Price

• *Functional Central Limit Theorems* (FCLT) and *Law of Large Numbers* (LLN) for GCHP

• *Some Numerical Examples* for GCHP

• *Monday, February 5, 2018*-Another 'Black Monday'?!

The End

Thank You!

Q&A time!

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