Change of Time Methods in Mathematical and Energy Finance *

Anatoliy Swishchuk
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada

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Outline of Presentation

1. Definition of Change of Time (CT) and Motivations
2. Change of Time Methods for Different Settings
3. Applications in Mathematical and Energy Finance
4. Delayed Heston Model: Var and Vol Swaps, Hedging
5. Discussion: Some Problems
6. Conclusion

Definition of Change of Time

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space with filtration $\mathcal{F}_t$, $t \in R_+ := [0, +\infty)$.

A *time change* is right-continuous increasing R_+-valued process T_t such that T_t is a stopping time for any $t \in R_+$.

The *inverse time change* $\hat{T}_t$ is defined as $\hat{T}_t := \inf\{s \in R_+ : T_s > t\}$. Each $\hat{T}_t$ is a stopping time as well.

By $\hat{\mathcal{F}}_t$ we define the time-changed filtration $\hat{\mathcal{F}}_t := \mathcal{F}_{T_t}$.

Change of Time Scale: Example

Let $\alpha(t)$ be a strictly positive measurable function on R and define the following strictly increasing adapted process

$$T_t = \int_0^t \alpha(x_s) ds,$$

where x_s is, e.g., a diffusion process. Then T_t is a *change of time*. We note that

$$\hat{T}_t := \inf\{s : T_s > t\}$$

is the *inverse time change*.

Change of Time Scale: Subordination

We could change time from t to a non-negative process $T(t)$ with non-decreasing sample paths.

If $X(t)$ and $T(t)$ are some processes, then $X(T(t))$ is subordinated to $X(t)$ and $T(t)$ is a *subordinator*.

Good examples of $T(t)$: *Poisson, α-stable, the Lévy, inverse Gaussian, gamma* subordinators.

Nice property for Lévy processes: if $X(t)$ is an arbitrary Lévy process and $T(t)$ is a subordinator on the same probability space then $X(T(t))$ is a Lévy process.

A Motivation to Use CT: Transformation of SDEs

We frequently transform an SDE into another which may be more amenable.

There are three general methods for transforming as SDE:

- *change of time* scale

- change of state space (e.g., by scale function)

- change of measure (e.g., by Girsanov's theorem)

We shall be considering the *change of time* scale to transform a *SDE driven by Wiener and Lévy processes*.

A Motivation to Use CT: Transformation of a Martingale Problem

Let a and b be a diffusion and drift coefficients for a diffusion process x_t such that P^x solves the *martingale problem* for (a, b) started at x. Then for a smooth function $f(x)$ on R we have

$$\widehat{C}_t^f := C_{\widehat{T}_t}^f := f(x_{\widehat{T}_t}) - f(x_0) - \int_0^{\widehat{T}_t} Af(x_s)ds$$

is an $\widehat{\mathcal{F}}_t$-local martingale, where A is a generator of x_t.

A Motivation to Use CT: Martingale Problem

If we *change variable in the integral*, writing $\widehat{x}_t := x_{\widehat{T}_t}$, we find that

$$\widehat{C}_t^f = f(\widehat{x}_t) - f(\widehat{x}_0) - \int_0^t \alpha^{-1}(\widehat{x}_s) Af(\widehat{x}_s) ds.$$

If α^{-1} is locally bounded, then $\widehat{C}_t^f$ is an $\mathcal{F}_{\widehat{T}_t}$-martingale and the law of $\widehat{x}_t$ is a solution to the martingale problem for $(\alpha^{-1}a, \alpha^{-1}b)$.

Literature Review on Time-Changing

Bochner (1949): introduced the notion of change of time (time-changed Brownian motion)

Feller (1956): introduced subordinated process $X(T(t))$ with Markov process $X(t)$ and $T(t)$ as a process with independent increments ($T(t)$ was called 'randomized operational time')

Literature Review on Time-Changing: Embedding Problem

The change of time method is closely associated with the *embedding problem*: to embed a process $X(t)$ in Brownian motion is to find a Wiener process process $W(t)$ and an increasing family of stopping times $T(t)$ such that $W(T(t))$ has the same joint distribution as $X(t)$.

Skorokhod (1965): first treated the embedding problem, showing that the sum of any sequence of independent r.v. with mean zero and finite variation could be embedded in Brownian motion using stopping times

Literature Review on Change of Time

Dambis (1965), Dubins & Schwartz (1965): independently showed that every continuous martingale could be embedded in Brownian motion

Huff (1969): showed that every process of pathwise bounded variation could be embedded in Brownian motion

Knight (1971): discovered multivariate extension of Dambis (1965), Dubins & Schwartz (1965) result

Literature Review on Change of Time

Meyer (1971), Papangelou (1972): independently discovered Knight's (1971) result for point processes

Monroe (1972): proved that every right continuous martingale could be embedded in a Brownian motion

Clark (1973): introduced Bochner's change of time into financial economics

Monroe (1978): proved that a process can be embedded in Brownian motion if and only if this process is a local semimartingale

Literature Review on Change of Time

Johnson (1979): introduced time-changed stochastic volatility model (SVM) in continuous time

Ikeda & Watanabe (1981): introduced and studied change of time for the solution of SDEs

Rosiński & Woyczyński (1986): considered time changes for integrals over a stable Lévy processes

Literature Review on Time-Changing

Johnson & Shanno (1987): studied pricing of options using time-changed SVM

Madan & Seneta (1990): introduced Variance Gamma (VG) process (Brownian motion with drift time changed by a gamma process)

Kallenberg (1992): considered time change representations for stable integrals

Literature Review on Change of Time

Lévy processes can also be used as a time change for other Lévy processes (*subordinators*)

Geman, Madan & Yor (2001): considered time changes for Lévy processes ('business time')

Kallsen & Shiryaev (2001): showed that Rosiński-Woyczyński-Kallenberg statement can not be extended to any other Lévy processes but symmetric α-stable

Literature Review on Change of Time

Barndorff-Nielsen, Nicolato & Shephard (2003): studied relatiomship between subordination and SVM using change of tuime ($T(t)$-'chronometer')

Carr, Geman, Madan, Yor (2003): used subordinated processes to construct SV for Lévy processes ($T(t)$-'business time')

Carr, Geman, Madan & Yor (2003): used change of time to introduce stochastic volatility into a Lévy model to achieve leverage effect and a long-term skew

Sw. (2004, 2007): applied change of time method for options and swaps pricing for Gaussian models

Change of Time Method for Martingales

Let $M(t)$ be a *martingale*, $\lim_{t\to+\infty}[M](t) = +\infty$ and $\hat{T}(t) := \inf\{s : [M](s) > t\}$.

Then $W(t) := M(\hat{T}(t))$ is a Brownian motion. Also, $M(t) := W([M](t))$ is a martingale.

Here, change of time $T(t) = [M](t)$.

Change of Time Method for Itô Integral

Let $M(t) := \int_0^t \sigma(s)dW(s)$ be Itô integral, $\lim_{t\to+\infty}[M](t) = \int_0^t \sigma^2(s)ds = +\infty$ and $\hat{T}(t) := \inf\{s : [M](s) > t\}$.

Then $W(t) := M(\hat{T}(t))$ is a Brownian motion. Also, $M(t) := W([M](t))$ is a martingale.

Here, change of time $T(t) = [M](t) = \int_0^t \sigma^2(s)ds$.

Change of Time Method for SDE driven by Brownian motion

We consider the following SDE driven by a *Brownian motion*:

$$dX(t) = a(t, X(t))dW(t),$$

where $W(t)$ is a Brownian motion and $a(t, X)$ is a continuous and measurable by t and X function on $[0, +\infty) \times R$.

Change of Time Method for SDE driven by Brownian motion (cntd)

Theorem. (*Ikeda and Watanabe* (1981), Chapter IV, Theorem 4.3)

Let $\widehat{W}(t)$ be an one-dimensional $\mathcal{F}_t$-Wiener process with $\widehat{W}(0) = 0$, given on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and let $X(0)$ be an $\mathcal{F}_0$-adapted random variable.

Define a continuous process $V = V(t)$ by the equality

$$V(t) = X(0) + \widehat{W}(t).$$

Change of Time Method for SDE driven by Brownian motion (cntd)

Let T_t be the change of time process:

$$T_t = \int_0^t a^2(T_s, X(0) + \hat{W}(s))ds.$$

If

$$X(t) := V(\hat{T}_t) = X(0) + \hat{W}(T_t)$$

and $\hat{\mathcal{F}}_t := \mathcal{F}_{\hat{T}_t}$, then there exists $\hat{\mathcal{F}}_t$-adapted Wiener process $W = W(t)$ such that $(X(t), W(t))$ is a solution of $dX(t) = a(t, X(t))dW(t)$ on probability space $(\Omega, \mathcal{F}, \hat{\mathcal{F}}_t, P)$, where $\hat{T}_t$ is the inverse to T_t time change.

Solutions to the One-Factor Gaussian Models Using CT

We use change of time method (see Ikeda and Watanabe (1981)) to get the solutions to the following below equations.

$W(t)$ below is an standard Brownian motion, and $\widehat{W}$ is a $(\hat{T}_t)_{t\in R_+}$-adapted standard Brownian motion on $(\Omega, \mathcal{F}, (\widehat{\mathcal{F}}_t)_{t\in R_+}, P)$.

Solutions to Some SDEs (Useful for Applications)

Geometric Brownian Motion. $dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$
Solution: $S(t) = e^{\mu t}[S(0) + \widehat{W}(T_t)]$,
where $T_t = \sigma^2 \int_0^t [S(0) + \widehat{W}(T_s)]^2 ds.$

Continuous-Time GARCH Process.
$dS(t) = \mu(b - S(t))dt + \sigma S(t)dW(t).$
Solution: $S(t) = e^{-\mu t}(S(0) - b + \widehat{W}(T_t)) + b,$
where $T_t = \sigma^2 \int_0^t [S(0) - b + \widehat{W}(T_s) + e^{\mu s} b)^2 ds.$

Cox-Ingersol-Ross Process. $d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dW(t).$
Solution: $\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \widehat{W}(T_t)) + \theta^2,$
where $T(t) = \gamma^2 \int_0^t [e^{kT(s)}(\sigma_0^2 - \theta^2 + \widehat{W}(T_s)) + \theta^2 e^{2kT(s)}]ds.$

Solutions to the Multi-Factor Gaussian Models (cntd)

Solutions to the *multi-factor models driven by Brownian motions* can be obtained using various combinations of solutions of the processes described by SDEs. We give one example of two-factor Continuous-Time GARCH model driven by Brownian motions:

$$\begin{cases} dS(t) &= r(t)S(t))dt + \sigma_1 S(t)dW^1(t) \\ dr(t) &= a(m - r(t))dt + \sigma_2 r(t)dW^2(t). \end{cases}$$

Solutions to the Multi-Factor Gaussian Models (cntd)

Solution, using CTM for the first and the second equations:

$$\begin{aligned} S(t) &= e^{\int_0^t r_s ds}[S_0 + \widehat{W}^1(T_t^1)] \\ &= e^{\int_0^t e^{-as}[r_0 - m + \widehat{W}^2(T_s^2)]ds}[S_0 + \widehat{W}^1(T_t^1)], \end{aligned}$$

where T^i and $\widehat{W}^i$ are inverse CT and Brownian motions defined for GBM and continuous-time GARCH model, respectively, $i = 1, 2$.

Can We Replace $W(t)$ by Lévy Process $L(t)$ and Use CT?

Any solution using CT method for

$$dX(t) = a(t, X(t-))dL(t),$$

where $L(t)$ is a Lévy process?

Kallsen & Shiryaev (2001) showed that this is only possible for symmetric α-stable Lévy process.

Lévy Processes

By *Lévy process* we define a stochastically continuous process with *stationary and independent increments*, Sato (1999), Applebaum (2003), Schoutens (2003).

Examples of Lévy Processes in Finance:

- Brownian motion with drift (only continuous Lévy process)
- Merton model=Brownian motion+drift+Gaussian jumps
- Kou model=Brownian motion+drift+exponential jumps
- VG, IG, NIG, GH processes
- *α-stable Lévy processes*

α-Stable Lévy Processes

Let $\alpha \in (0, 2]$. An *α-stable Lévy process* L such that L_1 (or equivalently any L_t) has an α-stable distribution (i.e., $L_1 \equiv S_\alpha(\sigma, \beta, \delta)$).

For values of $\alpha \in (1, 2]$ location parameter δ corresponds to the mean of the α-stable distribution, while for $0 < \alpha \leq 1$, δ corresponds to its median.

The parameter $\beta \in [-1, 1]$ determines the skewness of the distribution.

The dispersion parameter $\sigma \geq 0$ corresponds to the spread of the distribution around its location parameter δ.

The characteristic exponent α determines the shape of the distribution.

α-Stable Lévy Processes

We call L a *symmetric α-stable Lévy process* if the distribution of L_1 is symmetric α-stable (i.e., $L_1 \equiv S_\alpha(\sigma, 0, 0)$ for some $\alpha \in (0, 2], \quad \sigma \in R_+$.) (See Sato (2005)).

Characteristic function:

$$\phi(u) = e^{-\sigma^\alpha |u|^\alpha}.$$

α-Stable Lévy Processes

The probability density of an α-stable law is not known in closed form except in the following three cases:

the *Lévy* $(\alpha = 1/2)$, *Cauchy* $(\alpha = 1)$ and the *Gaussian* $(\alpha = 2)$ distributions.

However, power series expansions can be derived for any density function.

Its tails (algebraic tails) decay at a lower rate than the Gaussian density tails (exponential tails).

α-**Stable Lévy Processes**

- the only self-similar Lévy processes: $L(at) \stackrel{Law}{=} a^{1/\alpha} L(t), a \geq 0$

- either Brownian motion or pure jump

- characteristic exponent, Lévy-Khintchine triplet known in closed form

- 4 parameters

- infinite variance (except for Brownian motion)

α-Stable Lévy Processes

• α-stable Lévy Processes are semimartingales ($\int_0^t f_s dL_s$ can be defined)

• α-stable Lévy Processes are pure discontinuous Markov processes with generator

$$Af(x) = \int_{R-\{0\}} [f(x+y) - f(x) - yf'(y)\mathbf{1}_{|y|<1}(y)] \frac{K_\alpha}{|y|^{1+\alpha}} dy$$

α-Stable Lévy Processes

α-stable distributions on R never admit a second moment, and they only admit a first moment if $\alpha > 1$.

$E|L(t)|^p$ is finite or infinite according as $0 < p < \alpha$ or $p > \alpha$, respectively.

In particular, for an α-stable process $EL(t) = \delta t$ $(1 < \alpha \leq 2)$ (Sato (2005)).

SDE Driven by α-stable Lévy Processes

$$dX_t = b(X_{t-})dt + \sigma(X_{t-})dL(t)$$

Janicki, Michna & Weron (1996): there exists unique solution for continuous b, σ and α-stable Lévy process.

Zanzotto (1997): solutions of one-dimensional SDEs driven by stable Lévy motion

Cartea & Howison (2006): option pricing with Lévy-stable processes generated by Lévy-stable integrated variance

SDE Driven by α-stable Lévy Processes

One-Factor Lévy Models

$L(t)$ below is a symmetric α-stable Lévy process. We define below various processes via SDE driven by α-stable Lévy process.

1. *Geometric α-stable Lévy Motion*.
$dS(t) = \mu S(t-)dt + \sigma S(t-)dL(t)$.

2. *Continuous-Time GARCH Process Driven by α-stable Lévy Process*.
$dS(t) = \mu(b - S(t-))dt + \sigma S(t-)dL(t)$.

3. *Cox-Ingersoll-Ross Process Driven by α-stable Lévy Motion*.
$dS(t) = k(\theta - S(t-))dt + \gamma\sqrt{S(t-)}dL(t)$.

Multi-Factor Lévy Models

Multi-factor models driven by α-stable Lévy motions can be obtained using various combinations of above-mentioned processes. We give one example of two-factor continuous-time GARCH model driven by α-stable Lévy processes:

$$\begin{cases} dS(t) = r(t-)S(t-))dt + \sigma_1 S(t-)dL^1(t) \\ dr(t) = a(m - r(t-))dt + \sigma_2 r(t-)dL^2(t). \end{cases}$$

Change of Time Method for SDE Driven by Lévy Process

Can we solve these SDEs driven by Lévy processes using change of time methos?

The answer is 'Yes' for SDEs driven by α-stable Lévy processes.

Change of Time Method for SDE Driven by Lévy Process

We denote by $L^{\alpha}_{a.s.}$ the family of all real measurable $\mathcal{F}_t$-adapted processes a on $\Omega\times[0,+\infty)$ such that for every $T>0$, $\int_0^T |a(t,\omega)|^{\alpha}dt < +\infty$ a.s. We consider the following *SDE driven by an α-stable Lévy process*:

$$dX(t) = a(t, X(t-))dL(t).$$

Change of Time Method for SDE Driven by Lévy Process

Theorem. *(Rosinski and Woyczynski (1986)*, Theorem 3.1., p.277). Let $a \in L^{\alpha}_{a.s.}$ be such that $T(t) := \int_0^t |a|^{\alpha} du \to +\infty$ a.s. as $u \to +\infty$. If $\hat{T}(t) := \inf\{u : T(u) > t\}$ and $\hat{\mathcal{F}}_t = \mathcal{F}_{\hat{T}(t)}$, then the time-changed ctochastic integral $\hat{L}(t) = \int_0^{\hat{T}(t)} a dL(t)$ is an $\hat{\mathcal{F}}_t - \alpha$-stable Lévy process, where $L(t)$ is $\mathcal{F}_t$-adapted and $\mathcal{F}_t$-α-stable Lévy process.

Change of Time Method for SDE Driven by Lévy Process

Consequently, a.s. for each $t > 0$ $\int_0^t a dL = \hat{L}(T(t))$, i.e., the stochastic integral with respect to a α-stable Lévy process is nothing but another α-stable Lévy process with randomly changed time scale.

Solutions to the One-Factor Lévy Models Using CTM

$L(t)$ below is a symmetric α-stable Lévy process, and $\widehat{L}$ is a $(\widehat{T}_t)_{t\in R_+}$-adapted symmetric α-stable Levy process on $(\Omega, \mathcal{F}, \widehat{\mathcal{F}}_t, P))$.

Geometric α-stable Lévy Motion. $dS(t) = \mu S(t-)dt + \sigma S(t-)dL(t)$. Solution $S(t) = e^{\mu t}[S(0) + \widehat{L}(T_t)]$, where $T_t = \sigma^\alpha \int_0^t [S(0) + \widehat{L}(T_s)]^\alpha ds$.

Continuous-Time GARCH α-stable Lévy Process. $dS(t) = \mu(b - S(t-))dt + \sigma S(t-)dL(t)$. Solution $S(t) = e^{-\mu t}(S(0) - b + \widehat{L}(T_t)) + b$, where $T_t = \sigma^\alpha \int_0^t [S(0) - b + \widehat{L}(T_s) + e^{\mu s} b]^\alpha ds$.

Cox-Ingersoll-Ross α-stable Lévy Process. $dS(t) = k(\theta^2 - S(t-))dt + \gamma\sqrt{S(t-)}dL(t)$. Solution $S^2(t) = e^{-kt}[S_0^2 - \theta^2 + \widehat{L}(T_t)] + \theta^2$, where $T_t = \gamma^\alpha \int_0^t [e^{kT_s}(S_0^2 - \theta^2 + \widehat{L}(T_s)) + \theta^2 e^{2kT_s}]^{\alpha/2} ds$.

Change of Time Method for SDE Driven by Lévy Process

Solution to the Multi-Factor Lévy models Using CTM

Solution to the multi-factor models driven by α-stable Lévy motions can be obtained using various combinations of solutions of the above-mentioned processes and CTM. We give one example of two-factor continuous-time GARCH model driven by α-stable Lévy motions:

$$\begin{cases} dS(t) = r(t-)S(t-))dt + \sigma_1 S(t-)dL^1(t) \\ dr(t) = a(m - r(t-))dt + \sigma_2 r(t-)dL^2(t). \end{cases}$$

Change of Time Method for SDE Driven by Lévy Process

Solution to the Multi-Factor Lévy Models Using CTM

Solution, using CTM for the first and the second equations:

$$\begin{aligned} S(t) &= e^{\int_0^t r_s ds}[S_0 + \hat{L}^1(T_t^1)] \\ &= e^{\int_0^t e^{-as}[r_0 - m + \hat{L}^2(T_s^2)]ds}[S_0 + \hat{L}^1(T_t^1)], \end{aligned}$$

where T^i are defined above (i=1,2), respectevely.

Applications of CTM in Financial and Energy Markets

Change of Time Method will be applied to:

- Financial markets
- Energy markets

Applications of CTM in Financial Markets

We show how to apply CTM to get:

- Black-Scholes formula
- Variance and volatility swaps for Heston Model
- Variance swap for Lévy-based Heston model
- Lévy-based SABR model

Applications of CTM in Financial Markets: Black-Scholes Formula

$$dS(t) = rS(t)dt + \sigma S(t)dW^*(t)$$

where

$$W^*(t) := W(t) + \frac{\mu - r}{\sigma}.$$

Solution:

$$S(t) = e^{rt}[S(0) + \hat{W}^*(T_t)],$$

where

$$\hat{W}^*(T_t) = S(0)(e^{\sigma W^*(t) - \frac{\sigma^2 t}{2}} - 1)$$

Applications of CTM in Financial Markets: Black-Scholes Formula

Straightforward calculation gives the Black-Scholes formula:

$$
\begin{aligned}
C(T) &= e^{-rT} E^{P^*}[(S(T) - K)^+] \\
&= e^{-rT} E^{P^*}[e^{rT}(S(0) + \widehat{W}^*(T_t)) - K]^+ \\
&= e^{-rT} E^{P^*}[e^{rT} S(0) e^{\sigma \widehat{W}^*(T) - (\sigma^2 T)/2} - K]^+ \\
&= S(0)\Phi(d_+) - K e^{-rT} \Phi(d_-).
\end{aligned}
$$

Applications of CTM in Financial Markets: VarSwaps and VolSwaps

Why trade volatility (variance)?

- Volatility swaps allow investors to profit from the risks of an increase or decrease in future volatility of an index of securities or to hedge against these risks

- If you think current volatility is low, for the right price you might want to take a position that profits if volatility increases

Applications of CTM in Financial Markets: Var and Vol Swaps

• Volatility swaps are forward contracts on future realized stock volatility

• Variance swaps are similar contract on variance, the square of the future volatility

Payoff for VarSwap at expiration: $N(\sigma_R^2(S) - K_{var})$

Payoff for VolSwap at expiration: $N(\sigma_R(S) - K_{vol})$

Here, $\sigma_R(S)$ is the realized volatility over the life of contract: $\sigma_R(S) := \sqrt{\frac{1}{T}\int_0^T \sigma_s^2 ds}.$

Applications of CTM in Financial Markets: Pricing of Var-Swaps and VolSwaps

The value of a forward contract P on future realized variance with strike price K_{var} is the expected present value of the future payoff in the risk-neutral world:

$$P_{var} = e^{-rT}(E\sigma_R^2(S) - K_{var}),$$

where r is the risk-free discount rate corresponding to the expiration date T, and E denotes the expectation.

Thus, for calculating variance swaps we need to know only $E\{\sigma_R^2(S)\}$, namely, mean value of the underlying variance.

Applications of CTM in Financial Markets: Pricing of Var and Vol Swaps

To calculate volatility swaps we need more. From Brockhaus-Long (2000) approximation (which is used the second order Taylor expansion for function $\sqrt{x}$) we have:

$$E\{\sqrt{\sigma_R^2(S)}\} \approx \sqrt{E\{V\}} - \frac{Var\{V\}}{8E\{V\}^{3/2}},$$

where $V := \sigma_R^2(S)$ and $\frac{Var\{V\}}{8E\{V\}^{3/2}}$ is the convexity adjustment.

Thus, to calculate the value of volatility swaps

$$P_{vol} = \{e^{-rT}(E\{\sigma_R(S)\} - K_{vol})\}$$

we need both $E\{V\}$ and $Var\{V\}$.

Applications of CTM in Financial Markets: VarSwap and VolSwap for Heston Model

Assume that underlying asset S_t in the risk-neutral world and variance follow the following model, Heston (1993) model:

$$\begin{cases} \frac{dS_t}{S_t} &= r_t dt + \sigma_t dw_t^1 \\ d\sigma_t^2 &= k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dw_t^2, \end{cases}$$

where r_t is deterministic interest rate, σ_0 and θ are short and long volatility, $k > 0$ is a reversion speed, $\gamma > 0$ is a volatility (of volatility) parameter, w_t^1 and w_t^2 are independent standard Wiener processes.

Applications of CTM in Financial Markets: Var and Vol Swaps for Heston Model

The solution of the following equation

$$d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dw_t^2$$

has the following look

$$\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \widehat{w}^2(T_t)) + \theta^2,$$

where T_t is the change of time.

Applications of CTM in Financial Markets: Pricing of Var and Vol Swaps for Heston Model

The value (or price) P_{var} of variance swap is

$$P_{var} = e^{-rT}[\frac{1-e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2 - K_{var}]$$

The value (or price) P_{vol} of volatility swap is

$$\begin{aligned} P_{vol} &= e^{-rT}\{(\tfrac{1-e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2)^{1/2} \\ &- (\tfrac{\gamma^2 e^{-2kT}}{2k^3T^2}[(2e^{2kT} - 4e^{kT}kT - 2)(\sigma_0^2 - \theta^2) \\ &+ (2e^{2kT}kT - 3e^{2kT} + 4e^{kT} - 1)\theta^2])/[8(\tfrac{1-e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2)^{3/2}] \\ &- K_{vol}\}. \end{aligned}$$

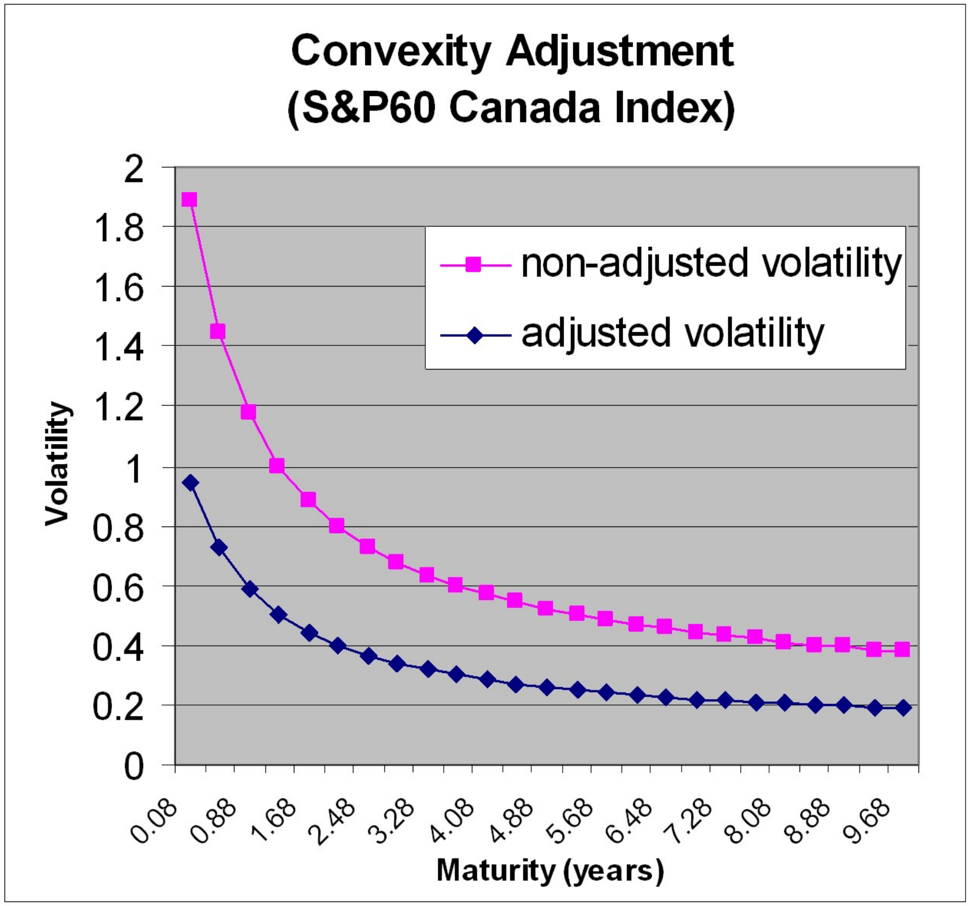
(See Brokhaus & Long (2000), Sw.(2004))

Applications of CTM in Financial Markets: Pricing of Var and Vol Swaps for Heston Model

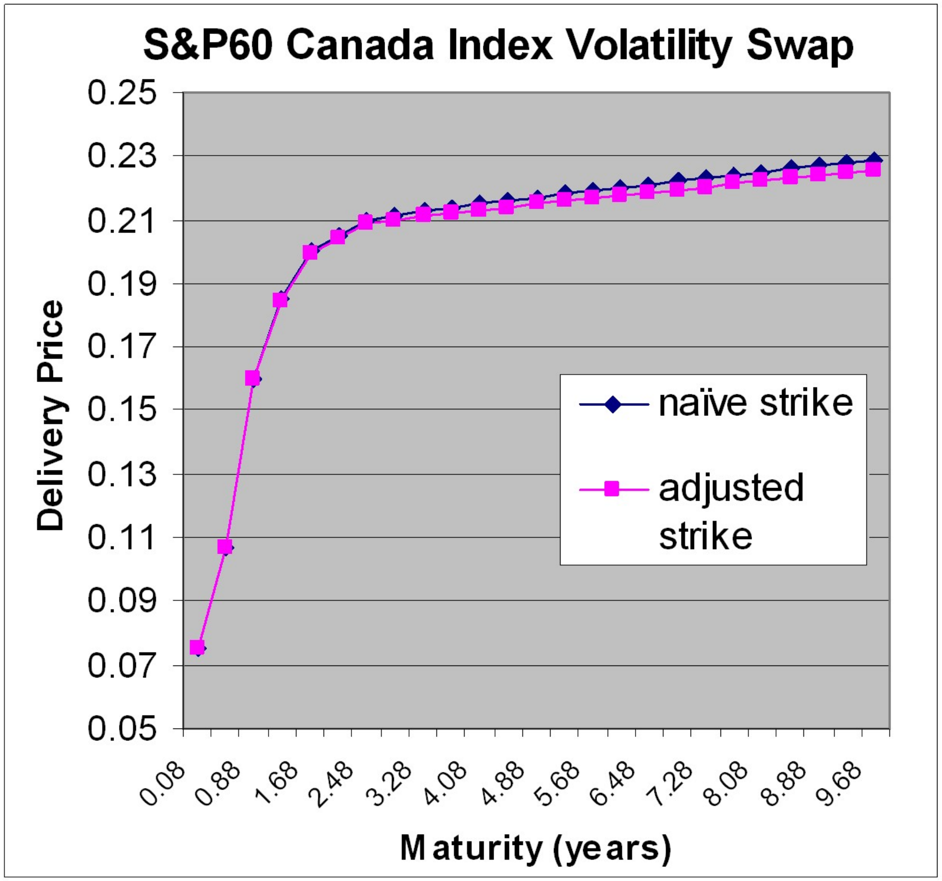
Numerical Example: S&P 60 Canada Index

The statistics on log returns $S\&P60$ Canada Index for 5 year (January 1997-February 2002) is presented in the following Table:

Statistics on Log-Returns $S\&P60$ Canada Index	
Series:	Log-returns $S\&P60$ Canada Index
Sample:	1 1300
Observations:	1300
Mean	0.000235
Median	0.000593
Maximum	0.051983
Minimum	-0.101108
Std. Dev.	0.013567
Skewness	-0.665741
Kurtosis	7.787327



Convexity Adjustment: Non-adjusted vs. Adjusted Vol



S&P60 Canada Index Vol Swap

Applications of CTM in Financial Markets: Variance Swaps for Lévy-based Heston Model

Assume that underlying asset S_t in the risk-neutral world and variance follow the following model:

$$\begin{cases} \frac{dS_t}{S_t} &= r_t dt + \sigma_t dw_t \\ d\sigma_t^2 &= k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dL_t, \end{cases}$$

where r_t is deterministic interest rate, σ_0 and θ are short and long volatility, $k > 0$ is a reversion speed, $\gamma > 0$ is a volatility (of volatility) parameter, w_t and L_t are independent standard Wiener and α-stable Lévy processes ($\alpha \in (1, 2]$).

Applications of CTM in Financial Markets: Variance Swaps for Lévy-based Heston Model

Solution:

$$\sigma^2(t) = e^{-kt}[\sigma_0^2 - \theta^2 + \hat{L}(T_t)] + \theta^2,$$

where $T_t = \gamma^\alpha \int_0^t [e^{kT_s}(\sigma_0^2 - \theta^2 + \hat{L}(T_s)) + \theta^2 e^{2kT_s}]^{\alpha/2} ds.$

Applications of CTM in Financial Markets: Variance Swaps for Lévy-based Heston Model

Realized Variance:

$$\sigma_R^2(S) := \frac{1}{T}\int_0^T \sigma^2(s)ds = \frac{1}{T}\int_0^T \{e^{-ks}[\sigma_0^2 - \theta^2 + \hat{L}(T_s)] + \theta^2\}ds,$$

Value of Variance Swap:

$$\begin{aligned} P_{var} &= E\{e^{-rT}(\sigma_R^2(S) - K_{var})\} \\ &= E\{e^{-rT}(\tfrac{1}{T}\int_0^T \{e^{-ks}[\sigma_0^2 - \theta^2 + \hat{L}(T_s)] + \theta^2\}ds - K_{var}) \end{aligned}$$

Applications of CTM in Financial Markets: Variance Swaps for Lévy-based Heston Model

Thus, for calculating variance swaps we need to know only $E\{\sigma_R^2(S)\}$, namely, mean value of the underlying variance, or $E[\widehat{L}(T_s)]$.

Only moments of order less than α exist for the non-Gaussian family of α-stable distribution. We suppose that $1 < \alpha \leq 2$ to find $E[\widehat{L}(T_s)]$.

Applications of CTM in Financial Markets: Variance Swaps for Lévy-based Heston Model

The value of variance swap for Lévy-based Heston Model:

$$P_{var} = e^{-rT}[\frac{1-e^{-kT}}{kT}(\sigma_0^2-\theta^2)+\theta^2+\frac{\delta T}{2}-K_{var}],$$

where δ is a location parameter.

When $\delta = 0$, then the value of the var swap for Lévy-based Heston model coincides with the value of the var swap for classical Heston model.

Applications in Financial Markets: Gaussian-based SABR or LMM Models

SABR model (see Hagan, Kumar, Lesniewski and Woodward (2002)) and the Libor Market Model (LMM) (Brace, Gatarek and Musiela (BGM, 1996)) have become industry standards for pricing plain-vanilla and complex interest rate products, respectively.

Gaussian-based SABR model, a stochastic volatility model in which the forward value satisfies:

$$\begin{cases} dF_t &= \sigma_t F_t^{\beta} dW_t^1 \\ d\sigma_t &= \nu \sigma_t dW_t^2, \end{cases}$$

Applications in Financial Markets: Lévy-based SABR

Lévy-based SABR model, a stochastic volatility model in which the forward value satisfies:

$$\begin{cases} dF_t &= \sigma_t F_t^{\beta} dW_t \\ d\sigma_t &= \nu \sigma_t dL_t, \end{cases}$$

Applications in Financial Markets: Lévy-based SABR

Lévy-Based SABR: solution using change of time

$$F_t = F_0 + \widehat{W}(T_t^1),$$

$$T_t^1 = \int_0^t \sigma^2_{T_s^1}(F_0 + \widehat{W}(s))^{2\beta} ds,$$

$$\sigma_t = \sigma_0 + \widehat{L}(T_t^2),$$

$$T_t^2 = \nu^\alpha \int_0^t (\sigma_0 + \widehat{L}(s))^\alpha ds.$$

Applications of CTM in Energy Markets

- Option Pricing Formula for a Mean-reverting Asset
- Pricing Futures and Forwards
- Variance and Volatility Swaps

Applications of CTM in Energy Markets: Option Pricing Formula for a Mean-Reverting Asset

Some commodity prices, like oil and gas, exhibit the mean reversion, unlike stock price. It means that they tend over time to return to some longterm mean. Here, we consider a risky asset S_t following the mean-reverting stochastic process given by the following stochastic differential equation (a.k.a. continuous-time GARCH process).

Applications of CTM in Energy Markets: Option Pricing Formula for a Mean-Reverting Asset

Continuous-time GARCH process for a mean-reverting asset:

$$dS_t = a(L - S_t)dt + \sigma S_t dW_t.$$

where W is a standard Wiener process, $\sigma > 0$ is the volatility, the constant L is called the 'long-term mean' of the process, to which it reverts over time, and $a > 0$ measures the 'strength' of mean reversion.

Applications of CTM in Energy Markets: Option Pricing Formula for a Mean-Reverting Asset

If λ is a market price of risk, then the latter equation in a risk-neutral world has a look:

$$dS_t = a^*(L^* - S_t)dt + \sigma S_t dW_t^*,$$

where

$$a^* := a + \lambda\sigma, \quad L^* := \frac{aL}{a + \lambda\sigma},$$

and $W_t^* := W_t + \lambda \int_0^t S(u)du$.

Applications of CTM in Energy Markets: Option Pricing Formula for a Mean-Reverting Asset

Solution using CTM:

$$S_t = e^{-a^* t}[S_0 - L^* + \widehat{W}^*(T_t)] + L^*.$$

Here, T_t is the CT for continuous-time GARCH process $S(t)$, introduced before.

Applications of CTM in Energy Markets: Option Pricing Formula for a Mean-Reverting Asset

$$\begin{aligned} C_T^* &= e^{-(r+a^*)T} S(0)\Phi(y_+) - e^{-rT} K\Phi(y_-) \\ &+ L^* e^{-(r+a^*)T}[(e^{a^*T} - 1) - \int_0^{y_0} z F_T^*(dz)], \end{aligned}$$

where y_0 is the solution of the following equation

$$y_0 = \frac{\ln(\frac{K}{S(0)}) + (\frac{\sigma^2}{2} + a^*)T}{\sigma\sqrt{T}}$$

$$-\frac{\ln(1 + \frac{a^*L^*}{S(0)} \int_0^T e^{a^*s} e^{-\sigma y_0 \sqrt{s} + \frac{\sigma^2 s}{2}} ds)}{\sigma\sqrt{T}},$$

Applications of CTM in Energy Markets: Option Pricing Formula for a Mean-Reverting Asset

$$y_+ := \sigma\sqrt{T} - y_0 \quad and \quad y_- := -y_0,$$

and $F_T^*(dz)$ is a probability distribution, that can be found using Yor's result (Yor (1992)) for exponential functions of Brownian motion . See Sw. (2008), J. Numer. Appl. Math., v. 1(96), pp. 216-233, for more details.

Applications of CTM in Energy Markets: Option Pricing Formula for a Mean -Reverting Asset

As we can see,

$$\begin{aligned} C_T^* = BS_T + L_T &= (Black - Scholes \quad Part) \\ &+ (Additional \quad Mean - revertion \quad Part). \end{aligned}$$

If $L^* = 0$ (or $L = 0$) and $a^* = -r$, then C_T^* coincides with Black-Scholes result!

Applications of CTM in Energy Markets: Energy Forwards and Futures

Random variables following α-stable distribution with small characteristic exponent are *highly impulsive*, and it is this heavy-tail characteristic that makes this density appropriate for modeling noise which is impulsive in nature, for example, energy prices such as electricity.

Applications of CTM in Energy Markets: Energy Forwards and Futures (Lévy-Based Schwartz-Smith Model)

$$\begin{cases} \ln(S_t) &= \kappa_t + \xi_t \\ d\kappa_t &= (-k\kappa_t - \lambda_\kappa)dt + \sigma_\kappa dL_\kappa \\ d\xi_t &= (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dW_\xi, \end{cases}$$

where S_t current spot price, κ_t is the short-term deviation in prices, ξ_t is the equilibrium price level.

Applications of CTM in Energy Markets: Energy Forwards and Futures (Lévy-Based Schwartz-Smith Model)

Let $F_{t,T}$ denotes the market price for a futures contract with maturity T, then:

$$\ln(F_{t,T}) = e^{-k(T-t)}\kappa_t + \xi_t + A(T-t),$$

where $A(T-t)$ is a deterministic function with explicit expression.‘ We note that κ_t, using change of time for α-stable processes can be presented in the following form:

$$\kappa_t = e^{-kt}[\kappa_0 + \frac{\lambda_\kappa}{k} + \widehat{L}_\kappa(T_t)],$$

$$T_t = \sigma_\kappa^\alpha \int_0^t (e^{-ks}[\kappa_0 + \frac{\lambda_\kappa}{k} + \widehat{L}_\kappa(T_s)] - \frac{\lambda_\kappa}{k})^\alpha ds$$

Applications of CTM in Energy Markets: Energy Forwards and Futures (Lévy-Based Schwartz-Smith Model)

In this way, the market price for a futures contract with maturity T has the following look:

$$\begin{aligned}\ln(F_{t,T}) &= e^{-kT}[\kappa_0 + \frac{\lambda_\kappa}{k} + \widehat{L}_\kappa(T_t)] \\ &+ \xi_0 + (\mu_\xi - \lambda_\xi)t + \sigma_\xi W_\xi + A(T-t),\end{aligned}$$

where Lévy process $\widehat{L}_\kappa$ and Wiener process W_ξ may be correlated.

Applications of CTM in Energy Markets: Energy Forwards and Futures (Lévy-Based Schwartz Model)

$$\begin{cases} d\ln(S_t) &= (r_t - \delta_t)S_t dt + S_t \sigma_1 dW_1 \\ d\delta_t &= k(a - \delta_t)dt + \sigma_2 dL \\ dr_t &= a(m - r_t)dt + \sigma_3 dW_2, \end{cases}$$

where W_1, W_2 are Wiener processes and L is an α-stable Lévy process. δ_t and r_t are instantaneous convenience yield and interest rate, respectively.

Applications of CTM in Energy Markets: Energy Forwards and Futures (Lévy-Based Schwartz Model)

We note that:

$$\begin{aligned} \delta_t &= e^{kt}(\delta_0 - a + \widehat{L}(\widehat{T}_t)), \\ \widehat{T}_t &= \sigma_2^{\alpha} \int_0^t (e^{ks}[\delta_0 - a + \widehat{L}(\widehat{T}_s)] + a)^{\alpha} ds \end{aligned}$$

and

$$\begin{aligned} r_t &= e^{at}(r_0 - m + \widehat{W}_2(T_t)), \\ T_t &= \sigma_3^2 \int_0^t (e^{as}[r_0 - m + \widehat{W}_2(T_s)] + m)^2 ds. \end{aligned}$$

Applications of CTM in Energy Markets: Energy Forwards and Futures (Lévy-Based Schwartz Model)

Solution for $\ln[S_t]$:

$$\ln[S_t] =$$

$$= e^{\int_0^t [e^{as}(r_0 - m + \widehat{W}_2(T_s^2)) - e^{ks}(\delta_0 - a + \widehat{L}(T_s))]ds} [\ln S_0 + \widehat{W}_1(T_t^1)].$$

Applications of CTM in Energy Markets: Energy Forwards and Futures (Lévy-Based Schwartz Model)

In this way, the futures contracts has the following form:

$$\begin{aligned}
\ln(F_{t,T}) &= \tfrac{1-e^{-k(T-t)}}{k}\delta_t + \tfrac{1-e^{-a(T-t)}}{a}r_t + \ln(S_t) + C(T-t) \\
&= \tfrac{1-e^{-k(T-t)}}{k}[e^{kt}(\delta_0 - a + \hat{L}(T_t))] \\
&+ \tfrac{1-e^{-a(T-t)}}{a}e^{at}(r_0 - m + \hat{W}_2(T_t^2)) \\
&+ \exp\{\textstyle\int_0^t(e^{as}(r_0 - m + \hat{W}_2(T_s^2)) \\
&- e^{ks}(\delta_0 - a + \hat{L}(T_s)))ds\}[\ln(S_0) + \hat{W}_1(T_t^1)] \\
&+ C(T-t),
\end{aligned}$$

where $C(T-t)$ is a deterministic explicit function.

Applications of CTM in Energy Markets: Var and Vol Swaps

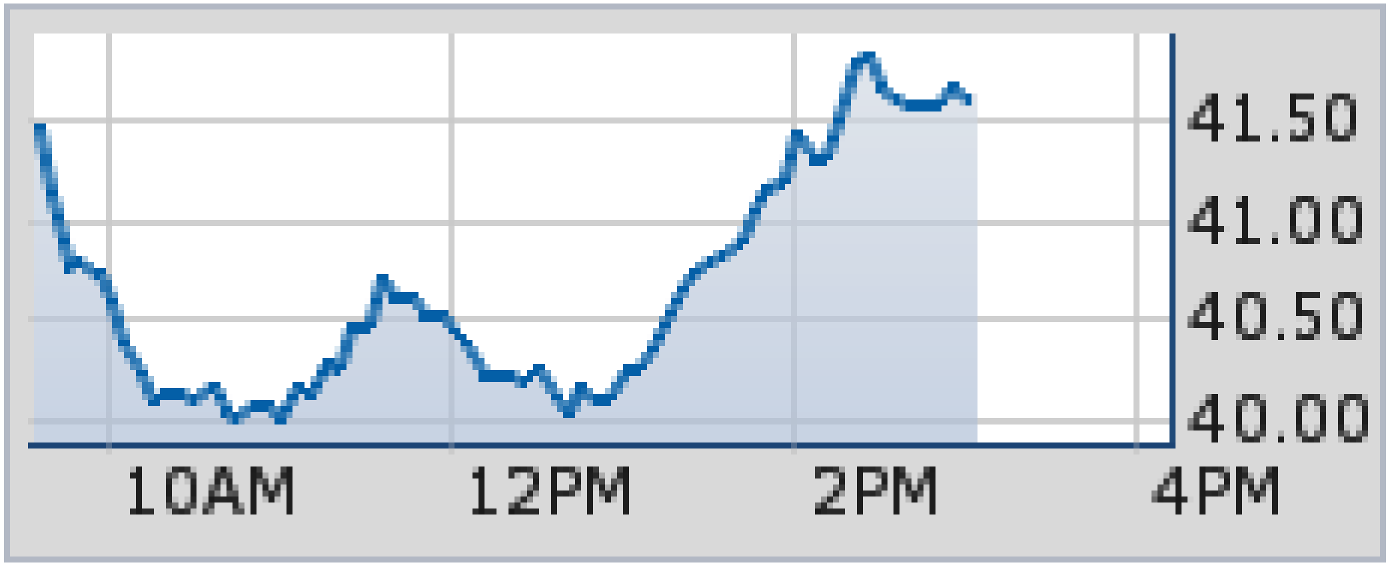
Variance swaps are quite common in commodity, e.g., in energy market, and they are commonly traded.

Energy is the most important commodity sector, and crude oil and natural gas constitute the largest components of the two most widely tracked commodity indices: the Standard & Poors Goldman Sachs Commodity Index (S&P GSCI) and the Dow Jones-AIG Commodity Index (DJ-AIGCI).

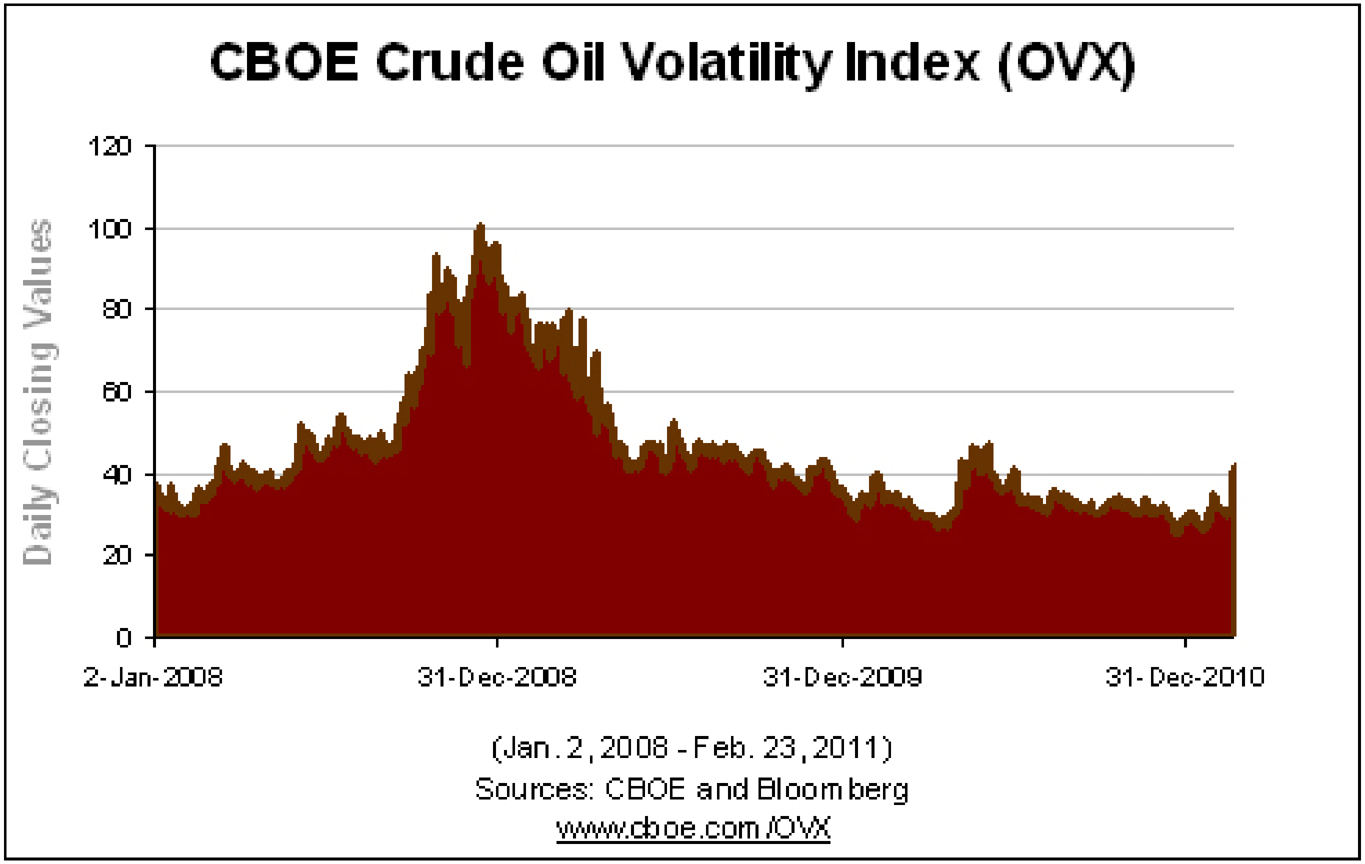
Applications of CTM in Energy Markets: Var and Vol Swaps

The CBOE Crude Oil ETF Volatility Index ('Oil VIX', Ticker - OVX) measures the market's expectation of 30-day volatility of crude oil prices by applying the VIX methodology to United States Oil Fund, LP (Ticker - USO) options spanning a wide range of strike prices (see Figures below. Courtesy-CBOE:

http://www.cboe.com/micro/oilvix/introduction.aspx).



OVX Bi-hourly



CBOE Crude Oil Volatility Index (OVX)

Applications of CTM in Energy Markets: Var and Vol Swaps

The following next slides are devoted to the pricing of variance and volatility swaps in energy market. We found explicit variance swap formula and closed form volatility swap formula (using change of time) for energy asset with stochastic volatility that follows continuous-time mean-reverting GARCH model. Numerical example is presented for AECO Natural Gas Index (1 May 1998-30 April 1999).

Applications of CTM in Energy Markets: Var and Vol Swaps

We consider a risky asset in energy market with stochastic volatility following a mean-reverting stochastic process the following stochastic differential equation:

$$d\sigma^2(t) = a(L - \sigma^2(t))dt + \gamma\sigma^2(t)dW_t,$$

where $a > 0$ is a speed (or 'strength') of mean reversion, $L > 0$ is the mean reverting level (or equilibrium level, or long-term mean), $\gamma > 0$ is the volatility of volatility $\sigma(t)$, W_t is a standard Wiener process.

Applications of CTM in Energy Markets: Var and Vol Swaps

In the risk-neutral world:

$$d\sigma^2(t) = a^*(L^* - \sigma^2(t))dt + \gamma\sigma^2(t)dW_t^*,$$

where

$$a^* := a + \lambda\gamma, \quad L^* := \frac{aL}{a + \lambda\gamma},$$

and $W_t^* := W_t + \lambda t.$

Applications of CTM in Energy Markets: Var and Vol Swaps

Solution:

$$\sigma^2(t) = e^{-a^* t}[\sigma^2(0) - L^* + \widehat{W}(T_t)] + L^*.$$

Applications of CTM in Energy Markets: Var and Vol Swaps

To calculate the variance swap for $\sigma^2(t)$ we need $E\sigma^2(t)$. From previous slide it follows that

$$E^*\sigma^2(t) = e^{-a^*t}[\sigma^2(0) - L^*] + L^*.$$

Then $E^*\sigma_R^2 := E^*V$ takes the following form:

$$E^*\sigma_R^2 := E^*V := \frac{1}{T}\int_0^T E\sigma^2(t)dt = \frac{(\sigma^2(0) - L^*)}{a^*T}(1 - e^{-a^*T}) + L^*.$$

Recall, that $V := \frac{1}{T}\int_0^T \sigma^2(t)dt$.

Applications of CTM in Energy Markets: Var and Vol Swaps

To calculate the volatility swap for $\sigma^2(t)$ we need $E\sqrt{V} = E\sqrt{\sigma_R}$ and it means that we more than just $E\sigma^2(t)$, because the realized volatility $\sigma_R := \sqrt{V} = \sqrt{\sigma_R^2}$. Using Brockhaus-Long approximation we then get

$$E^*\sqrt{V} \approx \sqrt{E^*V} - \frac{Var^*(V)}{8(E^*V)^{3/2}}.$$

The $Var^*(V)$ is calculating as follows:

$$\begin{aligned} Var^*(V) &= E^*V^2 - (E^*V)^2 \\ &= \tfrac{1}{T^2}\int_0^T\int_0^T e^{-a^*(t+s)}\{\gamma^2[(\sigma^2(0)-L^*)^2\tfrac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2} \\ &+ \tfrac{2L^*(\sigma^2(0)-L^*)(e^{a^*(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a^*-\gamma^2} + \tfrac{(L^*)^2(e^{2a^*(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{2a^*-\gamma^2}]\}dtds. \end{aligned}$$

Applications of CTM in Energy Markets: Var and Vol Swaps (Numerical Example)

We shall calculate the value of variance and volatility swaps prices of a daily natural gas contract. To apply our formula for calculating these values we need to calibrate the parameters $a, \quad L, \quad \sigma_0^2$ and γ (T is monthly). These parameters may be obtained from futures prices for the AECO Natural Gas Index for the period 1 May 1998 to 30 April 1999 (see Bos, Ware and Pavlov (2002)).

Applications of CTM in Energy Markets: Var and Vol Swaps (Numerical Example)

The parameters are the following:

Parameters			
a	γ	L	λ
4.6488	1.5116	2.7264	0.18

Applications of CTM in Energy Markets: Var and Vol Swaps (Numerical Example)

From this table we can calculate the values for risk adjusted parameters a^* and L^* :

$$a^* = a + \lambda\gamma = 4.9337,$$

and

$$L^* = \frac{aL}{a + \lambda\gamma} = 2.5690.$$

For the value of $\sigma^2(0)$ we can take $\sigma^2(0) = 2.25$.

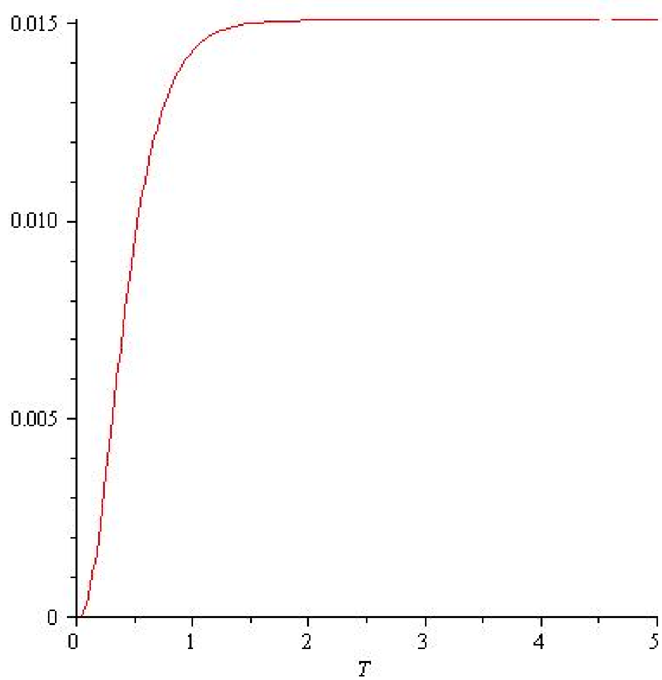


Fig. 1: Variance Swap

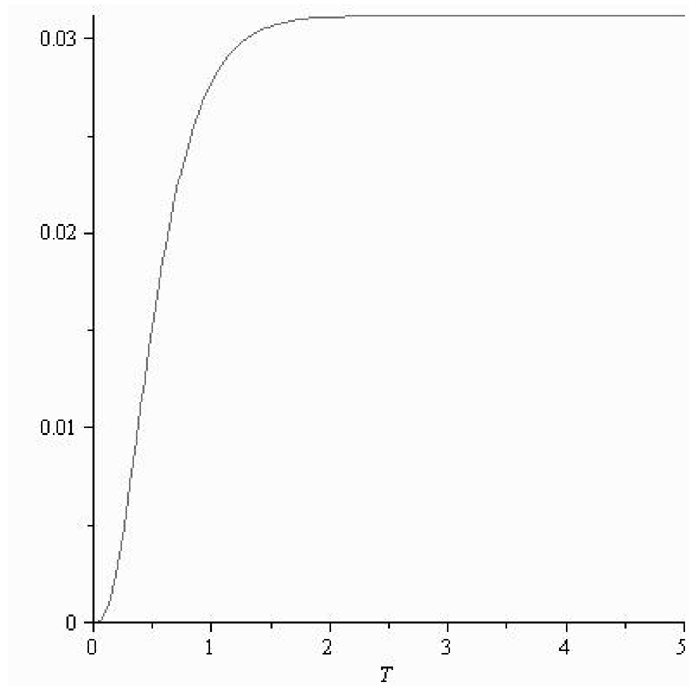


Fig. 2: Volatility Swap

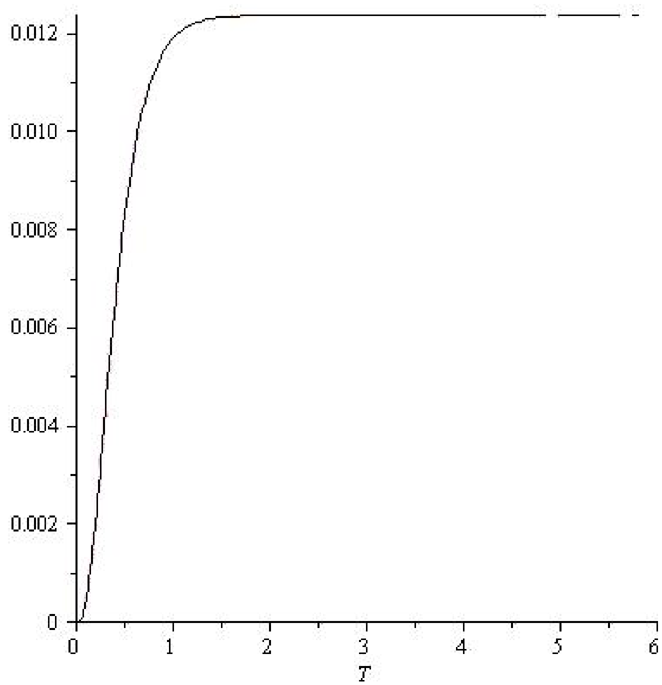


Fig. 3: Variance Swap (Risk Adjusted Parameters)

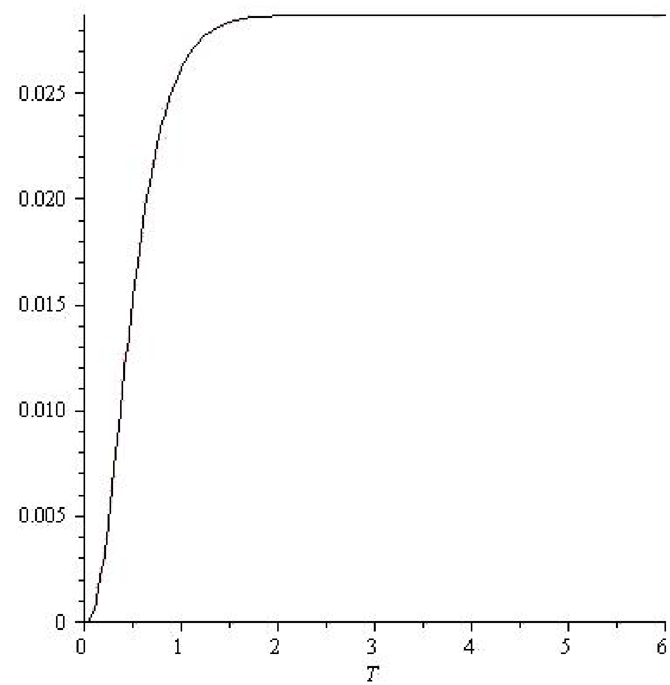


Fig. 4: Volatility Swap (Risk Adjusted Parameters)

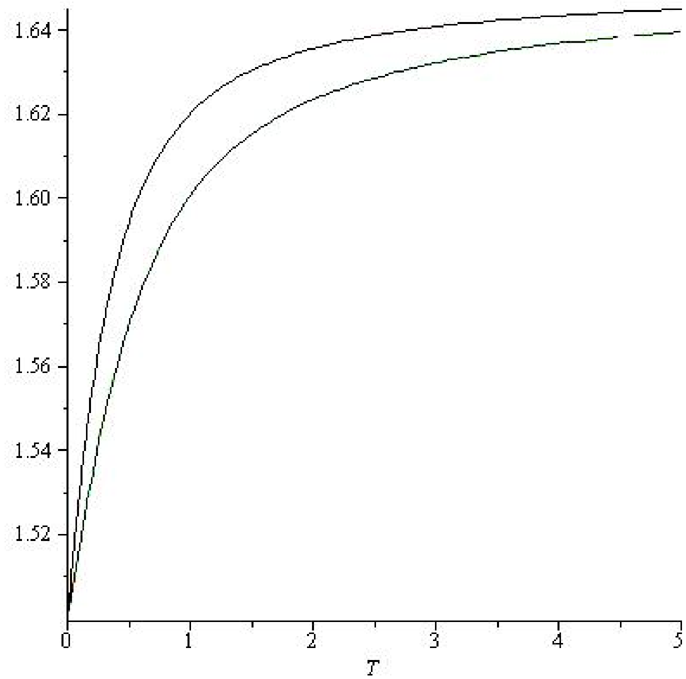


Fig. 5: Comparison: Adjusted and Non-Adjusted Price

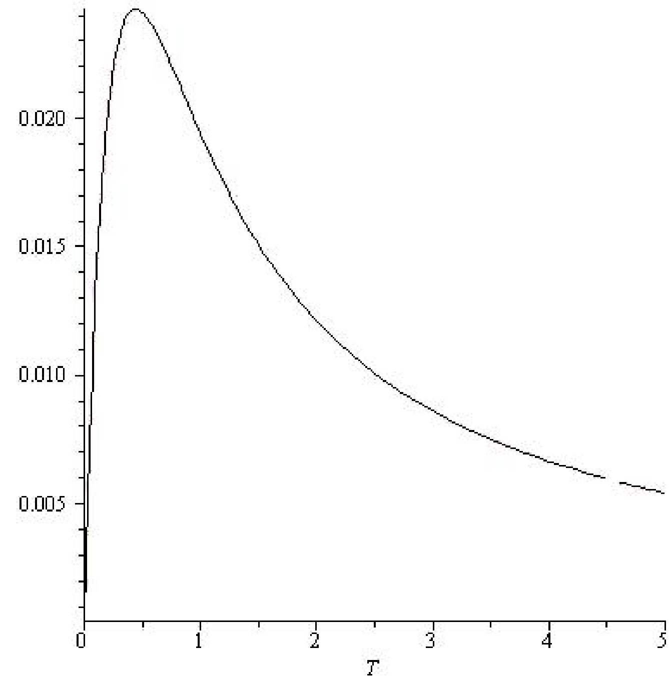


Fig. 6: Convexity Adjustment

Delayed Heston Model

• Motivation: to include past history (a.k.a. delay) of the variance (over some delayed time interval $[t-\tau, t]$)

• Advantage: Improvement of the Volatility Surface Fitting (44% reduction of the calibration error) compare with Classical Heston model

• Goal: to price and hedge volatility swaps

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

The Heston model is one of the most popular stochastic volatility models in the industry, as semi-closed formulas for vanilla option prices are available, few (five) parameters need to be calibrated, and it accounts for the mean-reverting feature of the volatility.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

We'd like to take into account not only its current state (as it is the case in the Heston model) but also its past history over some interval $[t-\tau, t]$, where τ is a positive constant and is called the delay. Namely, at each time t, the immediate future volatility at time $t+\epsilon$ will not only depend on its value at time t but also on all its history over $[t-\tau, t]$.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

Namely, at each time t, the immediate future volatility at time $t+\epsilon$ will not only depend on its value at time t but also on all its history over $[t-\tau, t]$. Starting from the well-known discrete-time GARCH(1,1) model, a continuous-time GARCH variance diffusion incorporating delay (let's refer to it as 'delayed vol') was introduced in a paper Sw. (2005). Unfortunately, the latter model doesn't lead to (semi-)closed formulas for the vanilla options, making it difficult to use for practitioners willing to calibrate on vanilla market prices (which can be a serious drawback).

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

Nevertheless, one can notice that the Heston model and 'delayed vol' are very similar in the sense that the expected values of the variances are the same - when we make the delay tends to 0 in 'delayed vol'. As mentioned before, the Heston framework is very convenient for practitioners, and therefore it is naturally tempting to adjust the Heston dynamics in order to incorporate - in some way - the delay introduced in 'delayed vol'.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

We considered in a first approach adjusting the Heston drift by a deterministic function of time so that the expected value of the variance under our new delayed Heston model is equal to the one under 'delayed vol'. Our approach can therefore be seen as a variance 1st moment correction of the Heston model, in order to account for the delay. It is important to note that our model is a generalization of the classical Heston model (the latter corresponding to the zero delay case $\tau = 0$ of our model).

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

We performed numerical tests to validate our approach. With recent market data (Sept. 30th 2011, underlying EURUSD), we performed the model calibration on the whole market vanilla option price surface (14 maturities from 1M to 10Y, 5 strikes ATM, 25 Delta Call/Put, 10 Delta Call/Put). The results show a significant (44%) reduction of the average absolute calibration error compared to the Heston model (i.e. average of the absolute differences between market and model prices).

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

Further, we considered variance and volatility swaps hedging and pricing in our delayed Heston framework. These contracts are widely used in the financial industry and therefore it is relevant to know their price processes (how much they worth at each time t) and how we can hedge a position on them, i.e. theoretically cancel the risk inherent to holding one unit of them.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

Using the fact that every continuous local martingale can be represented as a time-changed Brownian motion, as well as the Brockhaus & Long approximation (that allows to approximate the expected value of the square-root of an almost surely non negative random variable using a 2nd order Taylor expansion approach), we were able to derive closed formulas for variance and volatility swaps price processes.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Prose)

In addition, as variance swaps are relatively liquid instruments in the market (i.e. they can be easily bought and sold), we considered the question of hedging a position on a volatility swap using variance swaps in our framework.

We were able to derive a closed formula for the dynamic hedge ratio, i.e. the number of units of variance swaps to hold at each time in order to hedge a position on a volatility swap.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

• Motivation: past history of the varinace in its diffusion (over some delayed time interval $[t-\tau, t]$)

• Non-Markov continuous-time GARCH model (Sw. (2005))

$$\frac{dV_t}{dt} = \gamma(\theta^2 - V_t) + \alpha\left[\frac{1}{\tau}\left(\int_{t-\tau}^{t} \sqrt{V_s}dZ_s^Q - (\mu - r)\tau\right)^2 - V_t\right]$$

•

$$\begin{cases} dV_t &= [\gamma(\theta^2 - V_t) + \epsilon_\tau(t)]dt + \delta\sqrt{V_t}dW_t^Q \\ \epsilon_\tau(t) &:= \alpha\left[\tau(\mu - r)^2 + \frac{1}{\tau}\int_{t-\tau}^{t} E^Q(V_s)ds - E^Q(V_t)\right]. \end{cases}$$

We note, that $\lim_{\tau\to 0}\sup_{t\in R_+}|\epsilon_\tau(t)| = 0$.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Calibration Results

• Semi-closed formulas available for call options

• September 30th 2011 for underlying EURUSD on the whole volatility surface (14 maturities from 1M to 10Y, 5 strikes: ATM, 25D call/put, 10D call/put)

• 44% reduction of the average absolute calibration error: 46bp for delayed Heston, 81bp for Heston

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Variance & Volatility Swaps Pricing

- Realized variance: $V_R := \frac{1}{T}\int_0^T V_s ds$

- $K_{var} = E^Q[V_R]$, $\quad K_{vol} = E^Q[\sqrt{V_R}]$

- Brockhaus & Long approximation: $E[\sqrt{Z}] \approx \sqrt{E[Z]} - \frac{Var[Z]}{8E[Z]^{3/2}}$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Variance & Volatility Swaps Pricing

- Using time-changed Brownian motion representation for continuous local martingales, we get closed formula for VarSwap and VolSwap fair strikes

- $x_t := -(V_0 - \theta_\tau^2)e^{\gamma - \gamma_\tau t} + e^{\gamma t}(V_t - \theta_\tau^2)$

- $dx_t = f(t, x_t)dW_t^Q, \quad x_t = \widehat{W}_{T_t}, \quad T_t = < x >_t = \int_0^t f^2(s, x_s)ds$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Variance & Volatility Swaps Pricing

- $\theta_\tau^2 := \theta^2 + \frac{\alpha\tau(\mu-r)^2}{\gamma}$, $\quad \gamma_\tau := \alpha + \gamma + \frac{\alpha}{\gamma_\tau \tau}(1 - e^{\gamma_\tau \tau})$

- $V_t = \theta_\tau^2 + (V_0 - \theta_\tau^2)e^{-\gamma_\tau t} + e^{-\gamma t}\widehat{W}_{T_t} = E^Q[V_t] + e^{-\gamma t}\widehat{W}_{T_t}$

The parameter θ_τ^2 can be interpreted as the delayed-adjusted long-range variance. We note, that $\theta_\tau^2 \to \theta^2$ as $\tau \to 0$.

The parameter γ_τ can be interpreted as the delayed-adjusted mean-reverting speed. We note, that $\gamma_\tau \to \gamma$ as $\tau \to 0$.

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Volatility Swap Hedging

- Price Processes:
- VarSwap: $X_t(T) := E_t^Q[V_R]$,
- VolSwap: $Y_t(T) := E_t^Q[\sqrt{V_R}]$,
- $V_R := \frac{1}{T}\int_0^T V_s ds$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

- Portfolio containing 1 VolSwap and β_t VarSwaps:

$$\Pi_t = e^{-r(T-t)}[Y_t(T) - K_{vol} + \beta_t(X_t(T) - K_{var})]$$

- If $I_t := \int_0^t V_s ds$ is the accumulated varinace at time t, then:

$$\begin{aligned} X_t(T) &= E_t^Q[\frac{I_t}{T} + \frac{1}{T}\int_t^T V_s ds] := g(t, I_t, V_t) \\ Y_t(T) &= E_t^Q[\sqrt{\frac{I_t}{T} + \frac{1}{T}\int_t^T V_s ds}] := h(t, I_t, V_t) \end{aligned}$$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Formulas)

Volatility Swap Hedging

• We compute the infinitesimal variations (using the fact that $X_t(T)$ and $Y_t(T)$ are martingales):

$$\begin{aligned} dX_t(T) &= \frac{\partial g}{\partial V_t}\delta\sqrt{V_t}dW_t^Q \\ dY_t(T) &= \frac{\partial h}{\partial V_t}\delta\sqrt{V_t}dW_t^Q \\ d\Pi_t &= r\Pi_t dt + e^{-r(T-t)}[\frac{\partial h}{\partial V_t} + \beta_t\frac{\partial g}{\partial V_t}]\delta\sqrt{V_t}dW_t^Q \end{aligned}$$

$\Rightarrow$

$$\beta_t = -\frac{\frac{\partial h}{\partial V_t}}{\frac{\partial g}{\partial V_t}} = -\frac{\frac{\partial Y_t(T)}{\partial V_t}}{\frac{\partial X_t(T)}{\partial V_t}}$$

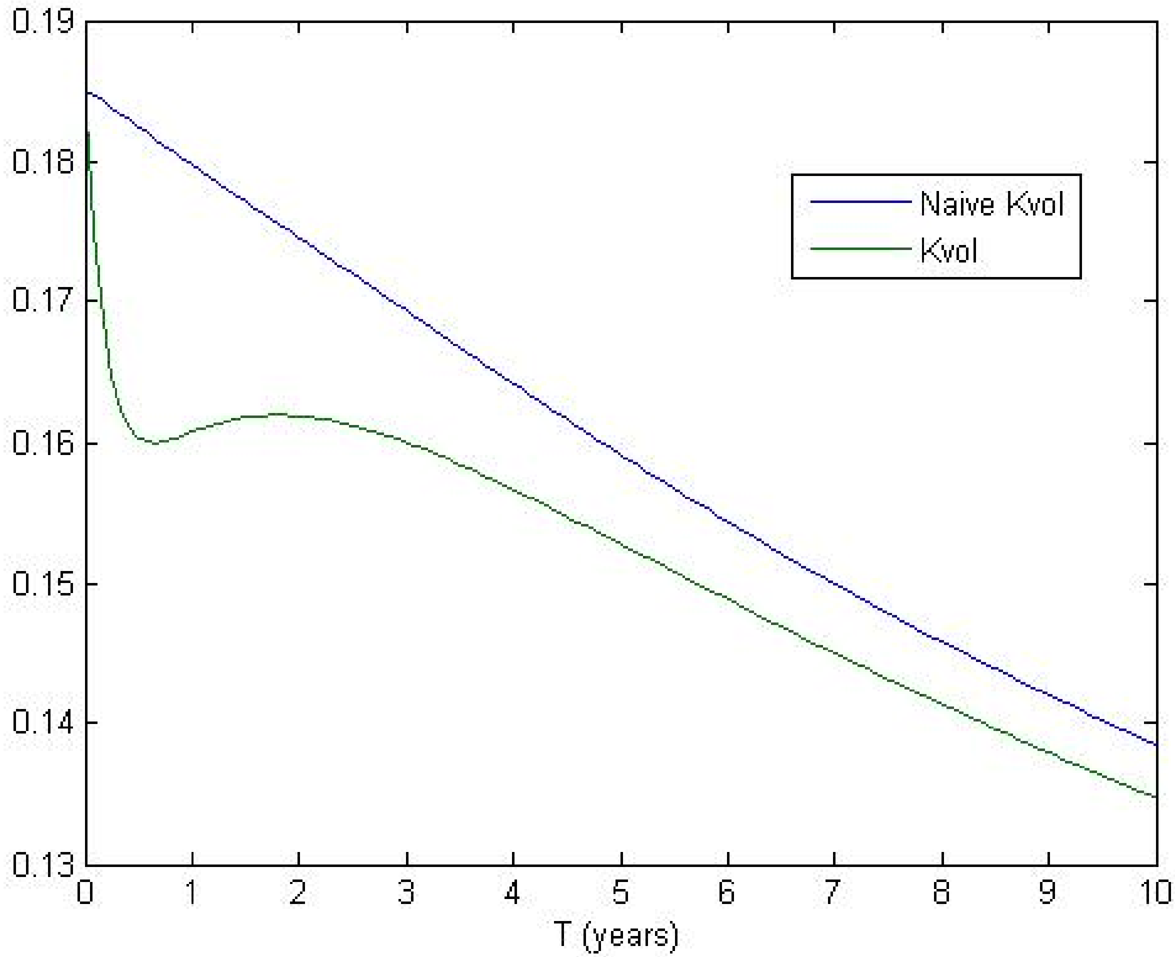
-hedge ratio

Delayed Heston Model: Pricing and Hedging of Volatility Swaps (Numerical Results)

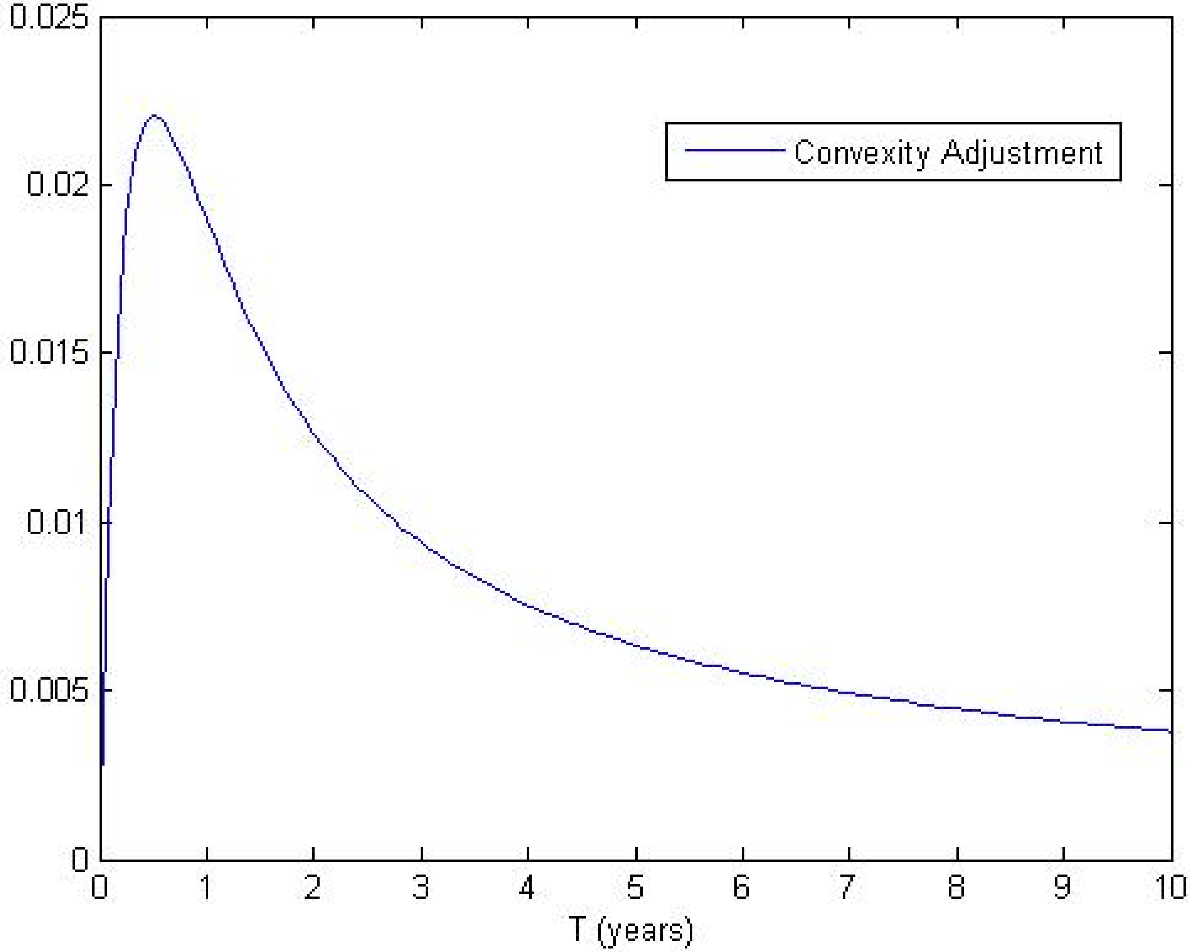
We take the parameters that have been calibrated above (vanilla options on September 30^{th} 2011 for underlying EURUSD, maturities from 1M to 10Y, strikes ATM, 25D Put/Call, 10D Put/Call), namely

$$(v_0, \gamma, \theta^2, \delta, c, \alpha, \tau) = (0.0343, 3.9037, 10^{-8}, 0.808, -0.5057, 71.35, 0.7821).$$

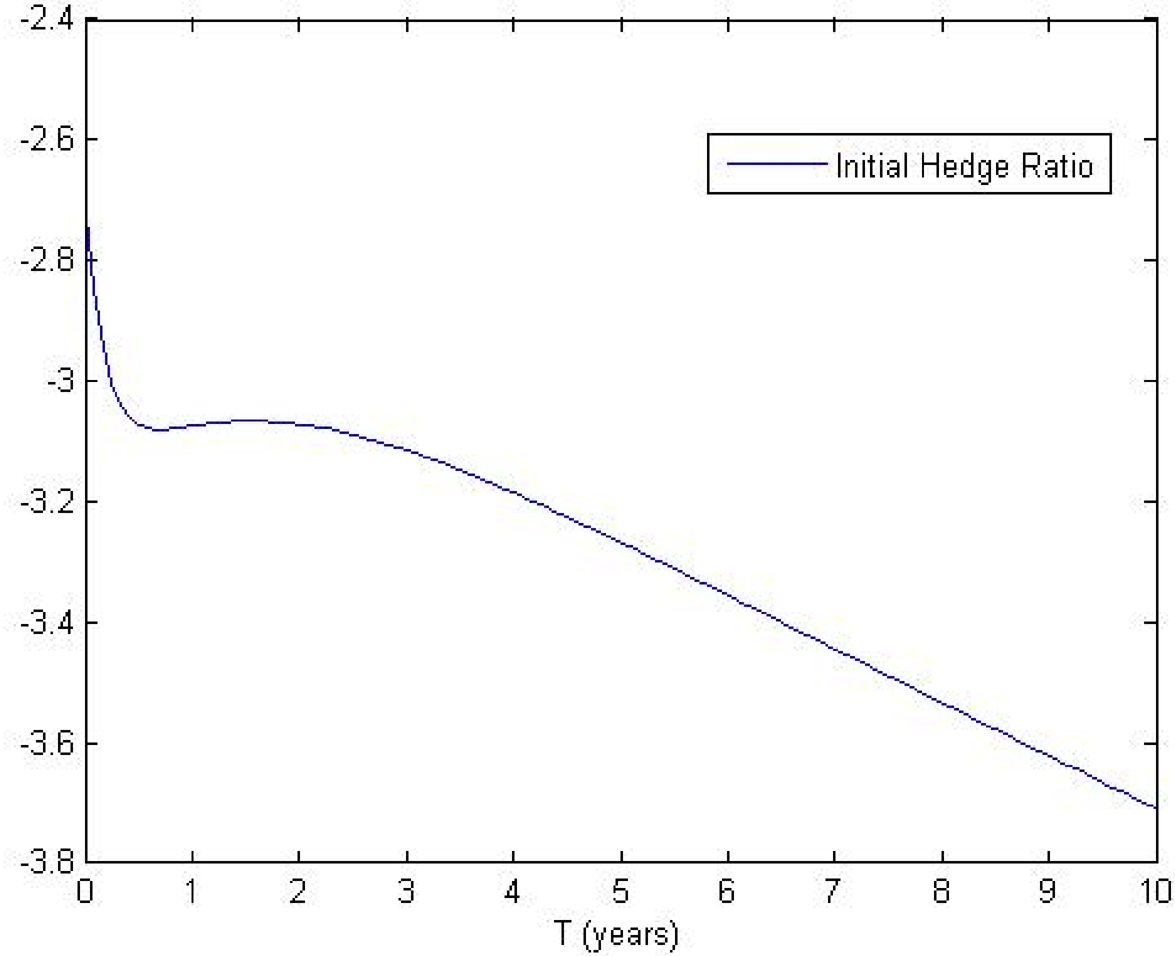
We plot the naive Volatility Swap strike $\sqrt{K_{var}}$ and the adjusted Volatility Swap strike $\sqrt{K_{var}} - \frac{Var^Q(V_R)}{8K_{var}^{\frac{3}{2}}}$ along the maturity dimension, as well as the convexity adjustment $\frac{Var^Q(V_R)}{8K_{var}^{\frac{3}{2}}}$:



Naive VolSwap vs. Adjusted VolSwap Strikes



Convexity Adjustment



Initial Hedge ratio $\beta_0(T)$

Delayed Heston Model: Pricing and Hedging of Volatility Swaps

These results had been obtained together with my PhD student Nelson Vadori and have been submitted to Wilmott J. as two papers:

1. 'Delayed Heston Model: Improvement of the Volatility Surface Fitting'

2. 'Pricing and Hedging of Volatility Swap in the Delayed Heston Model: Part 2'

Discussion: Some Problems

- Explicit Expression for $\widehat{W}(T_t)$ in classical Heston model?

CIR process: $d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dW(t)$.

Solution: $\sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \widehat{W}(T_t)) + \theta^2$,

It would be easier to calculate many volatility derivatives (Cov, Corr Swaps, etc.).

We know the expressions, though, for GBM and continuous-time GARCH models.

Discussion: Expression for $\widehat{W}(T(t))$ in GBM

GBM: $dS = \mu S dt + \sigma S dW.$

Solution using CT: $S(t) = e^{\mu t}[S_0 + \widehat{W}(T(t))]$, where

$$\widehat{W}(T(t)) = S_0(e^{\sigma W(t) - \sigma^2 t/2} - 1) -$$

martingale! Then, the solution looks like (familiar expression!):

$$S(t) = S_0 e^{\mu t} e^{\sigma W(t) - \sigma^2 t/2}$$

Discussion: Expression for $\widehat{W}(T(t))$ in GARCH

Continuous-time GARCH: $dS = a(L - S)dt + \sigma S dW(t)$.

Solution using CT: $S(t) = e^{-at}[S_0 - L + \widehat{W}(T(t))] + L$,

where

$$\begin{aligned}\widehat{W}(T(t)) &= S_0[e^{\sigma W(t) - \sigma^2 t/2} - 1] \\ &+ L[(1 - e^{at}) + a e^{\sigma W(t) - \sigma^2 t/2} \textstyle\int_0^t e^{as} e^{-\sigma W(s) + \sigma^2 s/2} ds)] -\end{aligned}$$

martingale (sum of two martingales)! Then, the solution looks like:

$$\begin{aligned}S(t) &= S_0 e^{-at} e^{\sigma W(t) - \sigma^2 t/2} \\ &+ aLe^{-at} e^{\sigma W(t) - \sigma^2 t/2} \textstyle\int_0^t e^{as} e^{-\sigma W(s) + \sigma^2 s/2} ds.\end{aligned}$$

Discussion: Some Problems

- Covariance and Correlation Swaps for Delayed Heston Model?

Discussion: Some Problems

- Comparison of VolSwap and other volatility derivatives for for Delayed Heston Model and Lévy-based stochastic volatility model?

- Delayed Heston Model has 7 parameters

- Lévy-based stochastic volatility model has only 5 parameters

Conclusion

- Definition of Change of Time (CT) and Motivations
- Literature Review on CT Methods
- Change of Time Methods for Different Settings
- Applications in Mathematical and Energy Finance
- Delayed Heston Model: Var and Vol Swaps, Hedging
- Discussion: Some Problems

The End

Thank You for Your Time and Attention!

E-mail: aswish@ucalgary.ca

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