Paper Review

*Hawkes Process: Fast Calibration, Application to Trade Clustering, and Diffusive Limit*

by Jose da Fonseca and Riadh Zaatour

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Introduction

1. The Analytical Framework
   • Describe the dynamics and affine structure of the moment-generating function
   • Computation of the moments and the autocorrelation function of the number of jumps over a given time interval
   • Moment estimation strategy

2. Applications
   • Present some data, various estimation results, and an impulse response analysis allowed by the model
   • A toy model for a stock for which we derive the limit properties
The Analytical Framework

Dynamics and Affine Structure of the Moment-Generating Function
• The point process is determined by the intensity process \((\lambda_t)_{t \geq 0}\) through the relations:

\[
P[N_{t+h} - N_t = 1|\mathcal{F}_t] = \lambda_t h + o(h),
\]
\[
P[N_{t+h} - N_t > 1|\mathcal{F}_t] = o(h),
\]
\[
P[N_{t+h} - N_t = 0|\mathcal{F}_t] = 1 - \lambda_t h + o(h)
\]

• The intensity follows the dynamic:

\[
d\lambda_t = \beta(\lambda_\infty - \lambda_t) \, dt + \alpha \, dN_t
\]

• Applying Ito’s lemma to \(e^{\beta t} \lambda_t\) yields:

\[
\lambda_t = e^{-\beta t}(\lambda_0 - \lambda_\infty) + \lambda_\infty + \int_0^t \alpha e^{-\beta(t-s)} \, dN_s
\]

• Observe that the impact on the intensity of a jump dies out exponentially as time passes
• Hawkes intensity is written as:
  \[ \lambda_t = \lambda_\infty + \int_{-\infty}^{t} \alpha e^{-\beta(t-s)} \, dN_s \]

• The presentation of intensity slightly differs in this paper due to the desire to perform stochastic differential calculus.

• The process \( X_t = (\lambda_t, N_t) \) is a Markov process in the state space \( D = \mathbb{R}_+ \times \mathbb{N} \). This property allows us to use the infinitesimal generator to investigate the distributional properties of the process.

• The infinitesimal generator of the process \( \mathcal{L} \), is the operator acting on a sufficiently regular function \( f : D \to \mathbb{R} \), such that:
  \[ \mathcal{L}f(x) = \lim_{h \to 0} \frac{\mathbb{E}^x_t[f(X_{t+h})] - f(x)}{h} \]

  with \( \mathbb{E}^x_t[\cdot] = \mathbb{E}^x[\cdot | \mathcal{F}_t] \) and \( X_t = x \) \( (\mathbb{E}_0[\cdot] = \mathbb{E}[\cdot]) \)
• The infinitesimal generator of the Hawkes process is:

\[ \mathcal{L}f(x) = \beta(\lambda_\infty - \lambda_t) \frac{\partial f}{\partial \lambda}(x) + \lambda_t [f(\lambda_t + \alpha, N_t + 1) - f(x)] \]

• For every function \( f \) in the domain of the infinitesimal generator, the process:

\[ M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_{tu}) \, du \]

is a martingale relative to its natural filtration. Thus, for \( s > t \):

\[ \mathbb{E}_t \left[ f(X_s) - \int_0^s \mathcal{L}f(X_{tu}) \, du \right] = f(X_t) - \int_0^t \mathcal{L}(X_{tu}) \, du \]

• From this and the martingale property, the Dynkin formula is obtained:

\[ \mathbb{E}_t [f(X_s)] = f(X_t) + \mathbb{E}_t \left[ \int_t^s \mathcal{L}f(X_{tu}) \, du \right] \]
• Dynkin formula allows for the computation of conditional expectation of functions of the Markov process \( X_t = (\lambda_t, N_t) \).

• \( X_t = (\lambda_t, N_t) \) is a Markov process that is affine, which implies that a closed form solution for the moment-generating function is available.

• Define the conditional moment-generating function of \( X_t = (\lambda_t, N_t) \) as:
  \[
  f(t, X_t) = \mathbb{E}_t^X[e^{u^T X_T}] = \mathbb{E}_t^X[e^{u_1 \lambda_T + u_2 N_T}]
  \]
  for \( u = (u_1, u_2)^T \in \mathbb{R}^2 \)

• \( f(t, X_t) \) is a martingale that satisfies \( \frac{\partial f}{\partial t}(t, X_t) + \mathcal{L}f(t, X_t) = 0 \) and boundary condition \( f(T, X_t) = e^{u^T X_T} \).

• We guess the solution of \( f(t, X_t) \) is an exponential affine form of the state variable because \( X_t = (\lambda_t, N_t) \) is a Markov affine point process:
  \[
  f(t, X_t) = e^{a(t) + b(t) \lambda_t + c(t) N_t}
  \]
• Two important observations can be obtained from this guess solution.

1. Setting this guess into \( \frac{\partial f}{\partial t}(t, X_t) + \mathcal{L}f(t, X_t) = 0 \) to obtain a system of ODE with terminal condition \( a(T) = 0, \quad b(T) = u_1, \quad \text{and} \quad c(T) = u_2 \)

\[
\begin{align*}
\frac{\partial a}{\partial t} &= -\beta \lambda_\infty b(t), \\
\frac{\partial b}{\partial t} &= \beta b(t) + 1 - e^{ab(t)+c(t)}, \\
\frac{\partial c}{\partial t} &= 0
\end{align*}
\]

2. The computation of the autocovariance function of the number of jumps increments, \( \mathbb{E}_t^x \left[ (N_{t_4} - N_{t_3})(N_{t_2} - N_{t_1}) \right] \) with \( t < t_1 < t_2 < t_3 < t_4 \), can be obtained from this guess solution by performing successive conditioning.
The Analytical Framework

Computing the Moments and the Autocovariance Function
Lemma 1. Given a Hawkes process $X_t = (\lambda_t, N_t)$ with dynamic given by $d\lambda_t = \beta(\lambda_\infty - \lambda_t)dt + \alpha dN_t$, the expected number of jumps $\mathbb{E}[N_t]$ and the expected intensity $\mathbb{E}[\lambda_t]$ satisfy the set of ODE:

$$
\begin{align*}
    d\mathbb{E}[N_t] &= \mathbb{E}[\lambda_t] \, dt, \\
    d\mathbb{E}[\lambda_t] &= (\beta \lambda_\infty + (\alpha - \beta)\mathbb{E}[\lambda_t]) \, dt
\end{align*}
$$

Lemma 2. Given a Hawkes process $X_t = (\lambda_t, N_t)$ with dynamic given by $d\lambda_t = \beta(\lambda_\infty - \lambda_t)dt + \alpha dN_t$, $\mathbb{E}[\lambda_t^2]$, $\mathbb{E}[\lambda_t N_t]$, $\mathbb{E}[N_t^2]$ satisfies the set of ODE:

$$
\begin{align*}
    d\mathbb{E}[N_t^2] &= 2\mathbb{E}[\lambda_t N_t] \, dt + \mathbb{E}[\lambda_t] \, dt, \\
    d\mathbb{E}[\lambda_t N_t] &= \beta \lambda_\infty \mathbb{E}[N_t] \, dt + (\alpha - \beta)\mathbb{E}[\lambda_t N_t] \, dt + \mathbb{E}[\lambda_t^2] \, dt + \alpha \mathbb{E}[\lambda_t] \, dt, \\
    d\mathbb{E}[\lambda_t^2] &= (\alpha^2 + 2\beta \lambda_\infty)\mathbb{E}[\lambda_t] \, dt + 2(\alpha - \beta)\mathbb{E}[\lambda_t^2] \, dt.
\end{align*}
$$
Proposition 1. Given a Hawkes process \( X_t = (\lambda_t, N_t) \) with dynamic given by \( d\lambda_t = \beta (\lambda_\infty - \lambda_t) dt + \alpha dN_t \), we have the following equalities

\[
\lim_{t \to \infty} \mathbb{E}[N_{t+\tau} - N_t] = \frac{\beta \lambda_\infty}{\beta - \alpha} \tau = \Lambda \tau
\]

with \( \Lambda = \frac{\lambda_\infty}{1 - \alpha/\beta} \) (the stationary regime expected intensity) gives the long-run expected value of the number of jumps during a time interval of length \( \tau \).
Proposition 2. Given a Hawkes process $X_t = (\lambda_t, N_t)$ with dynamic given by $d\lambda_t = \beta(\lambda_\infty - \lambda_t)dt + \alpha dN_t$, the autocorrelation function of the number of jumps over a given interval $\tau$ is:

$$Acf(\tau, \delta) = \lim_{t \to \infty} \frac{\mathbb{E}[(N_{t+\tau} - N_t)(N_{t+2\tau+\delta} - N_{t+\tau+\delta})] - \mathbb{E}[(N_{t+\tau} - N_t)]\mathbb{E}[(N_{t+2\tau+\delta} - N_{t+\tau+\delta})]}{\sqrt{\text{var}(N_{t+\tau} - N_t) \text{var}(N_{t+2\tau+\delta} - N_{t+\tau+\delta})}}$$

$$= \frac{e^{-2\beta\tau}(e^{\alpha\tau} - e^{\beta\tau})^2}{2(\alpha(\alpha - 2\beta))(e^{(\alpha - \beta)\tau} - 1) + \beta^2 \tau(\alpha - \beta))} \cdot e^{(\alpha - \beta)\delta}.$$
The Analytical Framework

Inference Strategies
Maximum likelihood estimation

• This estimation leads to a nonlinear optimization algorithm such as Nelder-Mead to find the maximum.
• For each set of parameters the evaluation of this estimation process requires a loop over the observations.
• For trade clustering, this looping process is very time consuming.
• Even with recent advancement, the calibration still takes a few minutes and a large number of function calls are performed.
Fast Hawkes process calibration

• With explicitly computed moments and the autocorrelation function for the Hawkes process, a natural estimation strategy is the generalized method of moments:

\[ \hat{\theta} = \text{argmin}\{(M - f(\theta))^T W (M - f(\theta))\} \]

where \( M \) is the vector of empirical moment, \( f(\theta) \) is the vector of corresponding theoretical moment, and \( W \) is a symmetric positive definite weighting matrix.

• The optimization problem can be solved very quickly by Levenberg-Marquardt algorithm.

• The optimization based on the mean and variance of number jumps during an interval \( \tau \), and autocorrelation function gives good results if calibration quality and speed are taken into account.
Fast Hawkes process calibration

- From a numerical point of view, it's simpler and more robust to work with normalized quantities, and the optimization problem becomes:

\[
\hat{\theta} = \arg\min \left\{ \left(1 - \frac{f(\theta)}{M}\right)^T W \left(1 - \frac{f(\theta)}{M}\right) \right\}
\]

where the components of the vector \(1 - \frac{f(\theta)}{M}\) are \(1 - \frac{f_i(\theta)}{M_i}\).

- The evaluation of the empirical moments is only made once during the optimization procedure.

- Very appealing procedure due to its run speed and robustness against data pollution.
Applications

Data
• Tick-by-tick data of trades and quotes timestamped in milliseconds
• Two stocks: BNP Paribas and Sanofi; and the futures on the Eurostoxx and the Dax
• Deals with trade time arrivals and statistics on the number of trades occurring on intervals of fixed length
• Many trades will have the same time to the nearest millisecond even if they did not take place at the same time
• This millisecond will count as a unique entry in the ML estimation procedure
• In the moment-based inference all the trades will be taken into account when computing the moments
Applications

Trade Clustering
• Explanations for clustering of trade arrival times
  • Liquidity takers splitting their orders so as to minimize their market impact
  • Insider traders reacting rapidly to take advantage from information they have before it is widespread in the market
  • Heterogeneity of market participants is responsible for the two-sided trade clustering
• To quantify this clustering, compute the correlation of the number of trades occurring during different time intervals of fixed length separated by a time lag:

\[
C(\tau, \delta) = \frac{\mathbb{E}[(N_{t+\tau} - N_{t})(N_{t+2\tau+\delta} - N_{t+\tau+\delta})] - \mathbb{E}[(N_{t+\tau} - N_{t})] \mathbb{E}[(N_{t+2\tau+\delta} - N_{t+\tau+\delta})]}{\sqrt{\text{var}(N_{t+\tau} - N_{t})} \text{var}(N_{t+2\tau+\delta} - N_{t+\tau+\delta})}
\]

• A plot of this autocorrelation as a function of the lag gives information about the degree of clustering.
• The absolute value of the correlation is higher for the two futures, which are far more liquid than the stocks.

• Nevertheless, the same decreasing shape is observed and the time life of this autocorrelation seems to be very close for all the symbols.

• These results justify the use of Hawkes process as modeling framework.
• To further reduce computational cost, we say the objective function only depends on the empirical and analytical autocorrelation function.

• We can then rely on Proposition 1 to obtain $\lambda_\infty$ from other parameters.

• We choose to fit the analytical autocorrelation function for $\tau = 60s$ and $\delta$ ranging from 0 to 600 seconds by step of 60 seconds.

• Also perform a daily calibration for each symbol, and report the mean and median estimated values, and standard deviations.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Measure</th>
<th>Poisson-λ</th>
<th>λ_∞</th>
<th>α</th>
<th>β</th>
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</table>
Applications

Branching Structure of Trading Activity
• The occurrence of a jump increases the intensity of the process, thereby the probability to observe another jump. There is a direct and indirect impulse response of the process intensity to a jump event.

• Denoting the expected increase of the process intensity at time $t$ as a response to a jump occurring at time 0 by $f(t)$, the following decomposition holds:
  
  • Direct response: an increase of the intensity by $\alpha$ that will decay exponentially as time passes
  
  • Indirect response: at any time $s$ between 0 and $t$, the direct increase of the intensity by $\alpha e^{-\beta s}$ leads to an indirect increase of the expected number of jumps at time $t$

\[
f(t) = \alpha e^{-\beta t} + \int_0^t \alpha e^{-\beta s} f(t - s) \, ds
\]

\[
f(t) = \alpha e^{-(\beta - \alpha) t}
\]
• The $N_{\text{response}}$, which is the expected number of jumps triggered by one jump occurring at time 0 if the process is observed indefinitely:

$$N_{\text{response}} = \int_0^\infty f(s) \, ds = \frac{\alpha}{\beta - \alpha}$$

• Can consider $N_{\text{response}}$ as a measure of liquidity and trading activity.

• Futures are more actively traded than the stocks due to a stronger branching structure.

• The formula also suggests the ratio $\frac{\alpha}{\beta}$ as the key quantity to evaluate the impulse response value.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Average $N_{\text{response}}$</th>
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<td>BNPP</td>
<td>14</td>
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<tr>
<td>Sanofi</td>
<td>10</td>
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</table>
Applications

Diffusive Limit and Signature Plot
Attempts to connect the microscopic price formation process observed at transaction level to its macroscopic properties at a coarser time scale.

Bacry et al. (2013a) introduces a model for microstructure price evolution based on mutually exciting Hawkes processes.

They connect the signature plot of volatility and Epps effect of asset correlations to the model parameters driving the price process.

This section uses the framework proposed by Bacry et al. (2013a) and the Hawkes process to develop a toy model for the movement of the mid price of a traded assets:

\[ S_t = S_0 + (N_{t}^{up} - N_{t}^{down}) \frac{\delta}{2} \]

where \( \delta \) is the tick value. The \( N_{t}^{up} \) and \( N_{t}^{down} \) are Hawkes processes capturing the up and down jumps of the mid price.
• Consider \( N_{t}^{up} \) and \( N_{t}^{down} \) independently but with the same parameters to avoid price explosion. In the stationary regime, their intensities are given by:

\[
\lambda_{t}^{up} = \lambda_{\infty} + \int_{0}^{t} \alpha e^{-\beta(t-s)} \, dN_{s}^{up},
\]

\[
\lambda_{t}^{down} = \lambda_{\infty} + \int_{0}^{t} \alpha e^{-\beta(t-s)} \, dN_{s}^{down}.
\]

• To relate this high-frequency description with low-frequency description, we need a limit theorem.

• Bacry et al. (2013b) relies on the martingale theory and limit theorems for semi-martingales to prove stability and convergence results for a general model with mutually exciting processes and a general kernel.

• The function \( g(t) = \alpha e^{-\beta t} \) is called the kernel of the Hawkes process.
• The process $X_t = (S_t, N_t^{up}, \lambda_t^{up}, N_t^{down}, \lambda_t^{down})$ is a Markov process. Its infinitesimal generator writes:

$$\mathcal{L}f(x) = \beta(\lambda_\infty - \lambda_t^{up}) \frac{\partial f}{\partial \lambda_t^{up}}(x) + \beta(\lambda_\infty - \lambda_t^{down}) \frac{\partial f}{\partial \lambda_t^{down}}(x)$$

$$+ \lambda_t^{up} \left[ f \left( S_t + \frac{\delta}{2}, N_t^{up} + 1, \lambda_t^{up} + \alpha, N_t^{down}, \lambda_t^{down} \right) - f(x) \right]$$

$$+ \lambda_t^{down} \left[ f \left( S_t - \frac{\delta}{2}, N_t^{up}, \lambda_t^{up}, N_t^{down} + 1, \lambda_t^{down} + \alpha \right) - f(x) \right]$$

• The explicit form of the infinitesimal generator allows us to apply Foster-Lyapounov techniques in order to establish stability results.
Define the function $V(x) = \frac{\lambda_{up} + \lambda_{down}}{2\lambda_\infty}$, then a simple calculation yields the geometric drift condition:

$$\mathcal{L}V(x) \leq (\alpha - \beta)V(x) + \beta$$

Write unit-time price increments:

$$\eta_i = \left[ (N_{i_{up}}^{up} - N_{i_{-1}}^{up}) - (N_{i_{down}}^{down} - N_{i_{-1}}^{down})\right] \times \frac{\delta}{2}$$

and consider the random sums $S_n = \sum_{i=1}^{n} \eta_i$, with $\{\eta_i; i = 1, \ldots, n\}$ being the price increments.

Focus on the asymptotic behavior of the rescaled price process:

$$\tilde{S}_t^n = \frac{S_{[nt]}}{\sqrt{n}}$$

The increments are geometrically mixing

$\tilde{S}_t^n$ converges to a Brownian motion in the sense of Skorokhod topology:

$$\tilde{S}_t^n \to \sigma W_t$$
• Calculations done before for the moments of the Hawkes process increments lead to a very simple expression for the volatility,

\[ \sigma^2 = \frac{\delta^2}{2} \frac{\lambda_\infty}{\left(1 - \frac{\alpha}{\beta}\right)^3} \]

• The larger the ratio \( \alpha / \beta \), the larger is the volatility. An upward (downward) chock is likely to trigger another upward (downward) chock if this ratio is large, and therefore it induces a positive autocorrelation for the mid price and a more persistent path with the effect of increasing asset’s volatility.
The Hawkes process can reproduce some stylized facts across time scales, such as the volatility signature plot, which depends on the realized variance over a period $T$ calculated by sampling the data by time intervals of length $\tau$. Within the toy model we have:

$$\hat{C}(\tau) = \frac{1}{T} \sum_{n=0}^{T/\tau-1} (S_{(n+1)\tau} - S_{n\tau})^2$$

$$= \frac{1}{T} \sum_{n=0}^{T/\tau-1} (\left(N_{(n+1)\tau}^{\text{up}} - N_{n\tau}^{\text{up}}\right) - \left(N_{(n+1)\tau}^{\text{down}} - N_{n\tau}^{\text{down}}\right))^2 \frac{\delta^2}{4}$$

$$= \frac{1}{T} \sum_{n=0}^{T/\tau-1} (N_{(n+1)\tau}^{\text{up}} - N_{n\tau}^{\text{up}})^2 \frac{\delta^2}{4} + \frac{1}{T} \sum_{n=0}^{T/\tau-1} (N_{(n+1)\tau}^{\text{down}} - N_{n\tau}^{\text{down}})^2 \frac{\delta^2}{4}$$

$$- 2 \frac{1}{T} \sum_{n=0}^{T/\tau-1} (N_{(n+1)\tau}^{\text{up}} - N_{n\tau}^{\text{up}})(N_{(n+1)\tau}^{\text{down}} - N_{n\tau}^{\text{down}}) \frac{\delta^2}{4}.$$
The mean signature plot is the expectation of the above quantity and can be computed explicitly,

**Proposition 3.** The mean signature plot, defined by $C(\tau) = \mathbb{E}[\hat{C}(\tau)]$, is given by:

$$C(\tau) = \mathbb{E}[\hat{C}(\tau)]$$

$$= \frac{\delta^2}{2\tau} V(\tau)$$

$$= \frac{\delta^2}{2} \Lambda \left( \kappa_-^2 + (1 - \kappa_-^2) \frac{(1 - e^{-\tau\gamma_-})}{\tau\gamma_-} \right),$$

where

$$\Lambda = \frac{\lambda_\infty}{1 - \alpha/\beta}, \quad \kappa_- = \frac{1}{1 - \alpha/\beta}, \quad \text{and} \quad \gamma_- = \beta - \alpha.$$
• The mean signature plot is an increasing function with respect to $\tau$ and this is due to the positive serial autocorrelation of the returns.

• Within this simple toy model we can determine the autocorrelation function of the price increments computed over intervals of size $\tau$ and lagged by $\delta$:

\[
CorrStock(\tau, \delta) = \frac{\mathbb{E}[(S_{t+\tau} - S_t)(S_{t+2\tau+\delta} - S_{t+\tau+\delta})]}{\sqrt{\mathbb{E}[(S_{t+\tau} - S_t)]\mathbb{E}[(S_{t+2\tau+\delta} - S_{t+\tau+\delta})]}} = \frac{e^{-2\beta\tau+\delta(\alpha-\beta)}(e^{\tau\beta} - e^{\tau\alpha})^2\alpha(2\beta - \alpha)}{2\beta^2(\beta - \alpha)}
\]
The paper provides a comparison between the toy model and the Bacry et al. (2013a)’s model, which is based on Hawkes processes that are mutually excited inside of self-excited.

In Bacry et al. (2013a)’s model, an upward chock will increase the down intensity and trigger a downward chock on the mid price, thereby leading to a mean reverting behavior for the mid price.

As a function of the sampling period, the signature plot is decreasing wrt $\tau$ because of this negative serial autocorrelation of the returns.

Bacry et al. (2013a)’s model is compatible with a decreasing pattern, whereas the toy model is compatible with an increasing pattern.

Due to the positive (negative) autocorrelation of the returns in the toy (Bacry et al.) model we have, for a given pair $(\alpha, \beta)$, the inequality $\sigma > \sigma_{BDHM}$. 
• Calibrate a Hawkes process to the mid price up-jumps and calculate the asymptotic volatilities for the two models.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>$\lambda_{\infty}$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma_{BDHM}$ (%)</th>
<th>Empirical $\sigma$ (%)</th>
<th>Toy model $\sigma$ (%)</th>
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<td>Eurostoxx</td>
<td>0.0184</td>
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<td>0.0228</td>
<td>0.0187</td>
<td>0.0288</td>
<td>2.45</td>
<td>6.92</td>
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<td>Schatz</td>
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<td>0.0223</td>
<td>0.0372</td>
<td>1.38</td>
<td>2.63</td>
<td>5.39</td>
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<td>JPY</td>
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<td>0.0764</td>
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<td>9.89</td>
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<td>0.0868</td>
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<tr>
<td>Crude Oil Brent</td>
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<td>0.0472</td>
<td>0.0528</td>
<td>6.22</td>
<td>41.25</td>
<td>126.29</td>
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<td>Sugar</td>
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<td>10.58</td>
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<td>73.30</td>
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<tr>
<td>CORN</td>
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<td>0.0694</td>
<td>9.34</td>
<td>43.73</td>
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<td>0.0763</td>
<td>10.59</td>
<td>57.17</td>
<td>94.53</td>
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</table>
Conclusion

• Explicitly compute the moments and the autocorrelation function of the number of jumps over an interval for the Hawkes process.

• Develop a method of moments estimation strategy that is extremely fast compared with the usual maximum likelihood estimation strategy.

• Use this estimation framework to calibrate the Hawkes process on trades for four stocks over a 2-year sample.

• Roll the daily estimation over 2 years to analyze the parameters stability.

• Explicitly compute the impulse response associated with the process, which determines the market impact of a trade.

• Compute the diffusive limit for the price process.
Conclusion

• Coping self- and mutually excited Hawkes process. Need to perform the computations in the multidimensional case.

• This paper connect the dynamic driving the trade process, using a Hawkes process, to the daily volatility. Apply this concept further at the microscopic level would also be interesting.

• The Hawkes process provides a natural modeling framework and would extend the interesting existing models based on the Poisson process. To compute the diffusive limit for a model based on the Hawkes process the moments as well as the autocorrelation are needed and they can be obtained using the computation strategy developed in this work.
Reference

