The Feynman-Kac formula establishes a link between linear partial differential equations and stochastic processes. Consider the heat equation,

\[ \partial_t h(t, x) + \frac{1}{2}\partial_{xx} h(t, x) = 0 \]

\[ h(T, x) = H(x) \]

Where \( H \in L^1 \). Although this PDE is deterministic, we find that \( x_t \) is a stochastic process called a Wiener Process,

\[ x_t - x_s \sim N(0, t - s) \quad s < t \]

The intuition of the connection lies in the example of heat diffusion: If \( h \) represents the heat of state \( x \) at time \( t \), then \( x_t \) represents the diffusive path of a heat "particle" with initial position \( x_0 \).
Formally, let’s define the Wiener process on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})\), where \(\mathcal{F}\) is the natural filtration generated by \(x\),

\[ \mathcal{F}_t = \sigma\{x_s^{-1}(A)|s \leq t, A \in \mathcal{F}\} \]

Consider the stochastic process,

\[ f_t = \mathbb{E}[H(x_t)|\mathcal{F}_t] \]

We can see that \(f_t\) is a martingale because, for \(s \leq t \leq T\),

\[ \mathbb{E}[f_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[H(x_t)|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[H(x_t)|\mathcal{F}_s] = f_s \]
We can also see that $f_t \in L^1$.

Consider Jensen’s Inequality: If $z$ is a random variable, and $g(z)$ is a convex function, then $\mathbb{E}[g(z)] \geq g(\mathbb{E}[z])$. It follows that,

$$\mathbb{E}[|f_t|] = \mathbb{E}[|\mathbb{E}[H(x_t)|\mathcal{F}_t]|] \leq \mathbb{E}[\mathbb{E}[|H(x_t)||\mathcal{F}_t]] < \infty$$

Let $Z$ be an independent standard normal random variable. Then,

$$f_t = \mathbb{E}[H(x_t)|\mathcal{F}_t] = \mathbb{E}[H((x_T - x_t) + x_t)|\mathcal{F}_t] = \mathbb{E}[H(\sqrt{T - t}Z + x_t)|\mathcal{F}_t] = g(t, x_t)$$

Where $g(t, x_t)$ is some deterministic function. We can see that $\mathbb{E}[H(x_t)|\mathcal{F}_t] = \mathbb{E}[H(x_t)|x_t]$ and so $f_t$ is Markovian.
Suppose $g(t, x)$ is twice differentiable. Then, applying Itô’s Lemma,

$$g(t+h, x_{t+h}) - g(x, x_t) = \int_t^{t+h} \left\{ \partial_t g(s, (x_s) + \frac{1}{2} \partial_{xx} g(s, x_s) \right\} ds + \int_t^{t+h} \partial_x g(s, x_s) dx_s$$

Recall that $g$ is a martingale. Then,

$$\mathbb{E}[g(t+h, x_{t+h}) - g(x, x_t)|x_t] = 0$$

$$\mathbb{E} \left[ \int_t^{t+h} \left\{ \partial_t g(s, (x_s) + \frac{1}{2} \partial_{xx} g(s, x_s) \right\} ds + \int_t^{t+h} \partial_x g(s, x_s) dx_s \bigg| x_t \right] = 0$$

$$\mathbb{E} \left[ \int_t^{t+h} \left\{ \partial_t g(s, (x_s) + \frac{1}{2} \partial_{xx} g(s, x_s) \right\} ds \bigg| x_t \right] = 0$$
Now, let’s divide by $h$ and take the limit $h \to 0$.

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \int_{t}^{t+h} \left\{ \partial_t g(s, (x_s) + \frac{1}{2} \partial_{xx} g(s, x_s) \right\} ds \bigg|_{x_t} \right] = 0$$

The expectation and integral vanish, leaving the identity,

$$\partial_t g(t, x) + \frac{1}{2} \partial_{xx} g(t, x) = 0$$

$$g(T, x) = H(x)$$

Thus, $g(t, x)$ satisfies the heat equation.
This is one example of the more general Feynman-Kac theorem.
Let $X$ denote an Itô Process satisfying the SDE,

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Where $W$ is a Wiener Process. Consider the function,

$$f(t, x) = \mathbb{E} \left[ \int_t^T e^{-\int_t^s \gamma(y, X_u)du} g(s, X_s)ds + e^{-\int_t^T \gamma(u, X_u)du} h(X_T) \bigg| X_t \right]$$

Where $h \in L^1$, $\int_0^T |g(s, X_s)|ds \in L^1$, and $\int_0^T \gamma(t, X_t)dt$ is bounded from below a.s., then $f(t, x)$ satisfies the PDE,

$$\partial_t f(t, x) + \mathcal{L}_t^X f(T, x) + g(t, x)f(t, x) = \gamma(t, x)f(t, x)$$

$$f(T, x) = h(x)$$

Where $\mathcal{L}_t^X$ is the infinitesimal generator of $X$. 
Suppose the price of a security $S_t$ follows the process $dS_t = rS_t dt + \sigma S_t dW_t$. Let $V(t, S_t)$ be the value of a derivative on $S_t$. Suppose $V(t, S)$ has an instantaneous cash flow of $g(t, S_t)$, a terminal cash flow $h(S)$, and a constant discount rate $r$. The value of $V(t, S)$ satisfies

$$V(t, S) = \mathbb{E} \left[ \int_t^T e^{-r(s-t)} g(s, S_s) ds + e^{-r(T-t)} h(S_T) \middle| S_t \right]$$

By the Feynman-Kac formula,

$$\partial_t V(t, S) + rS \partial_s V(t, s) + \frac{1}{2} \sigma^2 S^2 \partial_{ss} V(t, s) + g(t, S) V(t, S) = r V(t, S)$$

$$V(T, S) = h(S)$$

We recover the PDE for a general derivative in the Black-Scholes economy.