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Explicit Option Pricing Formula for a Mean-Reverting Asset in Energy Market

Some commodity prices, like oil and gas, exhibit the mean reversion, unlike stock price. It means that they tend over time to return to some long-term mean. In this paper we consider a risky asset $S_t$ following the mean-reverting stochastic process. The aim of this paper is to obtain an explicit expression for a European option price based on $S_t$, using a change of time method. A numerical example for the AECO Natural Gas Index (1 May 1998-30 April 1999) is presented.

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1. Introduction

Some commodity prices, like oil and gas, exhibit the mean reversion, unlike stock price. It means that they tend over time to return to some long-term mean. In this paper we consider a risky asset $S_t$ following the mean-reverting stochastic process given by the following stochastic differential equation

$$dS_t = a(L - S_t)dt + \sigma S_t dW_t,$$

where $W$ is a standard Wiener process, $\sigma > 0$ is the volatility, the constant $L$ is called the ‘long-term mean’ of the process, to which it reverts over time, and $a > 0$ measures the ‘strength’ of mean reversion.

This mean-reverting model is a one-factor version of the two-factor model made popular in the context of energy modelling by Pilipovic (1997). Black’s model (1976) and Schwartz’s model (1997) have become a standard approach to the problem of pricing options on commodities. These models have the advantage of mathematical convenience,
in that they give rise to closed-form solutions for some types of options (See Wilmott (2000)).

Bos, Ware and Pavlov (2002) presented a method for evaluation of the price of a European option based on $S_t$, using a semi-spectral method. They did not have the convenience of a closed-form solution, however, they showed that values for certain types of options may nevertheless be found extremely efficiently. They used the following partial differential equation (see, for example, Wilmott, Howison and Dewynne (1995))

$$C_t' + R(S,t)C_S' + \sigma^2 S^2 C_{SS}''/2 = rC$$

for option prices $C(S,t)$, where $R(S,t)$ depends only on $S$ and $t$, and corresponds to the drift induced by the risk-neutral measure, and $r$ is the risk-free interest rate. Simplifying this equation to the singular diffusion equation they were able to calculate numerically the solution.

The aim of this paper is to obtain an explicit expression for a European option price, $C(S,t)$, based on $S_t$, using a change of time method (see Swishchuk (2007)). This method was once applied by the author to price variance, volatility, covariance and correlation swaps for the Heston model (see Swishchuk (2004)).

2. Mean-Reverting Asset Model (MRAM)

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space with a sample space $\Omega$, $\sigma$-algebra of Borel sets $\mathcal{F}$ and probability $P$. The filtration $\mathcal{F}_t$, $t \in [0,T]$, is the natural filtration of a standard Brownian motion $W_t$, $t \in [0,T]$, such that $\mathcal{F}_T = \mathcal{F}$.

Some commodity prices, like oil and gas, exhibit the mean reversion, unlike stock price. It means that they tend over time to return to some long-term mean. In this paper we consider a risky asset $S_t$ following the mean-reverting stochastic process given by the following stochastic differential equation

$$dS_t = a(L - S_t)dt + \sigma S_t dW_t, \quad (1)$$

where $W_t$ is an $\mathcal{F}_t$-measurable one-dimensional standard Wiener process, $\sigma > 0$ is the volatility, constant $L$ is called the 'long-term mean' of the process, to which it reverts over time, and $a > 0$ measures the 'strength' of mean reversion.

3. Explicit Option Pricing Formula for European Call Option for MRAM under Physical Measure

In this section, we are going to obtain an explicit expression for a European option price, $C(S,t)$, based on $S_t$, using a change of time method and physical measure.
3.1. Explicit Solution of MRAM. Let

\[ V_t := e^{at}(S_t - L), \]  

(2)

Then, from (2) and (1) we obtain

\[
dV_t = ae^{at}(S_t - L)dt + e^{at}dS_t = \sigma(V_t + e^{at}L)dW_t. \tag{3}
\]

Using change of time approach to the equation (3) (see Ikeda and Watanabe (1981) or Elliott (1982)) we obtain the following solution of the equation (3)

\[
V_t = S_0 - L + \tilde{W}(\phi_t^{-1}),
\]

or (see (2)),

\[
S_t = e^{-at}[S_0 - L + \tilde{W}(\phi_t^{-1})] + L, \tag{4}
\]

where \( \tilde{W}(t) \) is an \( \mathcal{F}_t \)-measurable standard one-dimensional Wiener process, \( \phi_t^{-1} \) is an inverse function to \( \phi_t \):

\[
\phi_t = \sigma^{-2} \int_0^t (S_0 - L + \tilde{W}(s) + e^{a\phi_s}L)^{-2}ds. \tag{5}
\]

We note that

\[
\phi_t^{-1} = \sigma^2 \int_0^t (S_0 - L + \tilde{W}(\phi_t^{-1}) + e^{as}L)^2ds, \tag{6}
\]

which follows from (5) and the following transformations:

\[
d\phi_t = \sigma^{-2}(S_0 - L + \tilde{W}(t) + e^{a\phi_t}L)^{-2}dt \Rightarrow \sigma^2(S_0 - L + \tilde{W}(t) + e^{a\phi_t}L)^2d\phi_t = dt \Rightarrow
\]

\[
t = \sigma^2 \int_0^t (S_0 - L + \tilde{W}(s) + e^{a\phi_s}L)^2d\phi_s \Rightarrow
\]

\[
\phi_t^{-1} = \sigma^2 \int_0^{\phi_t^{-1}} (S_0 - L + \tilde{W}(s) + e^{a\phi_s}L)^2d\phi_s
\]

\[
= \sigma^2 \int_0^{\phi_t^{-1}} (S_0 - L + \tilde{W}(\phi_t^{-1}) + e^{as}L)^2ds.
\]

3.2. Some Properties of the Process \( \tilde{W}(\phi_t^{-1}) \). We note that process \( \tilde{W}(\phi_t^{-1}) \) is \( \mathcal{F}_t \)-measurable and \( \mathcal{F}_t \)-martingale.

Then

\[
E\tilde{W}(\phi_t^{-1}) = 0. \tag{7}
\]

Let’s calculate the second moment of \( \tilde{W}(\phi_t^{-1}) \) (see (6)):

\[
E\tilde{W}^2(\phi_t^{-1}) = E<\tilde{W}(\phi_t^{-1})> = E\phi_t^{-1}
\]

\[
= \sigma^2 \int_0^t E(S_0 - L + \tilde{W}(\phi_s^{-1}) + e^{as}L)^2ds
\]

\[
= \sigma^2[(S_0 - L)^2t + 2L(S_0 - L)(e^{at} - 1) + \frac{L^2(e^{at} - 1)}{2a}]
\]

\[
+ \int_0^t E\tilde{W}^2(\phi_s^{-1})ds. \tag{8}
\]
From (8), solving this linear ordinary nonhomogeneous differential equation with respect to $E\tilde{W}^2(\phi_t^{-1})$,

$$
\frac{dE\tilde{W}^2(\phi_t^{-1})}{dt} = \sigma^2[(S_0 - L)^2 + 2L(S_0 - L)e^{at} + L^2e^{2at} + E\tilde{W}^2(\phi_t^{-1})],
$$
we obtain

$$
E\tilde{W}^2(\phi_t^{-1}) = \sigma^2[(S_0 - L)^2 \frac{e^{\sigma^2t} - 1}{\sigma^2} + \frac{2L(S_0 - L)(e^{at} - e^{\sigma^2t})}{a - \sigma^2} + \frac{L^2(e^{2at} - e^{\sigma^2t})}{2a - \sigma^2}].
$$

(9)

3.3. Explicit Expression for the Process $\tilde{W}(\phi_t^{-1})$. It turns out that we can find the explicit expression for the process $\tilde{W}(\phi_t^{-1})$.

From the expression (see Section 3.1)

$$
V_t = S_0 - L + \tilde{W}(\phi_t^{-1}),
$$
we have the following relationship between $W(t)$ and $\tilde{W}(\phi_t^{-1})$:

$$
d\tilde{W}(\phi_t^{-1}) = \sigma \int_0^t [S(0) - L + Le^{at} + \tilde{W}(\phi_s^{-1})]dW(t).
$$

It is a linear SDE with respect to $\tilde{W}(\phi_t^{-1})$ and we can solve it explicitly. The solution has the following look:

$$
\tilde{W}(\phi_t^{-1}) = S(0)(e^{\sigma W(t) - \frac{\sigma^2t}{2}} - 1) + L(1 - e^{at}) + aLe^{\sigma W(t) - \frac{\sigma^2t}{2}} \int_0^t e^{as}e^{-\sigma W(s) + \frac{\sigma^2s}{2}}ds.
$$

(10)

It is easy to see from (10) that $\tilde{W}(\phi_t^{-1})$ can be presented in the form of a linear combination of two zero-mean martingales $m_1(t)$ and $m_2(t)$:

$$
\tilde{W}(\phi_t^{-1}) = m_1(t) + Lm_2(t),
$$

where

$$
m_1(t) := S(0)(e^{\sigma W(t) - \frac{\sigma^2t}{2}} - 1)
$$

and

$$
m_2(t) = (1 - e^{at}) + aLe^{\sigma W(t) - \frac{\sigma^2t}{2}} \int_0^t e^{as}e^{-\sigma W(s) + \frac{\sigma^2s}{2}}ds.
$$

Indeed, process $\tilde{W}(\phi_t^{-1})$ is a martingale (see Section 3.2), also it is well-known that process $e^{\sigma W(t) - \frac{\sigma^2t}{2}}$ and, hence, process $m_1(t)$ is a martingale. Then the process $m_2(t)$, as the difference between two martingales, is also martingale. In this way, we have

$$
Em_1(t) = 0,
$$
since

$$Ee^{\sigma W(t)-\frac{\sigma^2 t}{2}} = 1.$$  

As for $$m_2(t)$$ we have

$$Em_2(t) = 0,$$

since from Itô’s formula we have

$$d(ae^{\sigma W(t)-\frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{\sigma^2 s}{2} ds) = a\sigma \int_0^t e^{as} e^{-\sigma W(s)} + \frac{\sigma^2 s}{2} ds dW(t) + ae^{\sigma W(t)-\frac{\sigma^2 t}{2}} e^{at} e^{-\sigma W(t)} + \frac{\sigma^2 t}{2} dt$$

and, hence,

$$Eae^{\sigma W(t)-\frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{\sigma^2 s}{2} ds = e^{at} - 1.$$  

It is interesting to see that the last expression, the first moment for

$$\eta(t) := ae^{\sigma W(t)-\frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{\sigma^2 s}{2} ds,$$

does not depend on $$\sigma$$.

It is true not only for the first moment but for all the moments of the process $$\eta(t) = ae^{\sigma W(t)-\frac{\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{\sigma^2 s}{2} ds$$.

Indeed, using Itô’s formula for $$\eta^n(t)$$ we obtain

$$d\eta^n(t) = na^n \sigma e^{n\sigma W(t)-\frac{n\sigma^2 t}{2}} \int_0^t e^{as} e^{-\sigma W(s)} + \frac{\sigma^2 s}{2} ds dW(t) + an(\eta_2(t))^{n-1} e^{at} dt,$$

and

$$dE\eta^n(t) = nae^{at} E\eta^{n-1}(t) dt, \quad n \geq 1.$$  

This is a recursive equation with initial function $$(n = 1) E\eta(t) = e^{at} - 1$$. After calculations we obtain the following formula for $$E\eta^n(t)$$:

$$E\eta^n(t) = (e^{at} - 1)^n.$$  

3.4. Some Properties of the Mean-Reverting Asset $$S_t$$. From (4) we obtain the mean value of the first moment for mean-reverting asset $$S_t$$:

$$ES_t = e^{-at}[S_0 - L] + L.$$  

It means that $$ES_t \to L$$ when $$t \to +\infty$$.

Using formulae (4) and (9) we can calculate the second moment of $$S_t$$:

$$ES_t^2 = (e^{-at}(S_0 - L) + L)^2 + \sigma^2 e^{-2at}[(S_0 - L)^2 e^{\sigma^2 t} - 1] + 2L(S_0 - L)(e^{at} - e^{\sigma^2 t}) + \frac{L^2 e^{2at} - e^{2\sigma^2 t}}{2\sigma - \sigma^2}.$$
Combining the first and the second moments we have the variance of $S_t$:

$$Var(S_t) = ES_t^2 - (ES_t)^2$$

$$= \sigma^2 e^{-2at}[(S_0 - L)^2 e^{\sigma^2 t - 1 - \frac{a^2 t}{2}} + \frac{2L(S_0 - L)(e^{at} - e^{\sigma^2 t})}{\sigma^2} + \frac{L^2(e^{2at} - e^{2\sigma^2 t})}{2a - \sigma^2}].$$

From the expression for $\tilde{W}(\phi^{-1})$ (see (10)) and for $S(t)$ in (4) we can find the explicit expression for $S(t)$ through $\tilde{W}(t)$:

$$S(t) = e^{-at}[S_0 - L + \tilde{W}(\phi^{-1})] + L = e^{-at}[S_0 - L + m_1(t) + Lm_2(t)] + L$$

$$= S(0)e^{-at}e^{\sigma W(t) - \frac{\sigma^2 t}{2}} + aLe^{-at}e^{\sigma W(t) - \frac{\sigma^2 t}{2}} \int_0^t e^{as}e^{-\sigma W(s) + \frac{\sigma^2 s}{2}}ds,$$

where $m_1(t)$ and $m_2(t)$ are defined as in Section 3.3.

3.5 Explicit Option Pricing Formula for European Call Option for MRAM under Physical Measure. The payoff function $f_T$ for European call option equals

$$f_T = (S_T - K)^+ := \max(S_T - K, 0),$$

where $S_T$ is an asset price defined in (4), $T$ is an expiration time (maturity) and $K$ is a strike price.

In this way (see (11)),

$$f_T = [e^{-aT}(S_0 - L + \tilde{W}(\phi^{-1})) + L - K]^+$$

$$= [S(0)e^{-aT}e^{\sigma W(T) - \frac{\sigma^2 T}{2}} + aLe^{-aT}e^{\sigma W(T) - \frac{\sigma^2 T}{2}} \int_0^T e^{as}e^{-\sigma W(s) + \frac{\sigma^2 s}{2}}ds - K]^+. $$

To find the option pricing formula we need to calculate

$$C_T = e^{-rT}Ef_T$$

$$= e^{-rT}[e^{-aT}(S_0 - L + \tilde{W}(\phi^{-1})) + L - K]^+$$

$$= \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{-\infty}^{+\infty} \max[S(0)e^{-aT}e^{\sigma y\sqrt{T} - \frac{\sigma^2 T}{2}},$$

$$+ aLe^{-aT}e^{\sigma y\sqrt{T} - \frac{\sigma^2 T}{2}} \int_0^T e^{as}e^{-\sigma y\sqrt{s} + \frac{\sigma^2 s}{2}}ds - K, 0]e^{-\frac{y^2}{2}}dy.$$

Let $y_0$ be a solution of the following equation:

$$S(0) \times e^{-aT}e^{\sigma y_0\sqrt{T} - \frac{\sigma^2 T}{2}}$$

$$+ aLe^{-aT}e^{\sigma y_0\sqrt{T} - \frac{\sigma^2 T}{2}} \int_0^T e^{as}e^{-\sigma y_0\sqrt{s} + \frac{\sigma^2 s}{2}}ds = K$$

(13)

or

$$y_0 = \frac{\ln\left(\frac{K}{S(0)}\right) + \left(\frac{\sigma^2}{2} + a\right)T}{\sigma\sqrt{T}}.$$
Pricing formula for a mean-reverting asset

\[- \ln(1 + \frac{aL}{S(0)} \int_0^T e^{as} e^{-\sigma y_0 \sqrt{T} + \frac{a^2 s}{2}} ds) \over \sigma \sqrt{T} \]

(14)

From (12)-(13) we have:

\[
C_T = \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{-\infty}^{+\infty} \max[S(0)e^{-aT}e^{\sigma y \sqrt{T} - \frac{a^2 T}{2}} - K, 0] e^{-\frac{y^2}{2}} dy
\]

(15)

where

\[
BS(T) := \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{y_0}^{+\infty} [S(0)e^{-aT}e^{\sigma y \sqrt{T} - \frac{a^2 T}{2}} - \frac{y^2}{2}] e^{-rT} K[1 - \Phi(y_0)],
\]

(16)

\[
A(T) := e^{-(r+a)T} \int_{y_0}^{+\infty} (ae^{\sigma y \sqrt{T} - \frac{a^2 T}{2}} \int_0^T e^{as} e^{-\sigma y \sqrt{s} + \frac{a^2 s}{2}} ds) e^{-\frac{y^2}{2}} dy,
\]

(17)

and

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.
\]

(18)

After calculation of $BS(T)$ we obtain

\[
BS(T) = e^{-(r+a)T} S(0) \Phi(y_+) - e^{-rT} K \Phi(y_-),
\]

(19)

where

\[
y_+ := \sigma \sqrt{T} - y_0 \quad \text{and} \quad y_- := -y_0
\]

(20)

and $y_0$ is defined in (14).

Consider $A(T)$ in (17).

Let $F_T(dz)$ be a distribution function for the process

\[
\eta(T) = ae^{\sigma W(T) - \frac{a^2 T}{2}} \int_0^T e^{as} e^{-\sigma W(s) + \frac{a^2 s}{2}} ds,
\]

which is a part of the integrand in (17).

As M. Yor [15, 16] mentioned there is still no closed form probability density function for time integral of an exponential Brownian motion, while the best result is a function with a double integral.
We can use Yor’s result [15] to get $F_T(dz)$ above. Using the scaling property of Wiener process and change of variables, we can rewrite our expression for $S(t)$ in (11) in the following way

$$S(T) = S(0)e^{-2B^v_{T_0}} + \frac{4}{\sigma^2}aLe^{-2B^v_{T_0}}A^v_{T_0},$$

where $T_0 = \sigma^2/4$, $T, v = 2\sigma^2 a + 1$, $B_t = -\sigma^2/2W(4\sigma^2 t)$, $B^v_{T_0} = vT_0 + B_{T_0}$, $A^v_{T_0} = \int_0^{T_0} e^{2B^v_s}ds$.

Also, the process $\eta(T)$ may be presented in the following way using these transformations

$$\eta(T) = \frac{4ae^{-aT}}{\sigma^2}e^{-2B^v_{T_0}}A^v_{T_0}.$$  

We state here the result obtained by Yor [37] for the joint probability density function of $A^v_{T_0}$ and $B^v_{T_0}$.

**Theorem 4.3.-1.** (M. Yor [37]). The joint probability density function of $A^v_{T_0}$ and $B^v_{T_0}$ satisfies

$$P(A^v_{T_0} \in du, B^v_{T_0} \in dx) = e^{uv - v^2t/2} \exp\left(\frac{1}{2u} + \frac{e^x}{u} t\right)\frac{dxdx}{u},$$

where $t > 0, u > 0, x \in R$ and

$$\theta(r, t) = \frac{r}{(2\pi^2 t)^{1/2}} e^{\frac{r^2}{2t}} \int_0^{+\infty} e^{-\frac{s^2}{2t} \cos \phi \sin \phi} \sin(s / t)ds.$$ 

Using this result we can write the distribution function for $\eta(T)$ in the following way

$$P(\eta(T) \leq u) = P(\frac{4ae^{-aT}}{\sigma^2}e^{-2B^v_{T_0}}A^v_{T_0} \leq u) = P(e^{-2B^v_{T_0}}A^v_{T_0} \leq \frac{\sigma^2 e^{aT}}{4a} u) = F_T(u).$$

In this way, $A(T)$ in (17) may be presented in the following way:

$$A(T) = L e^{-(r+a)T} \int_{y_0}^{+\infty} zF_T(dz).$$

After calculation of $A(T)$ we obtain the following expression for $A(T)$ :

$$A(T) = L e^{-(r+a)T}[(e^{aT} - 1) - \int_0^{y_0} zF_T(dz)],$$

since $E\eta(T) = e^{aT} - 1$.

Finally, summarizing (12)-(21), we have obtained the following Theorem.
Theorem 3.1. Option pricing formula for European call option for mean-reverting asset under physical measure has the following look:

\[
C_T = e^{-(r+a)T}S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-) + Le^{-(r+a)T}[e^{aT} - 1] - \int_0^{y_0} zF_T(dz) - e^{-rT}K\Phi(y_-) + Le^{-(r+a)T}[e^{aT} - 1],
\]

where \( y_0 \) is defined in (14), \( y_+ \) and \( y_- \) in (20), \( \Phi(y) \) in (18), and \( F_T(dz) \) is a distribution function in (21).

Remar. From (21)-(22) we find that European Call Option Price \( C_T \) for mean-reverting asset lies between the following boundaries:

\[
BS(T) \leq C_T \leq BS(T) + Le^{-(r+a)T}[e^{aT} - 1],
\]
or (see (19)),

\[
e^{-(r+a)T}S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-) \leq C_T \leq e^{-(r+a)T}S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-) + Le^{-(r+a)T}[e^{aT} - 1].
\]

4. MEAN-REVERTING RISK-NEUTRAL ASSET MODEL (MRRNAM)

Consider our model (1)

\[
dS_t = a(L - S_t)dt + \sigma S_t dW_t.
\]

We want to find a probability \( P^* \) equivalent to \( P \), under which the process \( e^{-rt}S_t \) is a martingale, where \( r > 0 \) is a constant interest rate. The hypothesis we made on the filtration \( (\mathcal{F}_t)_{t \in [0,T]} \) allows us to express the density of the probability \( P^* \) with respect to \( P \). We denote this density by \( L_T \).

It is well-known (see Lamperton and Lapeyre (1996), Proposition 6.1.1, p. 123), that there is an adopted process \( (q(t))_{t \in [0,T]} \) such that, for all \( t \in [0,T] \),

\[
L_t = \exp\left[ \int_0^t q(s) dW_s - \frac{1}{2} \int_0^t q^2(s) ds \right] \quad a.s.
\]

In this case,

\[
\frac{dP^*}{dP} = \exp\left[ \int_0^T q(s) dW_s - \frac{1}{2} \int_0^T q^2(s) ds \right] = L_T.
\]

In our case, with model (17), the process \( q(t) \) is equal to

\[
q(t) = -\lambda S_t,
\]
where \( \lambda \) is the market price of risk and \( \lambda \in R \). Hence, for our model

\[
L_T = \exp[-\lambda \int_0^T S(u)dW_u - \frac{1}{2}\lambda \int_0^T S^2(u)du].
\]

Under probability \( P* \), the process \((W_t^*)\) defined by

\[
W_t^* := W_t + \lambda \int_0^t S(u)du
\]

is a standard Brownian motion (Girsanov theorem) (see Elliott and Kopp (1999)).

Therefore, in a risk-neutral world our model (23) takes the following look:

\[
dS_t = (aL - (a + \lambda\sigma)S_t)dt + \sigma S_t dW_t^*,
\]

or, equivalently,

\[
dS_t = a^*(L^* - S_t)dt + \sigma S_t dW_t^*,
\]

where

\[
a^* := a + \lambda\sigma, \quad L^* := \frac{aL}{a + \lambda\sigma},
\]

and \( W_t^* \) is defined in (25).

Now, we have the same model in (26) as in (1), and we are going to apply our method of changing of time to this model (26) to obtain the explicit option pricing formula.

5. EXPLICIT OPTION PRICING FORMULA FOR EUROPEAN CALL OPTION FOR MRRNAM

In this section, we are going to obtain explicit option pricing formula for European call option under risk-neutral measure \( P* \), using the same arguments as in sections 3-7, where in place of \( a \) and \( L \) we are going to take \( a^* \) and \( L^* \)

\[
a \to a^* := a + \lambda\sigma, \quad L \to L^* := \frac{aL}{a + \lambda\sigma},
\]

where \( \lambda \) is a market price of risk (See section 3).

5.1. EXPLICIT SOLUTION FOR THE MEAN-REVERTING RISK-NEUTRAL ASSET MODEL. Applying (2)-(6) to our model (26) we obtain the following explicit solution for our risk-neutral model (26):

\[
S_t = e^{-a^*t}[S_0 - L^* + \tilde{W}^*((\phi^*_t)^{-1})] + L,
\]

where \( \tilde{W}^*(t) \) is an \( \mathcal{F}_t \)-measurable standard one-dimensional Wiener process under measure \( P^* \) and \((\phi^*_t)^{-1}\) is an inverse function to \( \phi^*_t \):

\[
\phi^*_t = \sigma^{-2} \int_0^t (S_0 - L^* + \tilde{W}^*(s) + e^{a^*\phi^*_s}L^*)^{-2}ds.
\]
We note that
\[
(\phi^*)_t^{-1} = \sigma^2 \int_0^t (S_0 - L^* + \tilde{W}^*((\phi^*_t)^{-1}) + e^{a^*_t s} L^*)^2 ds,
\]
where \(a^*_t\) and \(L^*_t\) are defined in (27).

5.2. Some Properties of the Process \(\tilde{W}^*((\phi^*_t)^{-1})\). Using the same argument as in Section 4, we obtain the following properties of the process \(\tilde{W}^*((\phi^*_t)^{-1})\) in (25). This is a zero-mean \(P^*_t\)-martingale and

\[
E^*[\tilde{W}^*((\phi^*_t)^{-1})] = 0,
\]
\[
E^*[\tilde{W}^*((\phi^*_t)^{-1})]^2 = \sigma^2 [(S_0 - L^*)^2 e^{\sigma^2 t} - 1] + \frac{2L^*(S_0 - L^*)(e^{a^*_t - e^{\sigma^2 t}})}{a^*_t - \sigma^2} + \frac{(L^*)^2 (e^{2a^*_t} - e^{2\sigma^2 t})}{2a^*_t - \sigma^2},
\]
where \(E^*\) is the expectation with respect to the probability \(P^*_t\) and \(a^*_t, L^*_t\) and \((\phi^*_t)^{-1}\) are defined in (27) and (30), respectively.

5.3. Explicit Expression for the Process \(\tilde{W}^*((\phi^*_t)^{-1})\). It is turns out that we can find the explicit expression for the process \(\tilde{W}^*((\phi^*_t)^{-1})\).

From the expression
\[
V_t = S_0 - L + \tilde{W}^*((\phi^*_t)^{-1}),
\]
we have the following relationship between \(W(t)\) and \(\tilde{W}((\phi^*_t)^{-1})\):

\[
d\tilde{W}^*((\phi^*_t)^{-1}) = \sigma \int_0^t [S_0 - L + e^{a^*_t s} + \tilde{W}^*((\phi^*_s)^{-1})] dW^*(t).
\]

It is linear SDE with respect to \(\tilde{W}^*((\phi^*_t)^{-1})\) and we can solve it explicitly. The solution has the following look:

\[
\tilde{W}^*((\phi^*_t)^{-1}) = S_0(e^{\sigma W^*(t)} - e^{\sigma W^*(0)} - 1) + L(1 - e^{a^*_t}) + aL e^{\sigma W^*(t)} - \frac{e^{2\sigma W^*(t)} - 1}{2} \int_0^t e^{a^*_s} e^{-\sigma W^*(s)} + \frac{e^{2\sigma W^*(s)} - 1}{2} ds.
\]

It is easy to see from (32) that \(\tilde{W}^*((\phi^*_t)^{-1})\) can be presented in the form of a linear combination of two zero-mean \(P^*_t\)-martingales \(m^*_1(t)\) and \(m^*_2(t)\):

\[
\tilde{W}^*((\phi^*_t)^{-1}) = m^*_1(t) + L^* m^*_2(t),
\]
where

\[
m^*_1(t) := S_0(e^{\sigma W^*(t)} - e^{\sigma W^*(0)} - 1)
\]
and

\[
m^*_2(t) = (1 - e^{a^*_t}) + a^* e^{\sigma W^*(t)} - \frac{e^{2\sigma W^*(t)} - 1}{2} \int_0^t e^{a^*_s} e^{-\sigma W^*(s)} + \frac{e^{2\sigma W^*(s)} - 1}{2} ds.
\]
Indeed, process $\tilde{W}^*(\overline{\phi}_t^{-1})$ is a martingale (see Section 5.2), also it is well-known that process $e^{\sigma W^*(t)-\frac{\sigma^2}{2} t}$ and, hence, process $m_1^*(t)$ is a martingale. Then the process $m_2^*(t)$, as the difference between two martingales, is also martingale. In this way, we have

$$E_p m_1^*(t) = 0,$$

since

$$E_p e^{\sigma W^*(t)-\frac{\sigma^2}{2} t} = 1.$$ 

As for $m_2(t)$ we have

$$E_p m_2(t) = 0,$$

since from Itô’s formula we have

\[
\begin{align*}
    d \left( a^* e^{\sigma W^*(t)-\frac{\sigma^2}{2} t} \int_0^t e^{a^* s e^{-\sigma W^*(s)}} + \frac{\sigma^2}{2} ds \right) &= a^* \sigma e^{\sigma W^*(t)-\frac{\sigma^2}{2} t} \int_0^t e^{a^* s e^{-\sigma W^*(s)}} + \frac{\sigma^2}{2} dW^*(t) \quad + \quad a^* e^{\sigma W^*(t)-\frac{\sigma^2}{2} t} e^{a^* t} e^{-\sigma W^*(t)} + \frac{\sigma^2}{2} dt \\
    &= a^* \sigma e^{\sigma W^*(t)-\frac{\sigma^2}{2} t} \int_0^t e^{a^* s e^{-\sigma W^*(s)}} + \frac{\sigma^2}{2} dW^*(t) \quad + \quad a^* e^{a^* t} dt,
\end{align*}
\]

and, hence,

$$E_p a^* e^{\sigma W^*(t)-\frac{\sigma^2}{2} t} \int_0^t e^{a^* s e^{-\sigma W^*(s)}} + \frac{\sigma^2}{2} ds = e^{a^* t} - 1.$$ 

It is interesting to see that in the last expression, the first moment for

$$\eta^*(t) := a^* e^{\sigma W^*(t)-\frac{\sigma^2}{2} t} \int_0^t e^{a^* s e^{-\sigma W^*(s)}} + \frac{\sigma^2}{2} ds,$$

does not depend on $\sigma$.

This is true not only for the first moment but for all the moments of the process $\eta^*(t) = a^* e^{\sigma W^*(t)-\frac{\sigma^2}{2} t} \int_0^t e^{a^* s e^{-\sigma W^*(s)}} + \frac{\sigma^2}{2} ds$.

Indeed, using the Itô’s formula for $(\eta^*(t))^n$ we obtain

\[
\begin{align*}
    d(\eta^*(t))^n &= n(a^*)^n \sigma e^{n \sigma W^*(t)-\frac{n \sigma^2}{2} t} \left( \int_0^t e^{a^* s e^{-\sigma W^*(s)}} + \frac{\sigma^2}{2} ds \right)^{n-1} dW^*(t) \\
    &\quad + a^* n(\eta^*_2(t)) \left( \int_0^t e^{a^* s e^{-\sigma W^*(s)}} + \frac{\sigma^2}{2} ds \right)^{n-1} e^{a^* t} dt,
\end{align*}
\]

and

$$dE(\eta^*(t))^n = na^* e^{a^* t} E(\eta^*(t))^{n-1} dt, \quad n \geq 1.$$ 

This is a recursive equation with initial function $(n = 1)$ $E\eta^*(t) = e^{a^* t} - 1$.

After calculations we obtain the following formula for $E(\eta^*(t))^n$:

$$E(\eta^*(t))^n = (e^{a^* t} - 1)^n.$$
5.4. Some Properties of the Mean-Reverting Risk-Neutral Asset $S_t$. Using the same argument as in Section 5, we obtain the following properties of the mean-reverting risk-neutral asset $S_t$ in (18):

\[
E^*S_t = e^{-a^*t}[S_0 - L^*] + L^*
\]

\[
Var^*(S_t) := E^*S_t^2 - (E^*S_t)^2
\]

\[
= \sigma^2 e^{-2a^*t}([S_0 - L^*])^2 + \frac{2L^*(S_0 - L^*)(e^{a^*t} - e^{a^*t})}{a^* - \sigma^2} + \frac{(L^*)^2(e^{2a^*t} - e^{2a^*t})}{2a^* - \sigma^2},
\]

where $E^*$ is the expectation with respect to the probability $P^*$ and $a^*, L^*$ and $(\phi_t^*)^{-1}$ are defined in (27) and (30), respectively.

From the expression for $\tilde{W}^*(\phi_t^{-1})$ (see (32)) and for $S(t)$ in (28) (see also (29)-(30)) we can find the explicit expression for $S(t)$ through $W^*(t)$:

\[
S(t) = e^{-a^*t}[S_0 - L^* + \tilde{W}^*(\phi_t^{-1})] + L^*
\]

\[= e^{-a^*t}[S_0 - L^* + m_1^*(t) + L^*m_2^*(t)] + L^*
\]

\[= S(0)e^{-at}e^{\sigma W^*(t)} + \frac{a^2}{2} - aLe^{-at}e^{\sigma W^*(t)} - \frac{a^2}{2} \int_0^t e^{as}e^{-\sigma W^*(s)} + \frac{a^2}{2} ds,
\]

where $m_1^*(t)$ and $m_2^*(t)$ are defined as in section 5.3.

5.5. Explicit Option Pricing Formula for European Call Option for MRAM under Risk-Neutral Measure. Proceeding with the same calculations (15)-(22) as in Section 3, where in place of $a$ and $L$ we take $a^*$ and $L^*$ in (27), we obtain the following Theorem.

**Theorem 5.1.** Explicit option pricing formula for European call option under risk-neutral measure has the following look:

\[
C^*_T = e^{-(r+a^*)T}S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-) + L^*e^{-(r+a^*)T}[(e^{a^*T} - 1) - \int_0^{y_0} zF^*_T(dz)],
\]

where $y_0$ is the solution of the following equation

\[
y_0 = \frac{\ln\left(\frac{K}{S(0)}\right) + (\frac{a^2}{2} + a^*)T}{\sigma\sqrt{T}}
\]

\[-\frac{\ln(1 + \frac{a^*L^*}{S(0)})^{\frac{a^*}{2}} e^{-\sigma y_0\sqrt{T}} + \frac{a^2}{2} ds}{\sigma\sqrt{T}},
\]

\[y_+ := \sigma\sqrt{T} - y_0 \quad \text{and} \quad y_- := -y_0,
\]

\[a^* := a + \lambda\sigma, \quad L^* := \frac{aL}{a + \lambda\sigma},
\]

and $F^*_T(dz)$ is the probability distribution as in (21), where instead of $a$ we have to take $a^* = a + \lambda\sigma$. 
Remark. From (35) we can find that European Call Option Price $C_T^*$ for mean-reverting asset under risk-neutral measure lies between the following boundaries:

$$e^{-(r+a^*)T}S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-) \leq C_T \leq e^{-(r+a^*)T}S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-) + L^* e^{-(r+a^*)T}[e^{a^*T} - 1],$$

(38)

where $y_0, y_-, y_+$ are defined in (36)-(37).

5.6. BLACK-SCHOLES FORMULA FOLLOWS: $L^* = 0$ and $a^* = -r$. If $L^* = 0$ and $a^* = -r$ we obtain from (35)

$$C_T = S(0)\Phi(y_+) - e^{-rT}K\Phi(y_-),$$

(39)

where

$$y_+ := \sigma\sqrt{T} - y_0 \quad \text{and} \quad y_- := -y_0,$$

(40)

and $y_0$ is the solution of the following equation (see (36))

$$S(0)e^{-rT}e^{\frac{\sigma^2}{2}T} = K$$

or

$$y_0 = \frac{\ln\left(\frac{K}{S(0)}\right) + \frac{\sigma^2}{2}T - rT}{\sigma\sqrt{T}}.$$  

(41)

But (39)-(41) is exactly the well-known Black-Scholes result!

6. NUMERICAL EXAMPLE: AECO NATURAL GAS INDEX (1 May 1998-30 April 1999)

We shall calculate the value of a European call option on the price of a daily natural gas contract. To apply our formula for calculating this value we need to calibrate the parameters $a, L, \sigma$ and $\lambda$. These parameters may be obtained from futures prices for the AECO Natural Gas Index for the period 1 May 1998 to 30 April 1999 (see Bos, Ware and Pavlov (2002), p.340). The parameters pertaining to the option are the following:

<table>
<thead>
<tr>
<th>Price and Option Process Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
</tr>
<tr>
<td>6 months</td>
</tr>
</tbody>
</table>

From this table we can calculate the values for $a^*$ and $L^*$:

$$a^* = a + \lambda\sigma = 4.9337,$$
and

\[ L^* = \frac{aL}{a + \lambda\sigma} = 2.5690. \]

For the value of \( S_0 \) we can take \( S_0 \in [1, 6] \).

Figure 1 (see Appendix) depicts the dependence of mean value \( ES_t \) on the maturity \( T \) for AECO Natural Gas Index (1 May 1998 to 30 April 1999).

Figure 2 (see Appendix) depicts the dependence of mean value \( ES_t \) on the initial value of stock \( S_0 \) and maturity \( T \) for AECO Natural Gas Index (1 May 1998 to 30 April 1999).

Figure 3 (see Appendix) depicts the dependence of variance of \( S_t \) on the initial value of stock \( S_0 \) and maturity \( T \) for AECO Natural Gas Index (1 May 1998 to 30 April 1999).

Figure 4 (see Appendix) depicts the dependence of volatility of \( S_t \) on the initial value of stock \( S_0 \) and maturity \( T \) for AECO Natural Gas Index (1 May 1998 to 30 April 1999).

Figure 5 (see Appendix) depicts the dependence of European Call Option Price for MRRNAM on the maturity (months) for AECO Natural Gas Index (1 May 1998 to 30 April 1999) with \( S(0) = 1 \) and \( K = 3 \).

7. Future Work

As we could see in Section 3. 4 (see also Figures 1, 3 and 4) the main drawbacks of one-factor mean-reverting models are: 1) the long-term mean \( L \) remains fixed over time: needs to be recalibrated on a continuous basis in order to ensure that the resulting curves are marked to market; 2) the biggest drawback is in option pricing: results in a model-implied volatility term structure that has the volatilities going to zero as expiration time increases (spot volatilities have to be increased to non-intuitive levels so that the long term options do not lose all the volatility value-as in the marketplace they certainly do not).

To eliminate these drawbacks we are going to consider two-factor mean-reverting model

\[
\begin{align*}
    dS_t &= \alpha(L_t - S_t)dt + \sigma S_t dW^1_t \\
    dL_t &= \xi L_t dt + \eta L_t dW^2_t
\end{align*}
\]

or

\[
\begin{align*}
    dS_t &= \alpha(L_t - S_t)dt + \sigma S_t dW^1_t \\
    dL_t &= \xi(N - L_t)dt + \eta L_t dW^2_t
\end{align*}
\]

(processes \( W^1_t \) and \( W^2_t \) may be correlated) and the change of time method to obtain explicit option pricing formula for these models. We note, that one of a possible way to eliminate these drawbacks is to consider the mean-reverting Markov switching model, see CHEN AND FORSYTH (2006). Another way is to consider above-mentioned SDEs with Lévy processes as driven processes in place of Wiener processes, see SCHOUTENS (2003) for applications of Lévy processes in finance.
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Bibliography


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9. Appendix: Figures
Fig. 1. Dependence of $ES_t$ on $T$ (AECO Natural Gas Index (1 May 1998-30 April 1999))

Fig. 2. Dependence of $ES_t$ on $S_0$ and $T$ (AECO Natural Gas Index (1 May 1998-30 April 1999))

Fig. 3. Dependence of variance of $S_t$ on $S_0$ and $T$ (AECO Natural Gas Index (1 May 1998-30 April 1999))

Fig. 4. Dependence of volatility of $S_t$ on $S_0$ and $T$ (AECO Natural Gas Index (1 May 1998-30 April 1999))

Fig. 5. Dependence of European Call Option Price on Maturity (months) ($S(0) = 1$ and $K = 3$) (AECO Natural Gas Index (1 May 1998-30 April 1999))