1. (30 points)

(a) (10 points) Use Mathematica or Maple to plot the probabilities of obtaining \( n \) heads for a fair (i.e. equally weighted) two-sided coin thrown \( N \) times, as a function of the fraction of heads (i.e. against \( n/N \)), for increasing values of \( N \). In particular, plot the probabilities for \( N = 10, 100, 1000, \) and 10 000. Each of these plots will correspond to the *distribution of macrostates*. Your computer might have trouble calculating the exponentials, so you might need to make use of Stirling’s approximation \( n! \approx \sqrt{2\pi n}(n/e)^n \), valid for large \( n \). Comment on the figures; what do they tell you about your confidence in the outcome of flipping many coins?

(b) (10 points) Calculate the maxima and widths of the distributions as a function of \( N \), for \( 10 \leq N \leq 1000 \) plot the results (the width of the distribution corresponds to the separation between the points at which the values of the function are equal to half its maximum). Obtain the best fits to the maxima and widths, assuming that they follow a power law. If the fits are performed only for very large systems, how do the results change? Infer the \( N \)-dependence of the distribution maxima and width for large \( N \).

(c) (10 points) Use the results in (b) to make a good guess for the functional form of the binomial distribution. Using this functional form, plot the distribution and compare the result with the actual binomial results on the same figure. Do the results match?

2. (30 points)

(a) (10 points) Using Maple or Mathematica write a routine that will perform a series of \( N \) random coin flips. On any given instance of the routine, there will be some number \( n \) heads and \( N - n \) tails; store the data. Run this routine \( M \gg 1 \) times assuming \( N \gg 1 \), and plot the relative number of instances for each \( n \) (i.e. divide the value by \( M \)) as a function of \( n \). How do the results compare with those obtained in question (1)? Do the results change with \( M \)?

(b) (10 points) Repeat the calculations in (2a), assuming that the coin is false, i.e. that there is a probability \( p \) that the outcome of a random coin flip is heads. One way to do this is to generate a random real number in the interval \([0,1]\), and then assign an outcome of heads if the number is in the interval \([0,p]\). Describe the change in the distribution function (if any), and explain.

(c) (10 points) Now assume that you are flipping a three-sided coin, whose outcomes are heads, tails, and feet with probabilities \( p \), \( q \), and \( 1 - p - q \), respectively. Repeat the calculations in (b), and make a density or contour plot of the distribution function as a function of the number of heads \( n \) and tails \( m \). What can you say about the trinomial distribution relative to the binomial distribution? What are your inferences for a general multinomial distribution function?