1. Let's start the cycle at volume $V_1$ and pressure $P_1$. First we have an isobaric expansion, bringing the system to a new volume $V_2$ but at the same pressure $P_1$ (that’s what an isobar is!). The temperature clearly increases during this phase. The first adiabatic expansion brings the volume to $V_3$ and pressure to $P_3$. This is followed by an isochoric depressurization to a new pressure $P_4$ but the same volume $V_3$. Last, an adiabatic compression decreases the volume and increases the pressure up to $V_1$ and $P_1$, and the cycle starts over. Here are the various results:

(a) $1 \rightarrow 2$ isobar: For the constant pressure environment, the heat is the difference in the enthalpies: $Q = U_2 + P_2 V_2 - (U_1 + P_1 V_1)$. Because the working fluid is an ideal gas, $U = (f/2)PV$ so

$$Q = \frac{f + 2}{2} (P_2 V_2 - P_1 V_1) = \frac{f + 2}{2} P_1 (V_2 - V_1).$$

(b) $2 \rightarrow 3$ adiabat: $P_2 V_2^\gamma = P_1 V_1^\gamma = P_3 V_3^\gamma$. The heat is zero so it won’t contribute to the efficiency.

(c) $3 \rightarrow 4$ isochore: The work here is zero, and the heat is

$$Q = C_V (T_4 - T_3) = U_4 - U_3 = \frac{f}{2} (P_4 V_4 - P_3 V_3) = \frac{f}{2} V_3 (P_4 - P_3).$$

(d) $1 \rightarrow 2$ adiabat: $P_1 V_1^\gamma = P_1 V_1^\gamma$. The heat is zero so it won’t contribute to the efficiency.

The efficiency is therefore:

$$e = 1 - \frac{\left| \frac{f}{2} V_3 (P_4 - P_3) \right|}{\frac{f \gamma P_1 (V_2 - V_1)}{2}} = 1 - \frac{\left| V_3 \left( \frac{P_4}{P_1} \right)^\gamma - \frac{P_1}{V_1} \left( \frac{V_3}{V_1} \right)^\gamma \right|}{\gamma P_1 (V_2 - V_1)},$$

because $\gamma = (f + 2)/f$. We also know that $P_3/P_1 = (V_2/V_3)^\gamma$ and $P_4/P_1 = (V_1/V_3)^\gamma$. So:

$$e = 1 - \frac{\left| V_3 \left[ \frac{P_1}{V_1} \left( \frac{V_3}{V_1} \right)^\gamma - \frac{P_1}{V_1} \left( \frac{V_3}{V_1} \right)^\gamma \right] \right|}{\gamma P_1 (V_2 - V_1)} = 1 - \left( \frac{V_2}{V_1} \right) \left( \frac{V_3}{V_1} \right)^\gamma \frac{1 - \left( \frac{V_3}{V_1} \right)^\gamma}{\gamma \left( \frac{V_3}{V_1} - 1 \right)} = 1 - \left( \frac{V_1}{V_3} \right)^\gamma - 1 \left( \frac{V_3}{V_1} \right)^\gamma - 1 \left( \frac{V_3}{V_1} - 1 \right).$$

Here I have expressed the result in terms of the compression ratio $V_3/V_1$ and the so-called ‘cutoff ratio’ $V_2/V_1$. The result for the Otto cycle is simply the first term that depends on the compression ratio. For ordinary gases $\gamma = 7/5 > 1$. Also $V_2/V_1 > 1$. This means that generically the remaining contribution will be greater than one. As a result the diesel engine is fundamentally less efficient than the gasoline engine for the same compression ratios. The efficiencies for diesel engines are actually higher, though, because the compression ratios are higher than for gasoline-powered engines. If the compression ratio is 18 and the cutoff ratio is 2 then I get

$$e = 1 - \left( \frac{1}{18} \right)^{2/5} \left( \frac{2^{7/5} - 1}{5} \right) \approx 63\%,$$

which is not bad! In contrast, the efficiency of the Otto cycle for the same compression ratio would be

$$e_{\text{Otto}} = 1 - \left( \frac{1}{18} \right)^{2/5} \approx 69\%,$$

though this large a compression is very unlikely in a gas engine.
2. The household fridge works on the reversed Rankine cycle, but the phase diagram of HFC-134a is different from that of water. Suppose point 1 is at the bottom right of the cycle, at the beginning of the adiabat. This point lies exactly at the liquid-vapor/vapor phase boundary so we can assume that all of the working fluid is vapor. At the end of the compression cycle (point 2), the working fluid is high pressure and high temperature vapor. At point 3 at the end of the apparent isobar (more on this below), corresponding to the condenser, the working fluid is all liquid again. At point 4 after the throttle, the working fluid is a mixture of liquid and vapor. So in the fridge cycle, the mixture of liquid and gas occurs at the end of the throttle (which is assumed to isenthalpic) rather than at the end of the adiabat as was the case in the steam engine.

The coefficient of performance for a fridge is the ratio of the heat extracted from the cold reservoir versus the work needed to power the compressor (no work is associated with the throttle). Thus

\[ \text{COP} = \frac{H_1 - H_4}{H_2 - H_1}. \]

Let’s figure out the enthalpies at the relevant points. At point 1 the working fluid is a vapor at -10°C, and the steam table gives \( H_1 = 392.9 \text{ kJ} \) (we assume that there is one kg of working fluid in all calculations). At point 2 we have superheated vapor at 30°C and high pressure (a) \( H_2 = 427.3 \text{ kJ} \), (b) \( H_2 = 423.4 \text{ kJ} \), and (c) \( H_2 = 416.8 \text{ kJ} \). These are all quite close to one another.

Note that the entropy is not perfectly conserved in this process. The entropy at point 1 is 1.7341 kJ/K, while at point 2 it is 1.856 kJ/K, 1.79 kJ/K, and 1.7278 kJ/K for pressures (a) through (c), respectively. The entropy changes are all very small, however, reflecting the small change in entropy when heating and pressuring the working fluid as long as it remains a gas. That said, the change in entropy is smallest at the highest pressure considered. To keep the entropy constant one would need a final pressure around 670 kPa.

Now there are two strategies for solving for the remaining enthalpies.

- We can assume that point 2 is again exactly at the liquid-vapor/vapor phase boundary, so that for any lower volume the working fluid is undergoing a phase transition into a liquid, so that by the point 3 all that has happened is that all the vapor has turned into liquid. In this case, the enthalpy at point 3 is given by the 30°C saturation table: \( H_3 = 241.8 \text{ kJ} \). Note that there is only one possible pressure at this temperature, 771.02 kPa. This means that the condenser cannot operate at constant pressure, as is assumed in the phase diagram! Rather, the pressure at point 2 would have to be close to this value; the closest value of the enthalpy for the superheated vapor is \( H_2 \approx 417.6 \text{ kJ} \), interpolating between the values at \( P = 750 \text{ kPa} \) and 800 kPa. Under the assumption that \( H_4 = H_3 \) then gives the coefficient of performance

\[ \text{COP} \approx \frac{392.9 - 241.8}{171.6 - 392.9} \approx 6.1. \]

Let’s compare this with the Carnot COP= \( \frac{T_c}{T_h - T_c} \approx 263/40 \approx 6.6 \), so it is indeed smaller than the upper bound. Because there is only one value for \( H_2 \) consistent with the pressures at points 2 and 3, there is really no different answer for (a), (b), and (c) using this interpretation.

- We can instead assume that point 3 is truly connected to point 2 at the same pressure, so that the enthalpy at point 3 is found in the saturation table at that pressure. For (a), (b), and (c), this gives enthalpies \( H_3 \approx 186.7 \text{ kJ} \) at a temperature of -10°C which is clearly impossible (the hot and cold reservoirs would have to be at the same temperature), 212.2 kJ at a temperature of 9°C (requiring a very cold room!), and approximately 237 kJ at a temperature of approximately 27°C, respectively. Likewise, we assume that point 2 is more deeply in the vapor phase, so that it would need to cool down as the volume decreases (at constant pressure) before it begins to change its phase during the condensation phase. Consider first the case (a). Clearly, the vapor is condensing at a temperature of -10°C, which is ridiculous, and any value of the COP for this case will be garbage. In any case we have all the data we need:

(a) \[ \text{COP} \approx \frac{392.9 - 186.7}{427.3 - 392.9} \approx 6.0; \]

(b) \[ \text{COP} \approx \frac{392.9 - 212.2}{423.4 - 392.9} \approx 5.9; \]

(c) \[ \text{COP} \approx \frac{392.9 - 237}{416.8 - 392.9} \approx 6.5. \]
The last case is the one where point 2 is closest to the liquid-vapor/vapor phase boundary, and this is also the most efficient, giving a COP that is closest to the ideal Carnot value. Interestingly, this is the case for which the $1 \rightarrow 2$ process is closest to truly adiabatic (i.e. isentropic). The first two have similar COPs, but the first one essentially implies that the throttle does nothing, and that the evaporation and condensation phases happen at the same temperature, which is ridiculous! Also it would mean that the temperatures of the cold and hot reservoirs would be almost the same.

(d) Next let’s consider the fraction of liquid and vapor at the end of the throttle portion of the cycle (i.e. $3 \rightarrow 4$). Under the assumption that the throttle process is isenthalpic (i.e. enthalpy-preserving), then we have to perform the analysis depending on the strategies adopted above.

- The enthalpy at point 3 is $H_3 = 241.8 \text{ kJ}$, as it also must be at point 4. At $-10^\circ\text{C}$ the pressure is $200.6 \text{ kPa}$ and the enthalpies of the liquid and vapor are $186.7 \text{ kJ}$ and $392.9 \text{ kJ}$, respectively. If $x$ is the fraction of liquid then $186.7x + 392.9(1-x) = 241.8$ which gives $x \approx 0.73$, or almost three quarters liquid after the throttle.

- Using the second strategy, the right hand sides are (a) $186.7 \text{ kJ}$, (b) $212.2 \text{ kJ}$, and (c) $237 \text{ kJ}$, respectively. Solving for $x$ gives (a) $x = 1$, (b) $x \approx 0.88$, and (c) $x \approx 0.76$, respectively.

(e) For the deep freezer we would like the working fluid to go down to a temperature of $-35^\circ\text{C}$, and the high pressure region will be $400 \text{ kPa}$. We have $H_1 = 377.4$. Taking the first strategy, the enthalpy at point 3 is again $241.8 \text{ kJ}$. This is again at $771 \text{ kPa}$, so an upper pressure of $400 \text{ kPa}$ is impossible; so again, $H_2 \approx 418 \text{ kJ}$. The coefficient of performance is then $\text{COP} = (377.4 - 241.8)/(418 - 377.4) \approx 3.3$, which is considerably lower than that for the fridge. This is expected, because the COP decreases as the temperature differences increase. The maximum Carnot COP is $T_c/(T_h - T_c) \approx 3.66$, so this is quite close.

In any case the second strategy seems better. At $400 \text{ kPa}$ the enthalpy at point 2 is again $H_2 = 423.4 \text{ kJ}$. Again by the time it has cooled and condensed at point 3 the enthalpy is $H_3 = H_4 = 212.9 \text{ kJ}$ at only $9^\circ\text{C}$ (again we need the room to be very cold) which gives $\text{COP} = (377.4 - 212.9)/(423.4 - 377.4) \approx 3.58$, which is amazingly close to the Carnot value. This is probably misleading, though, because the maximum temperature of the room (hot reservoir) is only $9^\circ\text{C}$, so the Carnot COP $= T_c/(T_h - T_c) \approx 5.4$, which is quite a bit larger than the ‘actual’ value.

3. (a) If the efficiency is 40% then this means that 60% of the energy ends up as waste heat $Q_c$; this can be obtained using

$$0.4 = \frac{|W|}{|Q_h|} = \frac{|W|}{|Q_c| + |W|}$$

which can be inverted to yield $|Q_c| = (0.6/0.4)|W| = 1.5|W|$. If the plant generates $|\dot{W}| = 1 \text{ GW}$ of electricity, then the rate at which it dumps heat into the environment must be $|\dot{Q}_c| = 1.5 \text{ GW}$.

(b) We know that heat is dumped into the river at a rate of $1.5 \times 10^9 \text{ J/s}$. Each second $100 \text{ m}^3$ of water flows past the heat-dumping site, corresponding to $10^5 \text{ kg}$ of water. So each kilogram of water receives $1.5 \times 10^4 \text{ J}$. The heat capacity of water is $4186 \text{ J/K}$, so the temperature associated with this heat is $1.5 \times 10^4/4186 \approx 3.6 \text{ K}$. 