The theory and use of two fire history models

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Received April 18, 1984
Accepted September 10, 1984


The objective of this paper is to explain the distributions, assumptions, interpretations, and relationships of the two compatible, stochastic models of fire history: the negative exponential and the Weibull. For each model the “fire interval” and “time-since-fire” distributions are given. Both models apply to homogenous stationary stochastic processes. The negative exponential states that the instantaneous fire hazard rate is constant for all stand ages. The Weibull states that the instantaneous fire hazard rate increases with stand age when the shape parameter is >1 (the negative exponential is a special case of the Weibull when shape =1). An empirical method is given for separating from an observed fire history distribution, the pre- and post-fire suppression distributions. Four relationships are derived from the models and defined per study region (per stand): (i) the fire cycle (average fire interval), (ii) the annual percent burned area (fire frequency), (iii) the average age of the vegetation (average prospective life-time), and (iv) the renewal rate.


L’objectif de ce texte est d’expliquer les distributions, les hypothèses, les interprétations et les rapports entre deux modèles stochastiques compatibles de l’histoire des incendies: celui de l’exponentielle négative et celui de Weibull. Les distributions “intervalle entre les feux” et “temps écoulé depuis un incendie” sont données pour chaque modèle. Les deux modèles s’appliquent à des processus stochastiques homogènes et stationnaires. Le modèle de l’exponentielle négative établit que le niveau de risque de feu instantané est constant pour tous les âges d’un peuplement. Celui de Weibull montre que le niveau de risque de feu instantané augmente avec l’âge du peuplement, quand le paramètre de la forme est >1 (le modèle de l’exponentielle négative constitue un cas spécial du modèle de Weibull où la forme est =1). On présente une méthode empirique pour distinguer les distributions avant et après suppression du feu, à partir d’une distribution observée pour une historique connue. À partir des modèles, on a dérivé quatre sortes de rapports qu’on a défini par région d’étude (par peuplement): (i) le cycle du feu (intervalle moyen entre les feux), (ii) le pourcentage annuel de superficie brûlée (fréquence des feux), (iii) l’âge moyen de la végétation (durée moyenne de l’espérance de vie) et (iv) le niveau de renouvellement.

Introduction

The determination of disturbance history, particularly forest fire history, has passed from its largely informal beginning of simply annotating fires from fire scars (e.g., Spurr 1954) to today’s concern of developing a quantitative methodology which will allow a deeper understanding of disturbance and its implication for community dynamics and land management. Heinselman (1973) began this methodological development by mapping past fires as carefully as possible using written records, fire scars, and stand origins. This study laid out the research program which we are still exploring today. Van Wagner (1978) advanced this approach by demonstrating how maps or samples of the present age of the forest, since last fire, could be used to determine one of the statistical distributions of fire history. He also provided a quantitative foundation for estimates of the fire cycle. Van Wagner (1978) and Johnson (1979) suggested two compatible statistical distributions for forest fire history. Arno and Snep (1977) outlined an informal sampling design for areas in which complete inventories of fire history are not possible. Zackrisson (1977), Stokes (1981), and Madany et al. (1982) stressed the problems and techniques of establishing accurate dates of fires using dendrochronological methods.

The objective of this paper is to explain the basic mathematics, assumptions, and strategies for the use of two models describing the stochastic history of fires: the negative exponential (Van Wagner 1978) and the Weibull (Johnson 1979).

Some confusion has arisen in the intention, meaning, and uses of these models (see e.g., Heinselman 1981) which we attempt to clarify, or as the case may be, reveal where our understanding is still deficient. We do not attempt an overall review of fire history studies.

We begin by giving the distributions, parameters, and the present fire ecology interpretations of the two models. Next we state the assumptions which are necessary for the application of the models to fire history studies. This is followed by a discussion of parameter estimation and how to separate different fire histories (e.g., periods of fire suppression) when they are combined in the same distribution. Finally we define several useful fire history concepts that are derived in a consistent manner from the two models. Some basic terminology will be reviewed before beginning.

Differences in responses of forest ecosystems to fires are due to the variable frequency and effects of fire, even over short distances. To classify fires more carefully into categories of similar effects on ecosystems, the idea of fire regimes has developed. For our purpose, fire regime is a multivariate system characterized by (i) the fire history measured in fire
frequency or fire return period, (ii) fire intensity measured in kW m⁻¹, and (iii) depth of burn (duff removed) measured in kg m⁻¹, or percent (cf. Van Wagner 1980).

Fire history is our main concern in this paper. We recognize, however, that the covariance of fire intensity and depth of burn must be considered in fire history studies. In keeping with this, the terms "fire" and "burn" mean the occurrence of fire on the element (see definition below). "Fire" or "burn" thus puts no restriction on the kind of fire intensity (e.g., was the fire lethal or nonlethal to the trees?) or depth of burn (e.g., did recruitment occur because of exposed mineral soil?). Notice that by this definition, fire history studies are not simply the study of fire scars but of any evidence which records the occurrence of fire. This definition of fire or burn is necessary if the variables, fire intensity and depth of burn, are to take on their full range of values. The fullest range of fire intensity and depth of burn must be attempted since many regions have complex fire regimes. For example, many natural ecosystems appear to have both short frequency, low intensity burns and longer frequency, higher intensity burns (Cooper 1960, 1961). Also, changes in fire intensity with little change in frequency may be brought about by certain forestry management practices. Practically, of course, fires of low intensity and little depth of burn must be detectable.

The universe of a fire history study is the area and time period of interest. The universe has a similar fire regime and vegetation. The elements of the universe are the objects on which fire history measurements are made. Elements are often considered point samples, i.e., arbitrarily small areas which are uniform with respect to fire history. As is true in plant ecology (Greig-Smith 1964), the size of the element will determine the frequency (probability) or return period. That is, the larger the element the shorter the return period. Elements are non-overlapping and their complete collection is called the universe. At present the procedure for determining the best universe, element, and sampling design in a fire history study is still an important area of investigation. It may be useful to remember that a universe need not be geographically continuous. A universe could be a collection of dispersed sites, all of which have very similar fire regimes but have significantly different regimes from some of their contiguous sites. A good example in the boreal forest is the peatland and mineral soil division, resulting in a mosaic of areas with different fire regimes.

Models

This section introduces the three fundamental distributions of the Weibull and negative exponential fire history models. The place of the important hazard of burning function in the distributions is given, and its role as a measure of the proneness to fire as a function of age, explained. Finally, the empirical situations in which the different distributions have been used are reviewed.

The Weibull model is expressed as either the fire interval distribution or as the time-since-last-fire distribution (Fig. 1). The fire interval distribution is given in either a cumulative or a probability density form. The cumulative fire interval distribution is

\[ F(t) = 1 - \exp \left[ -(t/b)^c \right], \quad t > 0 \\
\]

F(y) = \begin{cases} 
1 - \exp \left[ -(t/b)^c \right] & b > 0 \\
0 & c = 1 
\end{cases}

where t is time measured as the interval between two fires; b and c, the scale and shape parameters. c is dimensionless while b is dimensioned in time (usually years) and has been called fire recurrence (cf. Johnson 1979). The latter is the fire interval which will be exceeded 36.79% of the time. The cumulative fire interval distribution gives the accumulated frequencies from the shortest interval up to interval t. It is the frequency or probability of having fires with intervals less than age t.

The probability density fire interval function is the frequency or probability of having fires with intervals of age t. It is

\[ f(t) = \frac{d}{dt} F(t) = ce^{-t/b} \exp \left[ -(t/b)^c \right] \]

Figure 1 shows the f(t) function with different shape parameters.

The time-since-fire distribution (Fig. 1) is always presented in its cumulative form

\[ A(t) = 1 - F(t) = \exp \left[ -(t/b)^c \right] \]

The t is interpreted here not as a fire interval but as the time which has passed since the last fire, the “running” time. Because A(t) is the complement of F(t), it is the frequency or probability of not having a fire up to t.

To get the age-class distribution of Van Wagner (1978) for the Weibull model, A(t) must be multiplied by the factor FF = 1/bΓ(1/c + 1) where FF is the fire frequency and Γ is a

\[ F(t) = 1 - \exp \left[ -(t/b)^c \right], t > 0 \\
\]

Fig. 1. The different distributions for the Weibull fire history model. Graphs show distributions with different shape parameter (c) settings and constant fire recurrence or scale parameter (b).

Note that although the dimension of the variable t is the same in different distributions, its interpretation changes.

The idea of an age-class distribution is old in forestry and was first applied to fire history by Van Wagner (1978). An age-class distribution is a proportion of the total universe in different age-classes.
\[ \lambda(t) = \frac{1}{A(t)} \frac{dA(t)}{dt} \]

\[ \lambda(t) = \left( \frac{c}{t} \right)^{c-1} / b^c \]

**Fig. 2.** The hazard function of the Weibull model. The shape parameter \( c \) values are the same as in Fig. 1 and the fire recurrence or scale parameter \( b \) is constant.

gamma function. The age-class distribution or the time-since-fire distribution as a proportion of the universe is

\[ A^*(t) = FF \exp \left[ -\left( t/b \right)^y \right] \]

The fire interval distribution \( f(t) \) can also be written as the product of two distributions:

\[ f(t) = \lambda(t) \cdot A(t) \]

where \( A(t) \) is the cumulative time-since-fire distribution and \( \lambda(t) \) can be called the hazard of burning (Johnson 1979) or simply the hazard distribution:

\[ \lambda(t) = \left( \frac{-1}{A(t)} \right) \frac{dA(t)}{dt} = ct^{-1} / b^y \]

The form of \( \lambda(t) \) is shown in Fig. 2 for various values of \( c \). The \( \lambda(t) \) is the density form of the hazard distribution, the cumulative form is

\[ \Lambda(t) = \int \lambda(t) dt = -\ln \left( 1 - F(t) \right) \]

The hazard of burning is interpreted as follows. The slope at any interval \( dt \) of the \( A(t) \) curve in Fig. 1 is related to the proportion of elements which have burned during that interval. The steeper the slope the greater the number of elements burning. The function \( \lambda(t) \) is then the rate of this loss by burning \( (dA(t)/dt) \) divided by the proportion of elements not having burned yet \( (A(t)) \). That is, the hazard of burning is independent of the number of elements having survived to age \( t \). For this reason it is often called the instantaneous rate of burning. The function \( \lambda(t) \) can be considered as the environmental factors which cause burning. The importance of the hazard function is that it indicates whether the hazard of burning rate stays constant or increases with age.

Notice that we have excluded shape parameter values less than \( c = 1 \) as they make little fire ecology sense. Shape values \( <1 \) indicate a decreasing hazard rate with time.

When the shape parameter \( c = 1 \), the Weibull model is also the negative exponential model and

\[ F(t) = 1 - \exp \left[ -\left( t/b \right) \right] \]

\[ f(t) = \frac{1}{b} \exp \left[ -\left( t/b \right) \right] \]

\[ A(t) = \exp \left[ -\left( t/b \right) \right] \]

\[ A^*(t) = b \exp \left[ -\left( t/b \right) \right] \]

\[ \lambda(t) = \frac{1}{b} \]

The negative exponential is thus a special case of the Weibull model.

In the derivation of the negative exponential model, the discrete variable form is often presented first (see Van Wagner 1978). That is \( f(t) = b(1 - b)^{y-1} \) and \( A(t) = b(1 - b)^y \) where \( b \) is the probability of fire per unit time and \( (1 - b)^{y-1} \) or \( (1 - b)^y \) is the conditional probability of, respectively, not having a fire with intervals \( >t \) or \( \geq t \). The random variable only takes on two forms: fire occurrence or nonoccurrence. When fire intervals or the time since fire are used instead of fire occurrence, the random variable is continuous and [8] to [11] are the correct forms.

The studies by Van Wagner (1978) and Johnson (1979) demonstrate not only the use of two models of fire history, but also two strategies for collecting data. Johnson (1979) worked in areas which had been burned only a short time before. Therefore he was able to observe directly fire interval data and fitted the fire interval distribution \( f(t) \) as the appropriate distribution.

Van Wagner (1978) on the other hand, had data which gave the present age of the forest since fire. Fire intervals were not observed, but instead the length of time the elements had survived since their last fire to the arbitrary present time. In this case the appropriate fitted distribution is \( A(t) \) or \( A^*(t) \). The use of either the \( f(t) \) or \( A^*(t) \) distribution, depending on the data collected, has caused confusion when the difference between probability density and cumulative form of the fire history models was not appreciated (e.g., Heinselman 1981).

**Fire history ideas in mathematical form**

Fire history models should not be judged only as having good empirical fits to some distribution. Empirical fitting is not definitive because tests of goodness of fit often cannot separate different models conclusively. Fire history models must attempt to translate specific ecological ideas into mathematical form. This is why assumptions are so important. If the ecological ideas are useful and the translation into the mathematics is sound, then the models suggest areas for further research and the conclusions advance our understanding of fire history.

Models of fire history are still in the stage of defining the fire ecology ideas and their mathematical form. Thinking clearly about fire history is still difficult for most of us. It is however illusionary to believe that the stochastic processes which create fire history can be dealt with in a storytelling tradition.

**Assumptions**

The negative exponential and Weibull models apply to homogenous stochastic processes. Two important stability criteria must be approximately met for their use in fire history studies.

First, collectively the elements in the study must all have the same fire regime. For example, consider a study area in which a part has been subject to fire protection and the rest has not.
A fire history of this total area would not fit any model which assumes stability of the fire regime. On the other hand, if the area was divided into two parts corresponding to the protected and unprotected areas, the stability criteria may be met.

Second, each element in the study must have on average a constant fire regime during the time-span of the study. An example of a situation where this is not so could be an area which came under fire protection only in the last 50 years. If the time frame of the study is 300 years, at least two fire regimes exist in the study data. How to deal with data of this form is discussed in a later section (see different fire histories in the same distribution).

Both of these assumptions are necessary so as to ensure that the parameters of the models are indeed descriptive of the fire history process over the whole universe during the time span studied.

A fire history study which is believed to meet these stability assumptions has an interesting property. It can be viewed either as the fire history of a hypothetical cohort of elements starting from the same fire and subject to gradual loss by reoccurrence of fire, or it can be viewed as the fire history of the universe viewed at one instance in time.

How hard it is to meet these two assumptions is still unclear, but a few hints may minimize problems. The time period covered by the study should be long enough so that the periodicity of particularly severe or light fire years is small compared with the period of study. Analogously the size of the universe should be large compared with the largest fires recorded in the time frame of the study. For example, a universe is not large enough if it was completely burned by a single fire. Also, time frames longer than 400 years are usually too long since they often contain climatic changes. Forest management, fire suppression, improved access, etc., will also probably change the fire history and thereby invalidate the stability assumption.

Fire model interpretations

The fire ecology justification for the negative exponential and Weibull models are at the present time given as follows. The negative exponential is a "random selection" model. It states that the burning of elements in the universe is Poisson, that is independent of the element's age. A constant proportion of the elements in each cohort is burned each time interval. Thus, no matter how long it has been since the last fire for any element, the instantaneous rate of burning (hazard function) is always the same. This results in a density fire interval distribution \( f(t) \) which has a monotonically decreasing number of elements in each older time (see Fig. 1). The greater number of younger compared with older elements is not due to any change in the rate of burning \( (t) \) at different ages. It is due instead to the annual depletion of each cohort (i.e., the \( A(t) \) distribution).

It follows that the old age tail of the negative exponential distribution \( A(t) \) is made up of stands that have survived by random chance alone, not because of any decrease in flammability with age.

The Weibull is an "age selection" model. It states that the burning of elements in the universe is a power function of the time since fire. The Weibull describes a continuum of fire ecology situations, all of which have a monotonic hazard rate. At the one extreme (when \( c = 1 \)) the model is the "ageless selection" model of the negative exponential, which has already been discussed.

When \( c \) is greater than 1, the instantaneous rate of burning is increasing with age, the rate depending on the value of \( c \). This results in a density fire interval distribution \( f(t) \) which has relatively fewer short intervals because of the low instantaneous rate of burning at younger ages. Then intermediate fire intervals are relatively more frequent because of the rising instantaneous rate of burning as age increases. The frequency of long intervals in turn decreases because of the decreasing number of surviving stands.

Parameter estimation

We have found the following approach to parameter estimation to be useful. (i) The data are plotted on either semilog graph paper for the negative exponential or Weibull probability paper to give a visual conformation of the appropriateness of the model. King (1971) gives a good discussion of how to do this. (ii) The parameters are estimated. (iii) The goodness of fit is tested. For the parameter estimation see below. For a discussion of how to do all three of these steps see Nelson (1982).

The maximum likelihood estimator (MLE) of the negative exponential parameter \( b \) is the sample average \( (\mu) \) of fire interval or time since fire.

The MLE of the Weibull parameters are given by Harter and Moore (1965) and require an interactive algorithm (e.g., Newton-Raphson). MLE of \( c \) and \( b \) are

\[
\hat{b} = \left( \frac{1}{n} \sum_{i=1}^{n} t_i \right)^{1/c}
\]

and

\[
\hat{c} = n \left( \frac{1}{b} \right)^{1/c} \sum_{i=1}^{n} t_i \log t_i - \sum_{i=1}^{n} \log t_i
\]

Note that \( t_1 \leq t_2 \leq t_3 \ldots \leq t_n \).

The method of moments for \( c \) gives

\[
\frac{\mu}{\sigma^2 + \mu} = \frac{\Gamma(1 + 1/c) \cdot \Gamma(1 + 1/c)}{\Gamma(1 + 2/c)}
\]

and

\[
\hat{b} = \frac{\mu}{\Gamma(1 + 1/c)}
\]

where \( \mu \) is the sample average, \( \sigma^2 \) is the sample variance, and \( \Gamma \) is the gamma function. Equation [13] uses the fact that the coefficient of variation of the Weibull is independent of \( b \).

Menon (1963) proposes a regression method which gives the following estimators:

\[
\hat{c} = \frac{6}{\pi} \left[ \frac{\sum_{i=1}^{n} (\log t_i)^2 - (\sum_{i=1}^{n} \log t_i)^2}{n - 1} \right]^{1/2}
\]

\[
\ln b = \frac{\sum_{i=1}^{n} \log t_i}{n} + 0.5772(1/c)
\]

Different fire histories in the same distribution

Most fire history studies today are in regions which have had several different hazard rates owing to periods of fire suppression, European settlement and changes in Indian land uses. These fire history distributions are therefore not stable (see Assumptions), but a combination of two or more distributions. These combination distributions will not fit any model of fire history which assumes a stable, homogeneous stochastic process. To separate these distributions we must first make the following assumptions. (i) The separate distributions are all Weibull (including the negative exponential). (ii) Each hazard
rate produces a measurable difference in the fire history and this effect was relatively similar throughout the region of study. The fire frequencies from the different hazard rates were statistically independent.

For a study in which these assumptions are met, the combined time-since-fire distribution is the product of the separate time-since-fire distributions:

\[ A(t) = A_1(t) \cdot A_2(t) \]

The hazard function is the sum of the different hazards:

\[ \lambda(t) = \lambda_1(t) + \lambda_2(t) \]

The best way to separate fire history distributions with different hazard rates is to plot the data on Weibull probability paper. The straight line parts of these graphs may indicate the different fire hazard rates. As an example, consider the time-since-last-fire distribution \( A(t) \) plotted on Weibull probability paper from the data given by Heinselman (1973) for the fire history in the Boundary Waters Canoe Area (Fig. 3). For the data to approximate a Weibull (including the negative exponential), the points should form a straight line. This is not possible for the whole series of data but three separate straight lines can be drawn. The top line corresponds to the period 1972–1911 called the suppression period by Heinselman (1973), the next line corresponds to the period 1910–1868 called the settlement period, and the last line corresponds to the pre-settlement period.

A plot such as Fig. 3 is useful for determining the data which belong to different hazard rates but it does not allow a comparison of these different distributions on a common basis. This can be done using the following procedure. Choose the transition dates which mark the beginning and the end of each period. Eliminate the data which do not fall between these dates. Recalculate the frequencies for this reduced data set as if it constituted 100% of the area. For examples, see Van Wagner (1978).

When plotting age-class data, two hints are worth noting. (i) When the age classes are not constant over the whole range of ages, the whole set must be normalized in terms of one chosen standard class width. Divide each frequency by its age-class width and multiply by the standard width. (ii) The entire area and age range must be accounted for in a rational manner. The old-age “tail” must not be ignored, and open-ended classes must be assigned an average age.

Fire history concepts

A number of concepts have arisen from fire history studies to describe relationships in the data. In this section we define concepts which are internally consistent with the negative exponential and Weibull models. The criteria for defining a concept are a mathematical relationship to the models and specific dimensions.

As we have already pointed out (see Assumptions), the fire history models can be interpreted on a per element basis or in terms of the proportion of the universe. Because of these different viewpoints, different terms have arisen for the area (universe) interpretation and for the point (element) interpretation. The fire cycle — average fire interval and annual percent burned — fire frequency represent two of these couples discussed here.

Since by definition frequency is the inverse of the return period, the fire cycle (average fire interval) is the inverse of the annual percent burned (fire frequency).

The fire cycle is the time required to burn an area equal to the universe. It is defined in this manner since during a fire cycle, some elements may not burn at all and some may burn more than once. Thus the fire cycle is not equivalent to burning each element once. For the Weibull model, it is

\[ \int_0^\infty A(t) \, dt = b \Gamma(1/c + 1) \]

where \( \Gamma \) is the gamma function. For the negative exponential it is simply \( b \), since \( c = 1 \) and \( \Gamma(2) = 1 \) in [21]. The derivation of [21] is found in the appendix.

The average fire interval is the expected return period per element and is the same as [21]. Both the fire cycle and average fire interval are dimensioned in time, usually years.

The annual percent burned is the proportion of the universe that burns per unit time and the fire frequency is the probability of an element burning per unit time. Both are the inverse of [21]:

\[ FF = 1/[b \Gamma(1/c + 1)] \]

One will remember that [22] was used as a coefficient in [4] to give the age-class distribution of Van Wagner (1978).

Another valuable concept is the average prospective lifetime of an element:

\[ APL = \int_0^\infty A(t) \, dt = b \Gamma(2/c) / \Gamma(1/c) \]

The lifetime, of course, refers to the time between fires. The derivation of this can be found in the appendix. The average prospective lifetime is identical to the centroid of the Weibull and thus could also be called the average age of the forest.

Finally, often an investigator wishes to know the expected number of times an element burns up to and including some time \( t \). This is specified by the renewal function which is defined as

\[ R(t) = F(t) + \int_0^t R(t-z) \, dz \]

Smith and Leadbetter (1963) give the arduous procedure for calculating this for the Weibull model.

For the negative exponential, the renewal function is

\[ R(t) = \frac{1}{n_1} (bt)^n \exp(-bt) \]
where \( n \) is the \( nth \) event. Thus \( R(t) \) gives the probability at time \( t \) of no fire. Of course the renewal function of the negative exponential is the Poisson distribution.

**Conclusion**

One obvious topic, the spatial aspect of fire history, has not been addressed. This subject is somewhat beyond this paper, primarily because at this time research is very limited. However, spatially changing hazard rates might be considered as the same kind of problem as temporally changing hazard rates.

Traditionally, spatial differences in fire histories have been thought of in terms of landscape units with different fire histories. These are mixed Weibull distributions which can be thought of in the form

\[
A(t) = p_1 A_1(t) + p_2 A_2(t)
\]

where \( p_i \) is the proportion of the different landscape units.

It might be as useful to still consider a landscape as consisting of a combination of Weibull distributions, as we did for temporal changes in hazard rates. Then [19] and [20] would suggest independent but temporally competing hazards.

**Acknowledgments**

We would like to thank Professor Dennis Parkinson, Head of the Biology Department, University of Calgary, for encouraging this collaboration by supporting a visit of C.E.V.W. to Calgary. The manuscript was improved by the comments of A. Gore, O. Loucks, and P. Zedler. E. A. Johnson thanks the Natural Sciences and Engineering Research Council of Canada for supporting his research.

**References**


**Appendix**

The following is the deviation of the fire cycle and average prospective lifetime for the Weibull distribution. We thank Duncan MacLeod of Environment Canada, Ottawa, for this deviation.

By definition the gamma function is

\[
\Gamma(n) = \int_0^\infty x^{n-1} \exp(-x) dx
\]

and

\[
\Gamma(n+1) = n \Gamma(n)
\]

The fire cycle is

\[
\int_0^\infty \exp(-t/b) \, dt = b \int_0^\infty \exp(-u) \, du
\]

where \( u = t/b \) and \( dt = b \, du \). Let \( v = u^n \), then \( du = dv/cu^{n-1} \) and \( u = v^{1/n} \). Then,

\[
\int_0^\infty \exp(-t/b) \, dt = b \int_0^\infty \exp(-v) \, dv/cu^{n-1}
\]

\[
= \frac{b}{c} \int_0^v \frac{1}{u} \exp(-v) \, dv
\]

\[
= \frac{b}{c} \int_0^v \frac{1}{v^{(\frac{n-1}{n})}} \exp(-v) \, dv
\]

\[
= \frac{b}{c} \int_0^v v^{-(\frac{n-1}{n})} \exp(-v) \, dv = b/c \int_0^\infty v^{(\frac{-1}{n}-1)} \exp(-v) \, dv
\]
which is \( b/c\Gamma(1/c) \) by definition of the gamma function. The average prospective lifetime is

\[
\frac{\int_0^\infty t \exp \left( -t/b^c \right) dt}{\int_0^\infty \exp \left( -t/b^c \right) dt}
\]

The denominator is equal to the fire cycle and the nominator is equal to

\[
\int_0^\infty t \exp \left( -t/b^c \right) dt = \int_0^\infty (bu) \exp \left( -u/b \right) bdu
\]

\[
= b^3 \int_0^\infty u \exp \left( -u/b \right) du = b^3 \int_0^\infty u \exp \left( -v/c \right) cdu
\]

\[
= b^3/c \int_0^\infty u^{1-c} \exp \left( -v/c \right) dv
\]

\[
= b^3/c \int_0^\infty u^{2-c} \exp \left( -v/c \right) dv = b^3/c \Gamma \left( \frac{2-c}{c} \right)
\]

Therefore, the average prospective lifetime is

\[
\frac{b^3/c\Gamma(2/c)}{b/c\Gamma(1/c)} = b\Gamma(2/c)/\Gamma(1/c)
\]