The Binomial Option Pricing Model

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Introduction

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Introduced by John C. Cox, Stephen A. Ross, and Mark Rubinstein in their 1979 paper: "Option Pricing: A Simplified Approach."

Provides a simple approach to pricing options

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Definitions

- European (aka Vanilla) Call option: This is a contract giving the holder (purchaser) of the option the right, but not the obligation, to buy the stock $S(T)$ at the expiry time $T$ for a fixed (strike) price $K$.

- Option payoff: the amount the option holder receives at time $T$; i.e., for a European Call option we have:

  \[ \text{payoff} = (S(T) - K)^+ = \max(S(T) - K, 0) \]
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Call Option Payoff and Profit

Definitions

![Diagram showing the relationship between Price at Maturity (S_T), Strike Price (K), Option Payoff, and Profit.]
Assumptions

- Constant (risk free, continuously compounded) interest rate $r$, i.e. the accumulated value at time $t = T$ of $1$ invested in a savings account at time $t = 0$ is $AV = e^{rT}$
- No transaction costs
- Liquid market
- No dividend payments
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Binomial model with 1 time step

Consider the following simple model:

- The stock price $S(T)$ can take one of two possible values at time $t = T$
- The probability of the two outcomes is known
- Zero interest rate for the period $t = 0$ to $t = T$
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We note that expected value of the $S(T)$ at time $t = 0$ is:

$$E[S(T)] = 110p^* + 90(1 - p^*)$$
$$= 108$$
Can we use this model to price Options?

Recall that the holder (purchaser) of a call option with maturity at time $T$ receives payoff given by $(S(T) - K)^+$ at time $T$.

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Binomial model with 1 time step

Suppose the option has strike price \( K = 95 \)

The expected option payoff at time \( t = 0 \) is:

\[
E[(S(T) - 95)^+] = 15p^* + 0(1 - p^*) = 13.5
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Is this a good “fair” price for our option?
The Binomial Option Pricing Model

Binomial model with 1 time step

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Let us suppose that Party A thinks that this is indeed a good method for valuing options.

- Party A is 90% sure that the stock will be worth $110 at time $T$.
- Party A would like to buy the stock for 100, however, he only has $15 in his pocket.
- In order to take advantage of his hunch, Party A decides to buy the option for $13.5.
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Party B, on the other hand, thinks that Party A has mispriced the option and sees an opportunity

- Party B agrees to sell Party A the option for $13.5
- After selling Party A the option, Party B takes out a loan for $86.5 (recall we are assuming zero interest)
- Party A then buys the stock (for 100) using her loan and the $13.5 received from Party A
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**Case I:** $S(T) = 110$

*Party A:* makes a profit

$\Rightarrow S(T) - K - 13.5 = 1.5$

*Party B:*
- Sells the stock for 110
- Pays Party A 15
- Pays back the loan of 86.5
- Makes a profit of 8.5

**Case II:** $S(T) = 90$

*Party A:* suffers a loss

$\Rightarrow 13.5$

*Party B:*
- Sells the stock for 90
- Pays back the loan of 86.5
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In order to compute a fair option price using the above setup we need the following condition:

\[ E_Q(S(T)) = S(0) \]

where \( E_Q(S(T)) = 110p + 90(1 - p) \) and \( p \) is the risk neutral probability.

- If we solve the above for \( p \), we may then compute the arbitrage free option price by calculating the expected payoff of the option.

- In this example, we have \( p = 1/2 \) and
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In a more general setting, the above risk neutrality condition still holds, regardless of the model.

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where \( r \) is the risk neutral interest rate.

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- However, in most realistic models stock price models, we will have infinitely many arbitrage free option prices
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- Wikipedia