LATTICE TENSOR PRODUCTS. II
IDEAL LATTICES

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Abstract. G. Grätzer and F. Wehrung has recently introduced the lattice tensor product, $A \boxtimes B$, of the lattices $A$ and $B$. In this note, for a finite lattice $A$ and an arbitrary lattice $B$, we compute the ideal lattice of $A \boxtimes B$, obtaining the isomorphism $\text{Id}(A \boxtimes B) \cong A \boxtimes \text{Id} B$. This generalizes an earlier result of G. Grätzer and F. Wehrung proving this isomorphism for $A = M_3$ and $B$ n-modular.

We prove this isomorphism by utilizing the coordinatization of $A \boxtimes B$ introduced in Part I of this paper.

1. Introduction

G. Grätzer and F. Wehrung [3] introduced the lattice tensor product, $A \boxtimes B$, of the lattices $A$ and $B$. In Part I of this paper, [2], we showed that for any finite lattice $A$, we can “coordinatize” $A \boxtimes B$, that is, represent $A \boxtimes B$ as a subset $A(B)$ of the direct power $B^A$ of $B$, and provide an effective criteria to recognize the $A$-tuples of elements of $B$ that occur in this representation.

In this note, we find a new application for this coordinatization:

**Theorem.** Let $A$ and $B$ be lattices. If $A$ is finite, then

$$\text{Id}(A \boxtimes B) \cong A \boxtimes \text{Id} B.$$  

Or, equivalently,

$$\text{Id}A(B) \cong A(\text{Id} B).$$

In G. Grätzer and F. Wehrung [4], a very special case of this isomorphism appears: $A = M_3$ and $B$ is n-modular.

To keep this paper short, we assume that the reader is familiar with the concepts introduced and reviewed in Part I [2]; Section 2 of Part I will be referenced as Section I.2; Lemma 3 of Part I will be referenced as Lemma I.3. We also refer the reader to [2] for the background of this research. For the general concepts and notation, consult G. Grätzer [1].

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The logical structure of our proof of this theorem is identical to the structure of the proof that the isomorphism $\text{Id}B^4 \cong (\text{Id}B)^4$ holds for a finite set $A$. However, in our argument, we must make additional claims to ensure that various maps that appear do belong to $A\langle B \rangle$ or to $A\langle \text{Id}B \rangle$. The coordinatization is of value not just because it facilitates computation and hence the proofs of these claims, but also because having the lattice tensor product represented as a subset of $B^4$ actually suggests this very natural line of argument.

We use the notation of Part I. In particular, the zero of $A$ will be denoted by 0.

2. Proof

Let $I : A \to \text{Id}B$, and let $\alpha \in B^4$. We call $\alpha$ a selection function for $I$, if $\alpha(a) \in I(a)$, for all $a \in A$. We shall also say that $\alpha$ selects $I$, in symbol, $\alpha \leadsto I$. Let $\text{Sel}I$ denote the set of all functions $\alpha$ selecting $I$. Note that $\text{Sel}I$ is an ideal of $B^4$. For $\alpha \in B^4$, the closure $\bar{\alpha} \in A\langle B \rangle$ was defined in Section I.2.2.

**Claim 1.** Let $I \in A\langle \text{Id}B \rangle$. Then $\alpha \leadsto I$ iff $\bar{\alpha} \leadsto I$.

**Proof.** Let $\alpha \leadsto I$. Then by Lemma I.3,

$$\bar{\alpha}(a) = \bigwedge_{x \in A} \bigvee_{y : x \notin y \neq x} \alpha(y) \in \bigwedge_{x \in A} \bigvee_{y : x \notin y \neq x} I(y) = \bar{I}(a) = I(a),$$

since $I$ is closed. So $\bar{\alpha} \leadsto I$. The reverse implication is trivial. $\square$

Let $\text{ClSel}I$ be the collection of all closed maps $\alpha$ selecting $I$, that is,

$$\text{ClSel}I = \text{Sel}I \cap A\langle B \rangle = \{\alpha \in A\langle B \rangle | \alpha \leadsto I\}.$$  

Let $\chi : A\langle \text{Id}B \rangle \to \text{Id}A\langle B \rangle$ be defined by $\chi : I \mapsto \text{ClSel}I$, for $I \in A\langle \text{Id}B \rangle$. We are going to prove that $\chi$ is an isomorphism of $A\langle \text{Id}B \rangle$ and $\text{Id}A\langle B \rangle$, verifying the Theorem.

First, we show that $\chi$ is well-defined.

**Claim 2.** If $I \in A\langle \text{Id}B \rangle$, then $\text{ClSel}I$ is an ideal of $A\langle B \rangle$.

**Proof.** Since $\text{Sel}I$ is hereditary in $B^4$, it follows that $\text{ClSel}I$ is hereditary in $A\langle B \rangle$. Let $\alpha, \beta \in \text{ClSel}I$. Since $\text{Sel}I$ is an ideal of $B^4$, it follows that $\alpha \vee_B \beta \in \text{Sel}I$. Thus, by Claim 1,

$$\alpha \vee_{A\langle B \rangle} \beta = \alpha \vee_B \beta \in \text{ClSel}I.$$  $\square$

It is clear that $\chi$ is order preserving. It follows trivially from the next claim that $\chi$ is one-to-one.
CLAIM 3. Let \( a \in A \) and \( I \in A(\text{Id} B) \). Then

\[
I(a) = \{ \alpha(a) \mid \alpha \in \text{ClSel} I \}.
\]

PROOF. By the definition of a selection function, it follows that

\[
I(a) \supseteq \{ \alpha(a) \mid \alpha \in \text{ClSel} I \}.
\]

Conversely, let \( x \in I(a) \). For each \( b \neq a \), let \( x_b \in I(b) \). Define the map \( \alpha \in B^A \) by

\[
\alpha(c) = \begin{cases} 
    x, & \text{if } c = a; \\
    x \land \bigwedge_{b \neq a} x_b, & \text{if } c \neq a.
\end{cases}
\]

Then \( \alpha \bowtie I \). By Claim 1, \( \bar{\alpha} \bowtie I \) also holds. By its construction, \( \alpha \leq \kappa_x \in A(B) \) (\( \kappa_x \) was defined in Section I.5.1), implying that \( \bar{\alpha} \leq \kappa_x \). Since \( \alpha(a) \leq \kappa_x(a) = x \), it follows that \( \bar{\alpha}(a) \leq x \) as well. So \( x = \alpha(a) \leq \bar{\alpha}(a) \leq x \), yielding that \( \bar{\alpha}(a) = x \). This verifies the reverse inclusion. \( \Box \)

COROLLARY. \( I \mapsto \text{ClSel} I \) is a one-to-one map of \( A(\text{Id} B) \) into \( \text{Id} A(B) \).

The next four claims shall verify that \( \chi \) is onto.

For \( \alpha \in B^A \), let \( \pi_a(\alpha) \) be the projection of \( \alpha \) onto its \( a \)-th coordinate.

CLAIM 4. Let \( W \) be an ideal of \( A(B) \). Then

\[
(2) \quad \pi_a(W) = \{ \alpha(a) \mid \alpha \in W \}
\]

is an ideal of \( B \).

PROOF. Let \( x \in \pi_a(W) \) and \( x' \leq x \). Find an element \( \alpha \in W \) such that \( \alpha(a) = x \). Let \( \beta = \alpha \land \kappa_x \). Then \( \beta \in W \), since \( W \) is an ideal of \( A(B) \). Also, \( \beta(a) = x' \), so \( x' \in \pi_a(W) \).

Now let us assume that \( x^1, x^2 \in \pi_a(W) \). Find \( \alpha_1, \alpha_2 \in W \) such that

\[
x^1 = \alpha_1(a) \quad \text{and} \quad x^2 = \alpha_2(a)
\]

and

\[
o = \bigwedge \alpha_1(A) \land \bigwedge \alpha_2(A).
\]

Let \( q = x^1 \lor x^2 \). Define

\[
\beta_i(y) = \begin{cases} 
    x^i, & \text{if } y \leq a; \\
    \alpha, & \text{otherwise},
\end{cases}
\]

for \( i = 1, 2 \). Again, for \( i = 1, 2 \), since \( \alpha_i \) is order reversing, it follows that \( \beta_i \leq \alpha \); we also have \( \beta_i \in A(B) \), since \( \beta_i = x^i \alpha \). (For \( p \in B \) and \( a < b \) in \( A \),

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Section I.5.1 defined the element $\bar{p}_{a,b}$ of $A(B)$. Since $W$ is an ideal of $A(B)$, it follows that

$$\beta = \beta_1 \vee_{A(B)} \beta_2 \in W.$$ 

But

$$\beta_1 \vee_{B^1} \beta_2 = \bar{q}_{\bar{a}} \in A(B).$$

Therefore,

$$\beta = \beta_1 \vee_{A(B)} \beta_2 = \beta_1 \vee_{B^1} \beta_2,$$

so

$$\beta(a) = q = x^1 \vee x^2 \in \pi_a(W).$$

We conclude that $\pi_a(W)$ is an ideal of $A(B)$.

For $W \subseteq \text{Id} A(B)$, define $\pi(W) \subseteq (\text{Id} B)^A$ by $\pi(W)(a) = \pi_a(W)$.

**Claim 5.** CLSel $\pi(W) = W$.

**Proof.**

(3) CLSel $\pi(W) = \{ \alpha \in A(B) \mid \alpha(a) \in \pi_a(W), \text{ for all } a \in A \}$

$$= \{ \alpha \in A(B) \mid \text{ for all } a \in A, \text{ there exists } \beta_a \in W \text{ with } \alpha(a) = \beta_a(a) \} \supseteq W;$$

indeed, if $\alpha \in W$, we take $\beta_a = \alpha$, for all $a \in A$.

Let $\alpha \in \text{CLSel} \pi(W)$. For each $a \in A$, find $\beta_a \in W$ with $\alpha(a) = \beta_a(a)$. As in the last proof, define

$$\alpha = \bigwedge \alpha_1(A) \land \bigwedge \alpha_2(A)$$

and

$$\beta^\alpha_a(x) = \begin{cases} \alpha(a), & \text{if } x \leq a; \\ o, & \text{otherwise.} \end{cases}$$

Since $\alpha$ is order reversing, it follows that $\beta^\alpha_a \leq \beta_a$, which implies that $\beta_a \in W$. Let $\beta = \bigvee_{B^1} \beta^\alpha_a$. We want to show that $\beta \in A(B)$. For $c \in A$, compute:

$$\beta(c) = \bigvee_{a \in A} \beta^\alpha_a(c) = \bigvee_{a \geq c} \alpha(c) = \alpha(c),$$

since $\alpha$ is order reversing. Thus $\alpha = \beta \in A(B)$ and $\beta = \bigvee_{B^1} \beta^\alpha_a \in W$, so $\alpha = \beta \in W$. □

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CLAIM 6. If \( \alpha \sim \pi(W) \), then \( \bar{\alpha} \sim \pi(W) \).

PROOF. Let \( \alpha \sim \pi(W) \). Then for each \( a \in A \), we can find \( \beta_a \in W \) such that \( \alpha(a) = \beta_a(a) \). Let \( \beta = \bigvee_{A \subseteq B} \beta_a \in W \). Since \( W \) is an ideal of \( A(B) \) and \( \alpha \leq \beta \in A(B) \), it follows that \( \beta \in W \). Therefore, \( \bar{\alpha} \leq \beta \) and, consequently, \( \bar{\alpha} \in W \). Thus \( \bar{\alpha} \sim \pi(W) \). \( \square \)

CLAIM 7. \( \pi(W) \in A(\text{Id } B) \).

PROOF. By Lemma 1.3, it suffices to show that \( p(\pi(W)) \subseteq \pi(W) \). For \( a \in A \), compute (in \( \text{Id } B \)):

\[
p_a(\pi(W)) = \bigwedge_{x \nmid a} \bigvee_{y \nleq x} \pi_a(W)
= \bigcap_{x \nmid a} \bigcup_{\alpha \in \text{Sel} \pi(W)} \left( \bigvee_{y \nleq x} \alpha(y) \right)
= \bigcup_{\nu : A \to \text{Sel} \pi(W)} \bigcap_{x \nmid a} \left( \bigvee_{y \nleq x} \nu(x)(y) \right).
\]

Let \( t \in p_a(\pi(W)) \). To define a map \( \nu : A \to \text{Sel} \pi(W) \), for each \( x \nmid a \), choose \( \alpha_x \in \text{Sel} \pi(W) \) with \( t \leq \bigvee_{y \nleq x} \alpha_x(y) \). Let \( \alpha = \bigvee_{B \subseteq A} \alpha_x \). Since \( \alpha_x \sim \pi(W) \), for each \( x \nmid a \), we also have \( \alpha \sim \pi(W) \). By Claim 6, \( \bar{\alpha} \sim \pi(W) \) also holds. Therefore, \( \bar{\alpha}(a) \in \pi_a(W) \). By the construction of \( \alpha \), we have the inequality \( t \leq \bigvee_{y \nleq x} \alpha(y) \), for each \( x \nmid a \), so

\[
t \leq \bigwedge_{x \nmid a} \bigvee_{y \nleq x} \alpha(y) = \bar{\alpha}(a) \in \pi_a(W).
\]

Now \( \pi_a(W) \) is an ideal of \( B \), so \( t \in \pi_a(W) \). Thus we have proved that \( p_a(\pi(W)) \subseteq \pi_a(W) \). \( \square \)

COROLLARY. The map \( \chi \) is onto.

PROOF. If \( W \in \text{Id } A(B) \), then \( \pi(W) \in A(\text{Id } B) \) and \( W = \chi \pi(W) \), by Claim 6. \( \square \)

Finally, \( \pi \) is order preserving and \( \pi = \chi^{-1} \), so \( \chi \) is an isomorphism, completing the proof of the Theorem.
References


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