American Option Pricing When the Underlying Assets Follow Augmented GARCH Processes and Their Diffusion Limits

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Abstract

Empirically the constant volatility model of Black & Scholes (1973) is found to suffer from a number of biases, and in the subsequent option pricing literature extensions have been made allowing for e.g. stochastic volatility (SV). In the time series literature time varying volatility has been modelled using the generalized autoregressive conditional heteroskedastic (GARCH) processes of Engle (1982) and Bollerslev (1986), and it is probably fair to say that from an econometric point these models are considerably easier to estimate than the continuous time SV models. In the present paper we use the Augmented GARCH framework of Duan (1997) to bridge the gap between the discrete time GARCH models and the continuous time SV models since important versions of the latter type can be obtained as diffusion limits of certain GARCH formulations. For the American option pricing problem a Monte Carlo study is performed to examine the differences between the discrete time models and the corresponding diffusion limits. Finally, a large scale empirical analysis is performed for three individual stock options comparing the estimated prices based on discrete time to the corresponding ones from continuous time models. The results indicates that the continuous time SV models generally perform better than the discrete time GARCH specifications although the differences are generally small.

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1 Introduction

In the seminal paper by Black & Scholes (1973) a closed form solution for the price of an European option is derived. The Black-Scholes formula has been celebrated as one of the major successes of modern financial economics, although empirical analysis has pointed towards several systematic pricing errors. In particular, numerous studies have documented smiles in the implied volatility as a function of moneyness as well as a tendency for the constant volatility model to underprice in particular short term out of the money options, and a number of alternative models have been developed in response to these “empirical regularities”. In particular, the assumptions underlying the Black-Scholes model have been widely criticized, and much effort has been put into extending the valuation framework. Apart from the assumption of continuous trade, a crucial assumption in the Black-Scholes model is that of constant volatility lognormality. However, the constant volatility lognormal model fails to explain a number of empirical regularities found in asset return series, the most important of which are leptokurtosis and the volatility clustering phenomenon (see Bollerslev, Engle & Nelson (1994)).

In the option pricing literature some of the earliest extensions to the Black-Scholes model are the continuous time stochastic volatility, or SV, models of Hull & White (1987), Wiggins (1987), Scott (1987), Stein & Stein (1991), and Heston (1993). More recent studies include, to name a few, Bakshi, Cao & Chen (1997) and Bates (2000) In these papers volatility is modelled as a separate stochastic process and this allow for a large degree of flexibility in the specification. A particularly appealing feature of these models is that under certain assumptions elegant solutions can be derived for the price of an European option. In some cases the pricing formulas are approximately closed form solutions of the same general type as the Black-Scholes formula with integrals of the stochastic volatility.

However, in real applications a problem in the continuous time SV models is that volatility is unobservable and hence estimation of this type of models is relatively complicated. Examples include the Efficient Method of Moments, or EMM, procedure of Gallant & Tauchen (1996) and more recently procedures using Markov Chain Monte Carlo, or MCMC, has been applied by Jacquier, Polson & Rossi (1994) and Jacquier, Polson & Rossi (2004). Moreover, when pricing claims for which numerical procedures are needed, e.g. as it is the case with an American option, future values of the unobservable volatility is needed. This variable is potentially very complicated to predict, and such procedure may thus require e.g. the full reprojection machinery associated with the EMM procedure (see Gallant & Tauchen (1998)).

In the time series literature several competing discrete time stock return models have been developed which can take account of these empirical regularities for the return series. A large number of these has been within the framework of autoregressive conditional heteroskedastic (ARCH) processes suggested by Engle (1982) and the generalized ARCH (GARCH) models introduced by Bollerslev (1986). A particular appealing feature with these models is that empirical data are readily available for estimation and this can be done with simple Maximum Likelihood, or ML, procedures. These models have been successfully applied to financial data such as stock return data as seen from Bollerslev, Chou & Kroner’s (1992) survey article with reference to several hundred papers.

However, in applications to option pricing the GARCH framework is somewhat complicated because generally no closed form solutions exist (although see Heston & Nandi (2000)). Thus, although the appro-
appropriate dynamics were derived in Duan (1995) numerical methods have to be used for the actual pricing and simulation methods have been used extensively. In particular, options on the Standard and Poor’s 500 stock market index have been the focus of a large part of this research, and various GARCH specifications have been used by Heston & Nandi (2000), Hsieh & Ritchken (2005), and Christoffersen & Jacobs (2004). Also, Bollerslev & Mikkelsen (1996) and Bollerslev & Mikkelsen (1999) have successfully used the GARCH option pricing framework together with fractionally integrated GARCH processes to price long-term European style equity anticipation securities (LEAPS) on this particular index.

In spite of the recent progresses in terms of available estimation techniques for continuous time SV models and in terms of option pricing models in the discrete time GARCH models there still appears to be a gap between the two strands of the literature. In particular, it is probably fair to say that it is far from the standard to report estimation results of both discrete time and continuous time return models with corresponding properties. Likewise, when pricing options with continuous time SV model the results are usually only compared to other models formulated in continuous time and when pricing options with discrete time GARCH models the results are usually only compared to other models formulated in discrete time. Thus, the question of which models is preferable remains an open one.

In the present paper we try to bridge this gap between the discrete time approach and the continuous time approach using the Augmented GARCH model of Duan (1997) and the option pricing model of Duan (1996) as the basic framework. To be specific, we carefully choose a number of well known special cases of the Augmented GARCH framework for which it can be shown that a number of well known stochastic volatility models extending the Black-Scholes framework are obtained in the limit. In doing so it is possible to compare option prices calculated in discrete time to those calculated from a number of well known continuous time models like those of Hull & White (1987), Wiggins (1987), Scott (1987), Stein & Stein (1991), and Heston (1993). First of all we provide estimation results for the GARCH models as well as for the SV models which is obtained by appropriately reparemeterizing the discrete time specifications. Secondly, the paper details how the pricing can be done for American options using the Least Squares Monte Carlo method of Longstaff & Schwartz (2001) and compare the properties of the discrete time models to their continuous time counterparts through a Monte Carlo study. Compared to the Black-Scholes case both types of model produce significantly different prices indicating the potential value of the extension allowing for time varying volatility, and the results also indicate that there are in fact differences between the various GARCH models and their continuous time counterparts. Finally, the paper contains a large scale empirical analysis using highly traded options on a number of individual stocks. The analysis shows that the continuous time models provides improvements in terms of pricing errors to what is obtained with the discrete time GARCH framework. However, the improvements are generally small and since the pricing is computationally more demanding it could be argued that the discrete time GARCH models provides the most reasonable alternative.

The rest of the paper is organized as follows: In Section 2 the option data and the return data are introduced and the empirical regularities are documented for the options and for the returns. In Section 3 the Augmented GARCH framework is introduced and estimation results are provided. Section 4 describes how option pricing can be performed using simulation methods. In Section 5 the results of an extensive Monte Carlo study of the properties of the Augmented GARCH option pricing model are reported. In particular,
the gains from allowing for different volatility processes within the Augmented GARCH framework compared to the constant volatility are analyzed together with what is obtained when the diffusion limits are used. In Section 6 the GARCH option pricing model is applied to the data, and Section 7 concludes. Tables and Figures can be found in the corresponding appendix.

2 Empirical findings and regularities

The option data we will be considering in the empirical part of the present paper have been extracted from the Berkeley Option DataBase (BODB). This database contains, among other observations, all quotes and trades during the day. The amount of data is substantial, and at the present we limit the period under consideration to January, 1991 through December, 1995. After this the Chicago Board Options Exchange (CBOE) discontinued the supply of data to BODB. We choose a sample of major US stocks from the Dow Jones Industrial Index for which options are actively traded. To be specific, we use General Motors (GM), International Business Machines (IBM), and Merck & Company Inc. (MRK). The reason for choosing these three stocks is that for the period under consideration options on these three stocks were the most traded in terms of actual trades as well as in terms of total volume.

2.1 Option characteristics and some empirical regularities

We take data on a weekly basis, selecting each Wednesday. If the Wednesday is a no trade day or no options for the particular asset were traded, we pick the Thursday immediately after. At any particular day we sample an end of day observation for all contracts, that is combinations of strike price and maturity, for which the traded volume during the day was at least five contracts. This is not a guarantee against thin trade effects, but it should go quite a way in terms of minimizing the problem. Furthermore, the choice of five contracts is more than what has been used in the previous literature. For each contract the observed end of day price is the bid-ask midpoint immediately before 3 PM. Valid quotes are the ones for which the bid-ask midpoint conforms to the simple arbitrage bounds in Hull (2000). We end up with 1835 options for GM, 4745 for IBM, and 1844 for MRK. The characteristics of the option data are shown in Table 1. We note that the sample considered corresponds to that used in Stentoft (2005).

To get an idea about the characteristics of the options we first split them into three categories of maturity denoted $T$. Short term, or ST, options are options with maturities of less than 42 trading days corresponding roughly to 2 months. The middle term, or MT, is maturities of more than 42 but less than 126 trading days. Long term, or LT, maturity options are the rest, that is options with more than 126 trading days, the equivalence of 6 months, to maturity. In terms of maturity, the individual stock option data is much alike across the stocks although slightly fewer of the options on IBM fall within the short term category. The major difference in terms of maturity occurs between the put and call options. For the put options around 60% are short term, but this proportion is around 10 percentage points smaller for the call options.

We also split the options into five different categories of moneyness denoted $Mon$ which we define as the ratio between the asset price, $S$, and the discounted strike price, $K$, for the call options. Thus, we set $Mon = S / (K \times \exp (-r \times T))$ where $r$ is the risk free interest rate. For the put options we define moneyness
inversely and set \( \text{Mom} = (K \ast \exp(-r \ast T))/S \). Options that are deep in the money, or D-ITM, will have relative high moneyness values and we choose 1.06 as the cut of point. Options that are in the money, or ITM, have a moneyness value between 1.02 and 1.06, and in the same manner we define deep out of the money, or D-OTM, options as having a moneyness of less than 0.94 and out of the money, or OTM, options as those options with a moneyness between 0.94 and 0.98. At the money, or ATM, options are defined broadly as options with a moneyness of between 0.98 and 1.02. Generally, the options are split equally between the different moneyness categories, although a relatively large fraction of the put options are deep out of the money instead of deep in the money. This holds for the individual options as well as when all the individual stock option are considered together.

Previous research has documented a systematic relationship between the mispricing observed empirically by the constant volatility models and moneyness, the so-called smile effect, as well as the maturity effect, which refers to a systematic relationship between time to maturity and the mispricing. These findings have become almost empirical regularities and are often analysed in terms of the implied volatilities obtained by inverting the constant volatility option pricing formula. For the present application it is however more convenient to use the relative bias. Thus, in order to examine these effects for the sample of options considered in this paper, Table 2 reports the relative bias of a constant volatility model estimated on the historical return data when compared with the option prices actually observed. The table indicates a clear pattern in the bias for particularly the individual stock options. This pattern is confirmed by Figure 1 which plots the relative bias surface as a function of the moneyness and the maturity. From this figure it is clear that at least for these options there is a systematic relation between the moneyness and the bias as well as between the maturity and the bias.

2.2 Return data and empirical regularities

The corresponding return series used for the individual stocks are the ones supplied by Center for Research in Security Prices (CRSP) in their 1997 Stock File (see CRSP (1998)). We use return data from January 2, 1976 through December 29, 1995 since this is as far back data on the individual stock return and dividend are available on a daily basis. Following the stock file guide, the time \( t \) return, \( R_{t,\text{CRSP}} \), from purchasing at time \( t' < t \) and selling at time \( t \) is specified as

\[
R_{t,\text{CRSP}} = \frac{S_t f_t + d_t}{S_{t'}} - 1,
\]

where \( S_t \) is the last sale price or closing bid-ask average at time \( t \), \( d_t \) is the cash dividends per share for time \( t \), and \( f_t \) is a price adjustment factor at time \( t \) for non cash dividends like spin-offs, mergers, exchanges, reorganizations, liquidations, and rights issues. To fit into the option pricing framework we work with the continuously compounded returns, and to limit potential numerical problems when estimating the different models we use returns in percentage terms. Thus, the series we work with for the individual returns are defined as

\[
R_t = 100 \ast \ln (1 + R_{t,\text{CRSP}}).
\]

Table 3 shows sample statistics for the chosen return series. From the table it is clear that any assumption of normality is rejected. In general, the individual stock returns are negatively skewed and leptokurtic,
although for MRK the skewness is insignificantly different from zero. This is confirmed by log-density plots for the standardized return series in the left hand panels of Figure 2 which all have fatter tails than the normal density. The right hand panels of Figure 2 show time plots of $R_t$. These indicate that, in addition to being non Gaussian, the returns are not independently identically distributed through time. On the contrary, it seems clear that periods of low volatility are followed by high volatility periods and vice versa. This finding is known as volatility clustering and the results are confirmed by the $Q^2(20)$ test statistics shown in Table 3. These statistics are all significant at a 5% level, which implies that the null hypothesis of no serial correlation in the squared standardized residuals can be rejected.

3 The Augmented GARCH framework of Duan (1997)

In the literature, findings of leptokurtosis and skewness together with volatility clustering are well documented for many other series than the ones in the present paper. The literature also presents an abundance of explanations for these empirical regularities. Although we recognize the large literature examining and seeking explanations of these regularities, the purpose of this paper is not to give yet another explanation. For the present we simply take them as given, and merely attempt to find a model able to accommodate the empirical regularities. We do this using the framework of generalized autoregressive conditional heteroskedastic processes as do many others (see Bollerslev et al. (1992)).

The Generalized ARCH (GARCH) process of Bollerslev (1986) restricts the conditional variance of a time series to depend upon past squared residuals and lagged values of the variance. When put together with a mean equation we talk about a GARCH (regression) model. A simple GARCH model for $R_t$ can be formulated as

$$
R_t = m_t + \sqrt{h_t} \epsilon_t
$$

$$
h_t = \omega + \alpha h_{t-1} \epsilon_{t-1}^2 + \beta h_{t-1}
$$

$$
\epsilon_t | \mathcal{F}_{t-1} \sim N(0, 1),
$$

where $\mathcal{F}_{t-1}$ is the time $t-1$ information set, $m_t$ is the conditional mean, and $\omega$, $\alpha$ and $\beta$ should be assumed positive in order to avoid any problems related to negativity of the variance process. With some care, the model can be considered an ARMA model for the variance (see Bollerslev (1986)), and therefore it has the potential to explain the volatility clustering often found in asset returns. Furthermore, if we assume existence of the appropriate moments, using Jensen’s inequality, we see that

$$
E \left[ \left( \sqrt{h_t} \epsilon_t \right)^4 \right] = E \left[ h_t^2 \right] E \left[ \epsilon_t^4 \right] \geq E \left[ h_t \right]^2 E \left[ \epsilon_t^4 \right],
$$

where equality holds if the conditional variance is constant. Thus, even if $\epsilon_t$ is normally distributed, from which it follows that $E \left[ \epsilon_t^4 \right] = 3$, the GARCH model could potentially explain the excess kurtosis found in the return series if $h_t$ varies. Although the GARCH model has been used extensively in financial econometrics since its introduction several extensions have been suggested. These call for a more general formulation of the GARCH framework which we present in the following.
To be precise, in this paper we consider a discrete time economy with the price of an asset denoted $S_t$ and the dividends of that asset denoted $d_t$. We will assume that the continuously compounded return process, $R_t$, can be modelled using the Augmented GARCH framework of Duan (1997) according to which we specify the dynamics for the one period continuously compounded return, $R_t = \ln (S_t + d_t) - \ln S_{t-1}$, as

$$R_t = r + \lambda \sqrt{h_t} - \frac{1}{2} h_t + \sqrt{h_t} \varepsilon_t,$$ (3)

$$\phi_t = \alpha_0 + \phi_{t-1} \left( \alpha_1 + \alpha_2 Z_t^{(2)} + \alpha_3 Z_t^{(3)} \right) + Z_t^{(4)},$$ (4)

$$h_t = \left\{ \begin{array}{ll}
|\xi \phi_t - \xi + 1|^{\frac{1}{\xi}} & \text{if } \xi \neq 0 \\
\exp(\phi_t - 1) & \text{if } \xi = 0,
\end{array} \right.$$ (5)

$$\varepsilon_t | F_{t-1} \sim N(0, 1),$$ (6)

where $F_{t-1}$ is the information set containing all information up to and including time $t - 1$. In (4) we further define:

$$Z_t^{(2)} = |\varepsilon_t - c\|^\delta$$
$$Z_t^{(3)} = \max(0, c - \varepsilon_t\|^\delta$$
$$Z_t^{(4)} = \alpha_4 f(|\varepsilon_t - c\|; \delta) + \alpha_5 f(\max(0, c - \varepsilon_t); \delta),$$ (7)

where $f(z; \delta) = \frac{z^{\delta-1}}{\delta}$ for any $z \geq 0$.

It follows from lognormality that one plus the conditional expected rate of return equals $\exp(\lambda \sqrt{h_t})$. $\lambda$ in (3) is readily interpreted as the unit risk premium when $r$ is the constant one period continuously compounded risk-free rate of return. We note that more general formulations for the mean in (3) can be used as long as these are measurable with respect to the information set $F_{t-1}$. Furthermore, alternative specifications could be used for the price of risk. However, changing the mean specification has generally little effect on the parameters of the volatility process and thus will also only have marginal effect on the dynamics used for option pricing. For this reason we maintain the proposed specification.

### 3.1 Some relevant special cases

The Augmented GARCH model nests a large number of well known GARCH specifications. In particular, the GARCH model above obtains if we set $\xi = 1$, $\delta = 2$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\alpha_5 = 0$, and $c = 0$. In particular, if these values are substituted into (4) and the relations in (7) are used it can be realized that $\phi_t$ is given by

$$\phi_t = \alpha_0 + \phi_{t-1} \left( \alpha_1 + \alpha_2 |\varepsilon_t|^2 \right)$$
$$= \alpha_0 + \alpha_1 \phi_{t-1} + \alpha_2 \phi_{t-1} \varepsilon_t^2.$$ (8)

If we further assume that $\alpha_0 > 0$, $\alpha_1 \geq 0$, and $\alpha_2 \geq 0$ it follows that $\phi_t$ is ensured to remain positive and by (5), $\phi_t = h_t$. Thus, we see that the GARCH process for the variance given by

$$h_t = \omega + \alpha h_{t-1} \varepsilon_t^2 + \beta h_{t-1},$$ (9)
is obtained when $\omega = \alpha_0, \alpha = \alpha_2, \text{and } \beta = \alpha_1$ to conform with the notation above. However, the Augmented GARCH model is much more general that this, and in particular a number of well known asymmetric models are also nested.

In the present paper we will, in addition to the GARCH specification, use three different specifications all of which can potentially accommodate asymmetric responses to negative and positive return innovations. Thus, these models allow for a leverage effect, which refers to the tendency for changes in stock prices to be negatively correlated with volatility. While the first is a straightforward generalization of the GARCH model, the two others specify the dynamics for the volatility and the logarithms of the variance, respectively.

The first asymmetric GARCH model we consider is the non-linear asymmetric GARCH model, or NGARCH for short, of Engle & Ng (1993). The particular specification of the variance process for this model is given by

$$ h_t = \omega + \alpha h_{t-1} (\varepsilon_{t-1} + \gamma)^2 + \beta h_{t-1}.$$  \hfill (10)

In the NGARCH model the leverage effect is modelled through the parameter $\gamma$, and if $\gamma < 0$ leverage effects are found. It is clear that this model nests the ordinary GARCH specification, which obtains when $\gamma = 0$, and the same parameter restrictions ensure that the process remains positive. The model may be obtained from the Augmented GARCH specification by setting $\xi = 1$, $\delta = 2$, $\alpha_3 = 0$, $\alpha_4 = 0$, and $\alpha_5 = 0$. In particular, this yields the following relation in (4) using the relations in (7)

$$ \phi_t = \alpha_0 + \phi_{t-1} (\alpha_1 + \alpha_2 |\varepsilon_t - c|^2) = \alpha_0 + \alpha_1 \phi_{t-1} + \alpha_2 \phi_{t-1} (\varepsilon_t - c)^2.$$  \hfill (11)

The specification in (10) obtains when $\omega = \alpha_0, \alpha = \alpha_2, \gamma = -c$, and $\beta = \alpha_1$.

The second asymmetric specification we will consider specifies the dynamics in terms of $\sqrt{h_t}$ and is referred to as the Threshold GARCH, or TGARCH. The particular specification we use is a version of the TGARCH model of Zakoian (1994) where we, instead of (9), specify the process as

$$ \sqrt{h_t} = \omega + \beta \sqrt{h_{t-1}} + \alpha \sqrt{h_{t-1}} (|\varepsilon_{t-1}| + \psi \varepsilon_{t-1}).$$  \hfill (12)

In the TGARCH model the leverage effect is modelled through the parameter $\psi$, and if $\psi < 0$ we say that leverage effects are found. The TGARCH model may be obtained from the Augmented GARCH model by setting $\xi = \frac{1}{2}, c = 0, \delta = 1, \alpha_4 = \alpha_2, \alpha_5 = \alpha_3$. In particular, this allows us to write (4) as

$$ \phi_t = \alpha_0 + \phi_{t-1} (\alpha_1 + \alpha_2 |\varepsilon_t| + \alpha_3 \max (0, c - \varepsilon_t)) + \alpha_2 f (|\varepsilon_t - c| ; \delta) + \alpha_3 f (\max (0, c - \varepsilon_t) ; \delta) = \alpha_0 + \alpha_1 \phi_{t-1} + \alpha_2 \phi_{t-1} |\varepsilon_t| + \alpha_3 \phi_{t-1} \max (0, -\varepsilon_t) + \alpha_2 (|\varepsilon_t| - 1) + \alpha_3 (\max (0, -\varepsilon_t) - 1).$$  \hfill (13)

If we further assume that $\alpha_1 \geq 0, \alpha_2 \geq 0$, and $\alpha_0 > \alpha_1 + \alpha_2 - 1$ the positivity of $\phi_t$ is ensured and (5) implies that $2\sqrt{h_t} - 1 = \phi_t$. Thus, we may rewrite (13) as

$$ 2\sqrt{h_t} - 1 = \alpha_0 + \alpha_1 (2 \sqrt{h_{t-1}} - 1) + \alpha_2 (2 \sqrt{h_{t-1}} - 1) |\varepsilon_t| + \alpha_3 (2 \sqrt{h_{t-1}} - 1) \max (0, -\varepsilon_t) + \alpha_2 (|\varepsilon_t| - 1) + \alpha_3 (\max (0, -\varepsilon_t) - 1),$$

or, after some reorganization,

$$ \sqrt{h_t} = (1 + \alpha_0 - \alpha_1)/2 + \alpha_1 \sqrt{h_{t-1}} + \alpha_2 \sqrt{h_{t-1}} |\varepsilon_t| + \alpha_3 \sqrt{h_{t-1}} \max (0, -\varepsilon_t).$$  \hfill (14)
Although it might not be immediately clear that this corresponds to the TGARCH specification, this is in fact the case. In particular, let \( \alpha = \alpha_2 + \frac{1}{2} \alpha_3 \) and \( \psi = \frac{-\alpha_3}{2 \alpha_2 + \alpha_3} \) from which it follows that
\[
\alpha \sqrt{h_{t-1}} (|\varepsilon_{t-1}| + \psi \varepsilon_{t-1}) = \left( \alpha_2 + \frac{1}{2} \alpha_3 \right) \sqrt{h_{t-1}} \left( |\varepsilon_{t-1}| + \frac{-\alpha_3}{2 \alpha_2 + \alpha_3} \varepsilon_{t-1} \right) \\
= \alpha_2 \sqrt{h_{t-1}} |\varepsilon_{t-1}| + \frac{1}{2} \alpha_3 \sqrt{h_{t-1}} |\varepsilon_{t-1}| - \frac{1}{2} \alpha_3 \sqrt{h_{t-1}} \varepsilon_{t-1} \\
= \alpha_2 \sqrt{h_{t-1}} |\varepsilon_{t-1}| + \frac{1}{2} \alpha_3 \sqrt{h_{t-1}} (\max(0, \varepsilon_{t-1}) + \max(0, -\varepsilon_{t-1}) - \max(0, \varepsilon_{t-1}) - \max(0, -\varepsilon_{t-1})) \\
= \alpha_2 \sqrt{h_{t-1}} |\varepsilon_{t-1}| + \alpha_3 \sqrt{h_{t-1}} \max(0, -\varepsilon_{t-1}).
\]
Substituting the left hand expression for the two last terms in (14) confirms the equivalence.

The third asymmetric process we consider is a version of the exponential GARCH model of Nelson (1991), or EGARCH for short. Instead of (9) the volatility process is now specified in logarithms as
\[
\ln(h_t) = \omega + \beta \ln(h_{t-1}) + \alpha (|\varepsilon_{t-1}| - E[|\varepsilon_{t-1}|] + \theta \varepsilon_{t-1}). \tag{15}
\]
In the EGARCH model the term \( |\varepsilon_{t-1}| - E[|\varepsilon_{t-1}|] \) is interpreted as the magnitude effect and \( \theta \varepsilon_{t-1} \) is called the sign effect. If \(-1 < \theta < 0\) a positive surprise increases volatility less than a negative one, and we say that leverage effects are present. We note that the EGARCH model does not nest the GARCH model. However, it has the nice feature that no restrictions need to be put on the parameters to ensure nonnegativity of the variance. The EGARCH model obtains from the Augmented GARCH model by setting \( \xi = 0, c = 0, \delta = 1, \alpha_2 = 0, \alpha_3 = 0 \) which yields
\[
\phi_t = \alpha_0 + \alpha_1 \phi_{t-1} + \alpha_4 (|\varepsilon_t| - 1) + \alpha_5 (\max(0, -\varepsilon_t) - 1), \\
h_t = \exp(\phi_t - 1).
\]
Since \( h_t = \exp(\phi_t - 1) \) implies that \( \phi_t = \ln h_t + 1 \) we get
\[
\ln h_t + 1 = \alpha_0 + \alpha_1 (\ln h_{t-1} + 1) + \alpha_4 (|\varepsilon_t| - 1) + \alpha_5 (\max(0, -\varepsilon_t) - 1),
\]
which corresponds to
\[
\ln h_t = (\alpha_0 + \alpha_1 - \alpha_4 - \alpha_5 - 1) + \alpha_1 \ln h_t + \alpha_4 |\varepsilon_t| + \alpha_5 \max(0, -\varepsilon_t). \tag{16}
\]
From this it is clear that if we set \( \alpha = \alpha_4 + \frac{1}{2} \alpha_5 \) and \( \theta = \frac{-\alpha_3}{2 \alpha_4 + \alpha_5} \) for the EGARCH model the same equivalence between (16) and (15) is easily shown using the same type of derivation as was the case for the TGARCH model.

### 3.1.1 Diffusion limits for some special augmented GARCH processes

The main reason for choosing the Augmented GARCH framework is that it has been shown to contain as its limit many of the bivariate diffusion processes that are used as the building blocks in many of the stochastic volatility option pricing models (see Duan (1997, Theorem 3)). In particular, this is the case for the GARCH,
NGARCH and EGARCH specifications considered above. Thus, the immediate benefit from working within the augmented GARCH framework is that the parameters of the diffusions can be implied from the discrete time data.

First of all, for the NGARCH specification in (10) the conditional variance process of the limiting diffusion model is

\[ dh_t = \alpha_0 dt + (\alpha_1 + \alpha_2 (1 + c^2) - 1) h_t dt - 2\alpha_2 h_t dW_{1t} + \sqrt{2}\alpha_2 h_t dW_{2t}, \]

where \( W_{1t} \) and \( W_{2t} \) are two independent Wiener processes. Appropriately re-parameterized this corresponds to the bivariate diffusion model used in Hull & White (1987). In the following we will refer to this as the C-NGARCH specification. Furthermore, the C-GARCH model, which obtains when \( c = 0 \), corresponds to the special case of the Hull & White (1987) model where the volatility process is independent of the return process.

Secondly, for the EGARCH specification in (15) the conditional variance process of the limiting diffusion model is

\[ d\ln h_t = \left( \alpha_0 + \alpha_1 - 1 + \alpha_4 \left( \frac{2}{\sqrt{2\pi}} - 1 \right) + \alpha_5 \left( \frac{1}{\sqrt{2\pi}} - 1 \right) \right) dt + (\alpha_1 - 1) \ln h_t dt - \frac{1}{2} \alpha_5 dW_{1t} + \alpha_4 + \frac{1}{2} \alpha_5 \sqrt{(\pi - 2) / \pi} dW_{2t}, \]

which can be related to the stochastic volatility model used in Wiggins (1987). This model will be referred to as the C-EGARCH model in the following.

Finally, for the TGARCH specification in (12) the conditional variance process of the limiting diffusion model is\(^1\)

\[ d\sqrt{h_t} = (\alpha_0 - \alpha_1 + 1) dt + \left( \alpha_1 + (2\alpha_2 + \alpha_3) / \sqrt{2\pi} - 1 \right) \sqrt{h_t} dt \\
- \frac{1}{2} \alpha_3 \sqrt{h_t} dW_{1t} + \left( \alpha_2 + \frac{1}{2} \alpha_3 \right) \sqrt{(\pi - 2) / \pi} \sqrt{h_t} dW_{2t}, \]

which we will refer to as the C-TGARCH specification in the following. As discussed in Duan (1996) and Duan (1997), the diffusion limit of the TGARCH model differs somewhat from the bivariate models used in Scott (1987), Stein & Stein (1991), and Heston (1993). However, although a model similar to that of the latter authors can be obtained from the Augmented GARCH model with slightly different parameter restrictions, in the following we work with the particular specification in (12) and the corresponding diffusion limit in (19).

The main reason for this is that this particular discrete time formulation has been used previously in the GARCH literature. Furthermore, in our experience this particular specification is much easier to estimate and better behaved empirically than the discrete time specification which would have as the diffusion limit e.g. the Heston (1993) model.

### 3.2 Estimation results

Table 4 in Appendix A report the QML estimation results for the return series with all computational work performed using Ox (see Doornik (2001)). We start by looking at column two containing the results

\(^1\)Note that there appears to be an error in equation (35) in Duan (1997). This has been corrected here.
for the constant volatility (CV) model. This model is obviously a special case nested by all the volatility specifications above and is specified as

$$h_t = \omega.$$  \hspace{1cm} (20)

Furthermore, it corresponds to the distributional assumption in the Black-Scholes model and thus serves as a natural benchmark.

The diagnostic tests for the CV model, however, indicate that the non-normality found in Table 3 is not the only problem this model faces. Another problem is the finding of ARCH type errors. This is indicated by the $Q^2(20)$ statistics which are significant at a five percent level. The null hypothesis of this test is that of no serial correlation of up to order twenty in the squared standardized residuals and it has been shown to be applicable by Bollerslev & Mikkelsen (1996) in the GARCH framework, when the degrees of freedom are corrected by the number of estimated ARCH parameters. The $Q(20)$ tests for serial correlation in the mean are also significant. However, one should note that the presence of ARCH may give rise to spurious significance of this type of tests for serial correlation in the mean (see Bollerslev & Mikkelsen (1996)).

The results from estimating the GARCH models are shown in column three in the table. Compared to these, the first thing to note is that the simpler CV model is easily rejected using a likelihood ratio test. Furthermore, for all series, both extra parameters are significant and the estimates are in line with what is usually found in the literature. In terms of serial correlation in the squared standardized residuals, the $Q^2(20)$ statistics show that for all but GM the null of no correlation cannot be rejected at a five or even at a one percent level. Thus, it seems that modelling volatility as a GARCH process goes quite a way in terms of eliminating the ARCH effects for IBM and MRK. Furthermore, for two of the three series the $Q(20)$ statistics are now insignificant at a one percent level.

Column four presents estimation results for the NGARCH model, and compared to the GARCH model the tables show that in all cases adding the leverage parameter increases the Log-Likelihood value significantly. Furthermore, for all the return series the estimated leverage parameter, $\gamma$, is significantly different from zero and has the expected sign. The same holds for the TGARCH and EGARCH models for which the estimation results are shown in columns five and six. Unfortunately, changes in Log-likelihood values cannot be used to compare these models. Instead, in the literature comparisons between such models are made by minimizing various information criteria. This procedure is used even though little is known about the statistical properties of these information criteria in a GARCH context. However, Bollerslev & Mikkelsen (1996) do show that the Schwarz Information Criteria, $SIC = -2 \frac{\mathcal{L}}{N} + \frac{2\ln(j)}{N}$, where $\mathcal{L}$ is the log-likelihood value, $N$ is the number of observations, and $j$ is the number of parameters, can be used to discriminate between the alternatives with good results. For the returns in Table 4 the SIC value is smaller for the asymmetric models than for the symmetric GARCH model, which indicates that asymmetries in the volatility specification are important features of the individual return data under consideration.
3.2.1 The implied parameters of the diffusion limits

Appropriately reparameterized the diffusion limit of the NGARCH specifications may be written as

\[ d\ln S_t = \left( r + \lambda \sqrt{h_t} - \frac{1}{2} h_t \right) dt + \sqrt{h_t} dW_{1t}, \quad (21) \]

\[ dh_t = \beta_0 dt + \beta_1 h_t dt + \beta_2 h_t dW_{1t} + \beta_3 h_t dW_{2t}, \quad (22) \]

where \( \beta_0 = \alpha_0, \beta_1 = (\alpha_1 + \alpha_2 (1 + c^2) - 1), \beta_2 = -2\alpha_2, \) and \( \beta_3 = \sqrt{2}\alpha_2. \) Furthermore, it follows immediately from this that the reparameterized diffusion limit for the GARCH specification is given by

\[ dh_t = \beta_0 dt + \beta_1 h_t dt + \beta_3 h_t dW_{2t}. \quad (23) \]

Obviously, the same can be done for the diffusion limits of the TGARCH and EGARCH models, and we simply write these as

\[ d\sqrt{h_t} = \beta_0 dt + \beta_1 \sqrt{h_t} dt + \beta_2 \sqrt{h_t} dW_{1t} + \beta_3 \sqrt{h_t} dW_{2t}, \quad (24) \]

and

\[ d\ln h_t = \beta_0 dt + \beta_1 \ln h_t dt + \beta_2 dW_{1t} + \beta_3 dW_{2t}, \quad (25) \]

respectively, with the appropriate parameter restrictions.

Table 5 reports the resulting estimated parameters of (22) – (25) which are comparable to the QMLE’s in Table 4. Overall the implied parameters seem very reasonable. In particular, the estimated diffusion limits show sign of strong persistence in shocks to the variance process and, when allowed for, asymmetries in the stock returns and variance. However, since a detailed econometric analysis of the results is beyond the scope of the present paper at this time we refrain from commenting further on the implied parameters, except but to say that this will be included in future research.

4 American option pricing with Augmented GARCH processes

A major advantage with discrete time return models is that data is readily available for estimation as shown above. However, when deriving a theoretical option pricing model, discreteness poses a potential problem as the asset market models easily become incomplete. This is indeed the case for the GARCH type models. Heuristically, the reason is that, unlike in the Binomial Model where the stock can take on only two different values next period conditional on the price today, in a GARCH model the number of future possible values is infinite. In some of the first attempts to derive a GARCH type option pricing model, e.g. Amin & Ng (1993), this problem was dealt with by simply assuming the existence and uniqueness of a state pricing density process \( \Pi_t, \) where \( \Pi_0 = 1, \) such that the price of any time \( t \) claim, \( C_t, \) to a cash flow \( C_{t+1} \) at time \( t + 1 \) is equal to

\[ C_t = E_t \left[ \frac{C_{t+1} \Pi_{t+1}}{\Pi_t} \right]. \quad (26) \]

Since the existence and uniqueness of the state pricing density process is equivalent to that of the existence of a unique equivalent martingale, following Harrison & Kreps (1979) we can price all securities as their expected future payoff discounted using the risk-free rate of interest. Thus, using the law of iterated expectations in
(26) a European put option with terminal payoff \( Z_T (S_T) = \max \{0, (K - S_T)\} \), where \( K \) is the strike price and \( S_t \) the price of the underlying asset at time \( t \), should have a time 0 price of

\[
p_0 = E^Q \left[ \exp \left\{ -\sum_{t=1}^{T} r_{t-1} \right\} Z_T (S_T) \mid \mathcal{F}_0 \right],
\]

where \( E^Q [\cdot \mid \mathcal{F}_0] \) means expectation under the equivalent martingale measure \( Q \) conditional on the time \( t = 0 \) information set. In the same manner we can express the time 0 price of an American put option with payoff \( Z_t (S_t) = \max \{0, (K - S_t)\} \) if exercised at time \( t \), as

\[
P_0 = \sup_{\tau} E^Q \left[ \exp \left\{ -\sum_{t=1}^{\tau} r_{t-1} \right\} Z_\tau (S_\tau) \mid \mathcal{F}_0 \right],
\]

where the supremum is over all stopping times \( \tau \) with \( 0 \leq \tau \leq T \).

As an alternative to simply assuming the existence and uniqueness of a state pricing density, further assumptions about preferences, apart from non-satiation, or about the risk premium have to be made in order to get to a risk-neutral valuation relationship. In Duan (1995) an extension of the risk-neutralization principle in Brennan (1979), referred to as the Locally Risk-Neutral Valuation Relationship (LRNVR), is used. The LRNVR can be shown to hold under some familiar assumptions on preferences and assumed conditional lognormality.

The model we use is the one of Section 3 which we for notational convenience specify as

\[
\ln \left( \frac{S_t + d_t}{S_{t-1}} \right) = r + \lambda \sqrt{h_t} - \frac{1}{2} h_t + \sqrt{h_t} \varepsilon_t, \quad \varepsilon_t \mid \mathcal{F}_{t-1} \sim N(0, 1) \text{ under measure } \mathcal{P},
\]

and where the variance process is specified as

\[
h_t = g(\Theta, h_{t'}, \varepsilon_{t'}; t' \leq t - 1).
\]

In (30), \( \Theta \) denotes a set of parameters and it is immediately clear that all the variance specifications of the previous section can be accommodated in this framework. The LRNVR derived in Duan (1995) stipulates that under the risk-neutralized pricing measure \( Q \) \( (S_t + d_t) / S_{t-1} \mid \mathcal{F}_{t-1} \) distributes lognormally with \( E^Q [(S_t + d_t) / S_{t-1} \mid \mathcal{F}_{t-1}] = \exp (r) \) and that the one-period ahead conditional variance is invariant with respect to the change to \( Q \) such that

\[
\text{Var} \left[ \ln \left( \frac{S_t + d_t}{S_{t-1}} \right) \mid \mathcal{F}_{t-1} \right] = \text{Var}^Q \left[ \ln \left( \frac{S_t + d_t}{S_{t-1}} \right) \mid \mathcal{F}_{t-1} \right].
\]

Thus, invoking the LRNVR it is relatively easy to derive the dynamics under the equivalent martingale measure for the model under consideration. First of all, the restriction on the mean return in the lognormal distribution of returns results in the following dynamics

\[
\ln \left( \frac{S_t + d_t}{S_{t-1}} \right) = r - \frac{1}{2} h_t + \sqrt{h_t} Z_t, \quad Z_t \mid \mathcal{F}_{t-1} \sim N(0, 1) \text{ under measure } Q.
\]
Secondly, to ensure (31) the variance process has to be specified as
\[ h_t = g(\Theta, h_t', Z_{t'} - \lambda; t' \leq t - 1). \] (33)

As an example, the risk-neutral dynamics for the NGARCH model are given by
\[ h_t = \omega + \alpha h_{t-1} (Z_{t-1} - \lambda + \gamma)^2 + \beta h_{t-1}, \] (34)

where \( Z_t \) has a standard normal distribution. Thus, while we may be lured into believing that we have successfully eliminated all preference related parameters this is not the case. However, the LRNVR is sufficient to reduce the preference considerations to the constant unit risk premium present in the variance equation.

### 4.1 Option pricing with the limiting diffusion

As it was the case under the data generating process, the Augmented GARCH process under the risk-neutralized pricing measure can be shown to converge to a bivariate diffusion system. This was shown in Theorem 2 in Duan (1996), and this general theorem can be used to derive the precise system corresponding to any risk-neutralized version of the GARCH variance specifications used above. However, it is also possible to show that the risk-neutral dynamics can be derived easily from the corresponding diffusion limits under the data generating process.

To fix ideas, assume that the diffusion limit of the Augmented GARCH system in (29) and (30) has been derived under measure \( \mathcal{P} \). With a slight abuse of notation we specify this as
\[
    d\ln S_t = \left( r + \lambda \sqrt{h_t} - \frac{1}{2} h_t \right) dt + \sqrt{h_t} dW_{1t},
\]
\[
    dh_t = g(\Theta, h_t, dW_{1t}, dW_{2t}),
\]

where \( W_{1t} \) and \( W_{2t} \) are the two independent Wiener processes adhered to above. It then follows from Duan (1996) that the limit under the corresponding risk-neutralized pricing measure, \( \mathcal{Q} \), becomes
\[
    d\ln S_t = \left( r - \frac{1}{2} h_t \right) dt + \sqrt{h_t} dW_{1t}^*,
\]
\[
    dh_t = g(\Theta, h_t, dW_{1t}^* - \lambda t, dW_{2t}^*),
\]

where \( W_{1t}^* = W_{1t} + \lambda t \) and \( W_{2t}^* = W_{2t} \) are two independent Wiener processes.

As an example, consider the limit of the NGARCH model, i.e. the model of Hull & White (1987), which we wrote in a reparameterized version as
\[
    dh_t = \beta_0 dt + \beta_1 h_t dt + \beta_2 h_t dW_{1t} + \beta_3 h_t dW_{2t}.
\]

Under the risk-neutralized pricing measure the variance process is given by
\[
    dh_t = \beta_0 dt + (\beta_1 - \lambda \beta_2) h_t dt + \beta_2 h_t dW_{1t}^* + \beta_3 h_t dW_{2t}^*,
\]

where \( W_{1t}^* \) and \( W_{2t}^* \) are the two independent Wiener processes. In the same manner the risk-neutral dynamics can be obtained for the limit of the EGARCH model, i.e. the model of Wiggins (1987), and the corresponding limit of the TGARCH model.
4.2 Implementation of the GARCH option pricing model using simulation

Although the pricing system in (32) and (33), or for that sake the system in (37) and (38), is completely self-contained, an actual application of the pricing system to even the simple European option formula in (27) is difficult because of the lack of a closed form expression for the time $T$ value of the underlying asset. However, it is immediately clear that it is possible to use the system in (32) and (33), respectively in (37) and (38), to generate a large number of paths with the risk-neutralized asset price possibly using some discretization scheme. From these, an estimate of the option value in (27) can be obtained by taking the average among the paths as

$$
\tilde{P}_0^M = \frac{1}{M} \sum_{j=1}^{M} e^{-rT} \max (X - S_T (j), 0),
$$

where $M$ is the number of simulated paths and $S_T (j)$ is the risk-neutralized value of the underlying stock at expiration of the option for path number $j$. Stemming back at least to Boyle (1977) estimates like (40) have been used to price European options.

Unfortunately, things are not quite as simple when American options are considered. The problem is the need to simultaneously determine the optimal exercise strategy and simply using

$$
\tilde{P}_0^M = \frac{1}{M} \sum_{j=1}^{M} \max \left[ e^{-r\tau} \max (X - S_\tau (j), 0) \right],
$$

to estimate the American put price in (28) would result in a biased estimate, as it is equivalent to assuming that the holder of the option has perfect foresight about future stock prices (see Broadie & Glasserman (1997)). Due to this difficulty, simulation methods were believed to be applicable to the pricing of European style options only until recently (see e.g. Hull (2000, p. 410)). However, the works by Carriere (1996), Longstaff & Schwartz (2001), and Tsitsiklis & Van Roy (2001) among others have shown otherwise and in these papers algorithms for pricing American style options using simulated values of the underlying asset under the risk-neutralized dynamics are proposed.

It is probably fair to say that the most important of these contributions is the Least Squares Monte Carlo (LSM) method of Longstaff & Schwartz (2001) in which expressions like (28) are evaluated using the cross-sectional information available at each step in the simulation. The numerical performance of this method has been examined in Moreno & Navas (2003) and in Stentoft (2004a) in the constant volatility Black-Scholes equivalent situation, and recently the mathematical foundation for the use of the method in derivatives research has been provided in Stentoft (2004b). Empirically the method is rapidly gaining importance, e.g. for real option valuation (see Gamba (2002)), and recently it was used in Stentoft (2005) to price options in a discrete time Gaussian GARCH context. In the present paper we will use the algorithm suggested in that paper.
5 A Monte Carlo study of the Augmented GARCH option pricing model

In Duan (1995) it is argued that for European call options the GARCH option pricing model can explain some of the regularities found in the empirical option pricing literature, like the smiles in implied Black-Scholes volatilities and the underpricing of short term options and out of the money options. In the paper by Stentoft (2005) these results are generalized to the situation with American options and asymmetries in the volatility specification. The latter paper further showed that the early exercise feature is important and that the pricing errors are consistent with what was found empirically for a number of individual stock options.

The purpose of our Monte Carlo study is to compare the results on American option pricing using the discrete time models with what is obtained with the corresponding diffusion limits, thus extending the results of Stentoft (2005) to the continuous time framework. In particular, we report results on American options with the discrete time GARCH and NGARCH specifications as well as the pricing results from their actual diffusion limits. While the latter results are interesting on their own, by carefully choosing the particular bivariate diffusion models and by using the same algorithm for pricing the study allows us to compare the two frameworks. Moreover, it is possible to gauge how well the family of discrete time GARCH models actually approximate the continuous time stochastic volatility process in terms of option pricing.

5.1 The Monte Carlo setup

To illustrate the pricing properties we use a set of artificial options with strike prices ranging from deep out of the money, say a moneyness equal to 0.90, to deep in the money, say a moneyness of 1.10, and with short (1 month), middle (3 months) and long maturities (6 months). We take a year to be 252 trading days, such that the options expire in 21, 63, and 126 trading days respectively. To a large extent this collection of options covers what is actually observed for traded options on individual stocks as detailed in Section 2. In line with what is found empirically we set the interest rate equal to 5% on an annual basis. Furthermore, corresponding to what is found for the American stocks we consider in the empirical analysis we choose to include dividend payments at a rate of 3% annually. The reported prices in the study are averages of 100 calculated estimates using different seeds in the random number generator, and in the tables the standard errors of these 100 estimates are reported in parenthesis below the corresponding price estimate. For the American price estimates, products and cross-products between the stock level and the volatility of total or ordinary stock are used in the cross-sectional regressions and in all the cases exercise is considered once a day at the end of every trading day.\(^2\)

When comparing the different volatility specifications within the Augmented GARCH framework it seems reasonable that we consider models with the same characteristics at least in terms of the level of persistence and the level of implied volatility. We do this by first choosing the persistence parameters, that is $\beta + \alpha$ in the GARCH model and $\beta + \alpha (1 + \gamma^2)$ in the NGARCH model, and then adjusting $\omega$ in order to yield the same unconditional variance, $\mathbb{E}[h]$, across the models. Thus, in the GARCH model we set $\omega_{GARCH} =$

\(^2\)Thus, to be precise we should refer to the options as being of “Bermudan” type instead of “American”.

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\[(1 - \beta - \alpha) \ast E[h] \text{ and in the NGARCH model we set } \omega_{\text{NGARCH}} = (1 - \beta - \alpha (1 + \gamma^2)) \ast E[h]. \] This method is the reverse of the variance targeting procedure which has been used in estimating.

A final request in our Monte Carlo study is that the parameter values for \(\lambda, \alpha, \beta, \gamma, \text{ and } E[h]\) be empirically plausible and again we choose values close to the actual averages of the estimated parameter values from the individual return series from 1976 through 1995 reported in Section 2. After rounding we decide to use the values in Table 6. Thus, we set \(\beta = 0.92\) and \(\alpha = 0.06\) in the GARCH specification, and \(\beta = 0.92, \alpha = 0.048, \text{ and } \gamma = -0.5\) in the NGARCH model. These values yield a persistence of 0.98 so in order to get an annualized unconditional volatility of 25% we set \(\omega = 4.96 \times 10^{-6}\). We sum up the models used and the parameter choice in Table 6 for the GARCH models and in Table 7 we show the implied models and the implied parameters used in the SV models.

### 5.2 Pricing American options with GARCH variance specifications

In Column four in Table 8 we report American put option prices calculated using a GARCH specification with empirically plausible parameter values from Table 6. Thus, realizations of the discretely observed stock price are calculated from

\[S_t = S_{t-1} \exp \left\{ \frac{(0.05 - 0.03) / 252 - \frac{1}{2} h_t + \sqrt{h_t} Z_t}{2} \right\}, \quad \text{where } Z_t \sim N(0, 1),\]

where \(h_t\) denotes the variance process which is specified as

\[h_t = 4.96 \times 10^{-6} + 0.92 \ast h_{t-1} + 0.06 \ast h_{t-1} \ast \varepsilon_{t-1}^2.\]

The variance process is initialized by setting \(h_1\) equal to the corresponding daily value of the unconditional volatility. In Column five, the relative bias arising if one were to use the constant volatility specification in the Binomial Model (BM) to price the put options, is shown. Thus, this bias indicates the mispricing by the CV model if the true model is the GARCH model. From this column it is clear that out of the money short maturity options would be particularly underpriced by the CV model, whereas at the money options would be overpriced. Although these effects persist as the maturity increases somewhat, the relative bias, particularly the underpricing of the out of the money options becomes less and less pronounced.

Column six in the table shows the NGARCH put prices, for which the variance process was specified as

\[h_t = 4.96 \times 10^{-6} + 0.92 \ast h_{t-1} + 0.048 \ast h_{t-1} \ast \varepsilon_{t-1} - 0.5^2,\]

and column seven the relative bias arising from using the CV model to price the options. From these columns it is clear that the underpricing of out of the money put options by the constant volatility model is even more pronounced than with the GARCH results in columns four and five. Furthermore, once asymmetries are introduced in the volatility model mispricing is present for even the longest maturities considered, too, and the underpricing by the CV model relative to the NGARCH for long term out of the money put options is 10%. We note that for the TGARCH and EGARCH models equivalent results obtains and thus these are not reported.

In the two top panels of Figure 3 the RBIAS is plotted against the moneyness and maturity for the GARCH and the NGARCH specifications respectively. Although both plots indicate a systematic mispricing
by the CV model when the true model has a GARCH type volatility specification particularly the right hand
one, which corresponds to the NGARCH model, shows an interesting pattern. To be precise, when this plot
is compared to the plots in Figure 1 the potential for the asymmetric GARCH volatility specification to
explain the mispricing observed empirically by the CV model becomes clear. Thus, we conclude that the
possibility of having asymmetry parameters in the variance process allows extra flexibility in the pricing
model when it comes to generating the particular patterns of mispricing observed empirically.

5.3 Pricing American options with GARCH diffusion limit variance specifications

In columns eight through eleven in Table 8 we present the results when the diffusion limits of the GARCH
and NGARCH models are used with the particular parameter values given in Table 7. In order to simulate
the stock and variance paths, an Euler discretization of the appropriate processes with 1024 intraday steps
is used. Thus, the mean specification has, in daily terms,

$$
d\ln S_t = \left((0.05 - 0.03)/252 - \frac{1}{2} h_t\right) dt + \sqrt{h_t} dW_{1t}^*.
$$

For the GARCH diffusion limit the variance process is specified as

$$
dh_t = 4.96 \times 10^{-6} dt - 0.02 h_t dt + 0.0679 h_t dW_{2t}^*,
$$

and for the NGARCH diffusion limit it is specified as

$$
dh_t = 4.96 \times 10^{-6} dt - 0.02 h_t dt - 0.048 dW_{1t}^* + 0.0679 h_t dW_{2t}^*,
$$

where $W_{1t}^*$ and $W_{2t}^*$ are the two independent Wiener processes. The assumption that exercise is possible
only at the end of every trading day is maintained for consistency.

When these columns are compared to columns four through seven, with the discrete time prices, we see
that generally the same pricing pattern results from using the diffusion limits of the GARCH model as with
the actual GARCH models. In particular, the underpricing of out of the money put options in a constant
volatility model remains very pronounced for the NGARCH diffusion limit. However, more importantly the
table shows that for the chosen specification these implied mispricings are in fact larger for the diffusion
limits in percentage terms than they were with the discrete time NGARCH specification. For the GARCH
diffusion limit the results reflect the lack of correlation between the return and the variance process in the
diffusion limit.

In the two bottom panels of Figure 3 the RBIAS is plotted against the moneyness and maturity for the
diffusion limits of the GARCH and the NGARCH specifications respectively. As expected the patterns are
much the same as with the discrete time models, with the diffusion limit of the NGARCH model particularly
distinct. Thus, before we turn to the empirical analysis we note that the empirically regularities observed
in Section 2 for the constant volatility pricing errors corresponds very well to what was found above when
the CV model is used on option prices from discrete time GARCH models as well as from continuous time
SV models. Furthermore, we note that our sample consists of a majority of short term options which again
are the ones for which the GARCH and SV models produce the largest differences relative to the CV model. Thus, we have reason to believe that our suggested extension can explain some of the mispricing of the CV model.

6 Empirical analysis of the Augmented GARCH option pricing model

In our empirical application of the GARCH option pricing model we use a total of $M = 20,000$ paths and we assume that the options can be exercised only once a day. This not only fits nicely into the discrete framework of the GARCH models, but it probably also serves as an adequate approximation to the possibility of continuous exercise. For the diffusion limits we use 32 discretizations per trading day to limit the amount of computational time. In estimating the conditional expectations, apart from a constant term, we use powers of and cross products between the asset price and the level of the volatility of order two or less. For computational convenience we use the fact that the option price is homogenous of degree one in the level of the underlying asset. Therefore, it is possible to use the same simulated paths to price all the options on a single day by scaling the simulated pathwise stock prices with the contemporaneously recorded level of the underlying asset. In our experience this can easily reduce the computation time by as much as 25% for a simple GARCH specification.

In the simulations we further make the following three assumptions about the effect of dividend payments: First, we assume that only cash dividend payments are important for our purpose. This assumption is validated by the fact that exchange traded options, in general, are protected against other forms of dividends like, say stock splits. Secondly, we make the assumption that both the ex-dividend day and the size of the dividends are known in advance. Of course, this is not strictly correct, however dividends seem to be paid regularly with fairly stable amounts through the period we consider. Finally, we assume that the effect of a cash dividend payment fully spills over on the stock price. Thus, if day $t$ is an ex-dividend day and the simulated risk-neutral continuously compounded return is $R_t$, the end of day stock price is calculated from the price at the previous day as

$$S_t = S_{t-1} \ast \exp (R_t) - d_{t-1}.$$  

We note that treating cash dividend payments as known both in size and timing and letting the payment spill over on the stock level seems to be the standard procedure.

The benchmark, with which we compare the pricing errors, is the model with constant volatility. However, due to the inclusion of dividends and the possibility of multiple such payments until expiration the Binomial Model is not easy to apply (see Hull (2000, Chapter 16.3)). Instead we use the CV formulation for $g(\cdot)$ and price options using simulation. We use all the historical observations available at any time $t$, $\mathcal{F}_{t-1}$, in the estimation part of the algorithm. Furthermore, we set $r$ in the mean equation equal to the present short rate as does Duan (1995). We use the one month LIBOR rate which was extracted from Datastream. This rate also corresponds to the one used in the risk-neutralized simulations.
6.1 Overall performance of the GARCH option pricing model

In the literature a number of different metrics has been used to gauge the performance of alternate option pricing models (see Bollerslev & Mikkelsen (1999)). In the present paper we choose to report results on two of these. To be specific, we use the relative mean bias, $RBIAS \equiv K^{-1} \sum_{k=1}^{K} \frac{(\bar{P}_k - P_k)}{P_k}$, and the relative mean squared error, $RSE \equiv K^{-1} \sum_{k=1}^{K} \frac{(\bar{P}_k - P_k)^2}{P_k}^2$, where $P_k$ denotes the $k$'th observed price and $\bar{P}_k$ denotes the $k$'th price estimate. Table 9 provides summary statistics for each of these metrics for the whole sample of individual stock options. The first thing to note from the table is that when the GARCH pricing results are compared to the CV model, this first model is the preferred one irrespective of which metric is used for the individual stock options as well as when call respectively put options are considered separately. In fact, for the individual options the CV specification is the worst performing model of all the volatility specifications, irrespective of which measure is used.

Table 9 also shows that within the discrete time framework as well as within the diffusion limits models asymmetries are important. Furthermore, when it comes to examining which model performs the best and thus minimizes the pricing errors for the individual options Table 9 shows that this is generally done with a specification based on the diffusion limit and not by a discrete GARCH specification. The panel shows that particularly the C-TGARCH and C-EGARCH specifications perform well, and thus confirm that asymmetries are important when pricing individual options. However, the actual pricing errors are very close for these models and their discrete counterparts. It is thus not immediately clear if the increase in computational complexity is reflected in added precision.

6.2 Maturity and moneyness effects for the options

In this section we analyze the performance of the models in terms of moneyness and maturity. We do this using the TGARCH and C-TGARCH models only since for the relative bias the results of Table 9 indicated that the best performing discrete model is the TGARCH model and the best performing model based on the diffusion limits is the C-TGARCH model. We note that the TGARCH model was not considered in Stentoft (2005). Instead that paper considered the GARCH, NGARCH, and EGARCH models along with two additional volatility specifications. However, since the best performing model was found to be the EGARCH specification in that paper this does not invalidate the conclusions of the present paper.

The results can be found in Table 10, in which Panel A shows the results for all the options together. The panel shows that both the TGARCH and the C-TGARCH model performs very well with the exception of the deep out of the money short term options. Comparing the two models the panel also shows that, with the exception of the long term options, the C-TGARCH model has the smallest pricing errors for all combinations of moneyness and maturity.

Panel B, with the results for the call options, and Panel C, with the results for the put options, shows that the general results hold true for the different types of options too. Once again the pricing errors are small except for the deep out of the money options. For the call options these options are underpriced by the TGARCH and C-TGARCH models irrespective of the maturity. However, for the put options only the short term options are underpriced whereas the middle and long term options are overpriced, and the small
pricing errors for the longer term options in Panel A should be seen in light of this. Figure 4, which plots the RBIAS against moneyness and maturity confirms these results.

In Table 11 the pricing errors in Table 10 are reported relative to those from the CV model in Table 2. In general, the improvement from allowing for GARCH specifications is impressive, and the overall pricing errors can be seen to be only around 19% of the CV error for the TGARCH model and 14% for the C-TGARCH model. The panel shows that improvements are found for all maturities and all categories of moneyness, although they are somewhat smaller for the deep out of the money and short run options. Again Panels B and C shows that the above conclusion holds when the call options are considered. For the puts the relative improvements are largest for the out of the money options and seen overall the pricing errors for the C-TGARCH is only around 2% of the errors with the CV model for the put options. Thus, in conclusion, the results shows that allowing for time varying volatility is important when pricing individual stock options, and although improvements are found for both call and put options these are particularly large for the put options.

7 Conclusion

In this paper we consider the problem of pricing options when the underlying asset exhibits time-varying volatility. We choose to model time-varying volatility in the context of Augmented GARCH processes, since these processes have been used extensively in the literature to model asset returns, and in many cases evidence has been found in favor of them. Furthermore, recent work has shown that a number of the bivariate models used in option pricing with stochastic volatility can be obtained as the diffusion limits of some special GARCH models and we focus on these in particular. We present a Monte Carlo study of the characteristics of the models and their corresponding diffusion limits. This study shows that the GARCH model can accommodate the smile in implied volatilities found empirically, and that the asymmetric models like the NGARCH model have the further ability to generate implied volatility smirks. The Monte Carlo study also shows that although the estimated prices from the GARCH specification and their diffusion limits are generally close to each other small differences can be found particularly for deep out of the money options. Finally, we use the method to price traded options on three individual stocks for which it is shown that GARCH effects and asymmetries are important. When comparing the particular GARCH models and their diffusion limits we find that the diffusion limits performs the best. However, in general only small differences occur and in an option pricing context the discrete time GARCH specifications may be considered to provide good approximations to the diffusion models considering the additional computational time required.

References


A Tables

Table 1: Sample Statistics for Option Data

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<thead>
<tr>
<th></th>
<th>GM</th>
<th>IBM</th>
<th>MRK</th>
<th>All stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total # of options</td>
<td>1689</td>
<td>4421</td>
<td>1722</td>
<td>7832</td>
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<tr>
<td>Call %</td>
<td>66.25%</td>
<td>61.25%</td>
<td>69.80%</td>
<td>64.21%</td>
</tr>
<tr>
<td>Put %</td>
<td>33.75%</td>
<td>38.75%</td>
<td>30.20%</td>
<td>35.79%</td>
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<tr>
<td>Total # of calls</td>
<td>1119</td>
<td>2708</td>
<td>1202</td>
<td>5029</td>
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<tr>
<td>ST %</td>
<td>51.83%</td>
<td>46.68%</td>
<td>52.66%</td>
<td>49.25%</td>
</tr>
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<td>MT %</td>
<td>42.45%</td>
<td>43.61%</td>
<td>39.93%</td>
<td>42.47%</td>
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<tr>
<td>LT %</td>
<td>5.72%</td>
<td>9.71%</td>
<td>7.40%</td>
<td>8.27%</td>
</tr>
<tr>
<td>D-ITM %</td>
<td>26.99%</td>
<td>18.94%</td>
<td>22.21%</td>
<td>21.52%</td>
</tr>
<tr>
<td>ITM %</td>
<td>15.01%</td>
<td>16.58%</td>
<td>17.64%</td>
<td>16.48%</td>
</tr>
<tr>
<td>ATM %</td>
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<td>19.61%</td>
<td>29.12%</td>
<td>21.73%</td>
</tr>
<tr>
<td>OTM %</td>
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<td>19.31%</td>
<td>18.05%</td>
<td>19.65%</td>
</tr>
<tr>
<td>D-OTM %</td>
<td>16.89%</td>
<td>25.55%</td>
<td>12.98%</td>
<td>20.62%</td>
</tr>
</tbody>
</table>

<table>
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<th>MRK</th>
<th>All stocks</th>
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<td>Total # of puts</td>
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<td>520</td>
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<td>59.54%</td>
</tr>
<tr>
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<td>38.76%</td>
<td>30.58%</td>
<td>36.50%</td>
</tr>
<tr>
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<td>4.38%</td>
<td>4.04%</td>
<td>3.96%</td>
</tr>
<tr>
<td>D-ITM %</td>
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<td>8.58%</td>
<td>12.69%</td>
<td>10.81%</td>
</tr>
<tr>
<td>ITM %</td>
<td>16.67%</td>
<td>14.42%</td>
<td>10.19%</td>
<td>14.09%</td>
</tr>
<tr>
<td>ATM %</td>
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<td>30.00%</td>
<td>23.33%</td>
</tr>
<tr>
<td>OTM %</td>
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<td>23.23%</td>
<td>25.19%</td>
<td>22.73%</td>
</tr>
<tr>
<td>D-OTM %</td>
<td>24.56%</td>
<td>32.69%</td>
<td>21.92%</td>
<td>29.04%</td>
</tr>
</tbody>
</table>

Notes: This table describes the option data. As specified in the text ST denotes short term options with less than 42 trading days to expiration, MT denotes middle term options with between 42 and 126 trading days to expiration, and finally LT denotes the long term options with more than 126 trading days to expiration. In terms of moneyness, an option is deep in the money (D-ITM) if it is more than 6\% in the money, where moneyness is calculated using the discounted strikeprice. An option is in the money if it is less than 6\% but more than 2\% in the money. For the out of the money (OTM) options the same applies. Finally, the at the money (ATM) options are less than 2\% in or out of the money.
Table 2: Performance for the individual stocks in terms of moneyness and maturity for the constant volatility specifications

### Panel A: All options

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>ST</th>
<th>MT</th>
<th>LT</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-ITM</td>
<td>-0.0331</td>
<td>-0.0645</td>
<td>-0.0662</td>
<td>-0.0465</td>
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<tr>
<td>ITM</td>
<td>-0.0567</td>
<td>-0.0927</td>
<td>-0.0836</td>
<td>-0.0693</td>
</tr>
<tr>
<td>ATM</td>
<td>-0.0640</td>
<td>-0.0913</td>
<td>-0.0877</td>
<td>-0.0743</td>
</tr>
<tr>
<td>OTM</td>
<td>-0.1845</td>
<td>-0.1368</td>
<td>-0.1599</td>
<td>-0.1658</td>
</tr>
<tr>
<td>D-OTM</td>
<td>-0.4519</td>
<td>-0.3147</td>
<td>-0.2993</td>
<td>-0.3548</td>
</tr>
<tr>
<td>All</td>
<td>-0.1363</td>
<td>-0.1743</td>
<td>-0.1703</td>
<td>-0.1539</td>
</tr>
</tbody>
</table>

### Panel B: Call options

<table>
<thead>
<tr>
<th>Moneyness</th>
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<th>MT</th>
<th>LT</th>
<th>All</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-ITM</td>
<td>-0.0359</td>
<td>-0.0681</td>
<td>-0.0697</td>
<td>-0.0508</td>
</tr>
<tr>
<td>ITM</td>
<td>-0.0560</td>
<td>-0.0902</td>
<td>-0.0780</td>
<td>-0.0698</td>
</tr>
<tr>
<td>ATM</td>
<td>-0.0616</td>
<td>-0.0957</td>
<td>-0.0819</td>
<td>-0.0755</td>
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<tr>
<td>OTM</td>
<td>-0.1852</td>
<td>-0.1410</td>
<td>-0.1761</td>
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<tr>
<td>D-OTM</td>
<td>-0.4202</td>
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<td>-0.3533</td>
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<tr>
<td>All</td>
<td>-0.1198</td>
<td>-0.1691</td>
<td>-0.1670</td>
<td>-0.1447</td>
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</table>

### Panel C: Put options

<table>
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<tr>
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</thead>
<tbody>
<tr>
<td>D-ITM</td>
<td>-0.0254</td>
<td>-0.0456</td>
<td>-0.0415</td>
<td>-0.0314</td>
</tr>
<tr>
<td>ITM</td>
<td>-0.0579</td>
<td>-0.1013</td>
<td>0.0000</td>
<td>-0.0684</td>
</tr>
<tr>
<td>ATM</td>
<td>-0.0672</td>
<td>-0.0817</td>
<td>-0.1193</td>
<td>-0.0723</td>
</tr>
<tr>
<td>OTM</td>
<td>-0.1837</td>
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<td>-0.0673</td>
<td>-0.1625</td>
</tr>
<tr>
<td>D-OTM</td>
<td>-0.4815</td>
<td>-0.2908</td>
<td>-0.2633</td>
<td>-0.3567</td>
</tr>
<tr>
<td>All</td>
<td>-0.1606</td>
<td>-0.1851</td>
<td>-0.1827</td>
<td>-0.1704</td>
</tr>
</tbody>
</table>

Notes: This table shows the performance of the constant volatility specifications in terms of moneyness and maturity. The reported results are for the relative bias metric thus, denoting the $k$'th price estimate by $\hat{P}_k$ and the $k$'th observed price by $P_k$ this is $RBIAS \equiv K^{-1} \sum_{k=1}^{K} \left( \frac{\hat{P}_k - P_k}{P_k} \right)$. Results are reported for different categories or moneyness and maturity. As specified in the text ST denotes short term options with less than 42 trading days to expiration, MT denotes middle term options with between 42 and 126 trading days to expiration, and finally LT denotes the long term options with more than 126 trading days to expiration. In terms of moneyness, an option is deep in the money (D-ITM) if it is more than 6% in the money, where moneyness is calculated using the discounted strikeprice. An option is in the money if it is less than 6% but more than 2% in the money. For the out of the money (OTM) options the same applies. Finally, the at the money (ATM) options are less than 2% in or out of the money.
Table 3: Sample statistics for return series

<table>
<thead>
<tr>
<th>Ticker</th>
<th>GM</th>
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<th>MRK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0366</td>
<td>0.0250</td>
<td>0.0669</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>1.6455</td>
<td>1.4677</td>
<td>1.4469</td>
</tr>
<tr>
<td>Skewness Statistic</td>
<td>−0.344 ([0.0000])</td>
<td>−0.940 ([0.0000])</td>
<td>−0.001 ([0.9879])</td>
</tr>
<tr>
<td>Ex. Kurt. Statistic</td>
<td>10.761 ([0.0000])</td>
<td>23.931 ([0.0000])</td>
<td>3.417 ([0.0000])</td>
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<tr>
<td>Normality Statistic</td>
<td>24490 ([0.0000])</td>
<td>121370 ([0.0000])</td>
<td>2460 ([0.0000])</td>
</tr>
<tr>
<td>$R_t$ ACF(1)</td>
<td>0.022</td>
<td>−0.041</td>
<td>0.041</td>
</tr>
<tr>
<td>$R_t$ ACF(2)</td>
<td>−0.040</td>
<td>−0.012</td>
<td>−0.016</td>
</tr>
<tr>
<td>$R_t$ ACF(3)</td>
<td>−0.019</td>
<td>0.008</td>
<td>−0.041</td>
</tr>
<tr>
<td>$R_t$ ACF(4)</td>
<td>−0.048</td>
<td>−0.030</td>
<td>−0.013</td>
</tr>
<tr>
<td>$R_t$ ACF(5)</td>
<td>−0.004</td>
<td>0.045</td>
<td>0.008</td>
</tr>
<tr>
<td>$</td>
<td>R_t</td>
<td>$ ACF(1)</td>
<td>0.130</td>
</tr>
<tr>
<td>$</td>
<td>R_t</td>
<td>$ ACF(2)</td>
<td>0.077</td>
</tr>
<tr>
<td>$</td>
<td>R_t</td>
<td>$ ACF(3)</td>
<td>0.088</td>
</tr>
<tr>
<td>$</td>
<td>R_t</td>
<td>$ ACF(4)</td>
<td>0.075</td>
</tr>
<tr>
<td>$</td>
<td>R_t</td>
<td>$ ACF(5)</td>
<td>0.048</td>
</tr>
<tr>
<td>$Q(20)$ Statistic</td>
<td>41.705 ([0.0030])</td>
<td>48.405 ([0.0004])</td>
<td>38.682 ([0.0073])</td>
</tr>
<tr>
<td>$Q^2(20)$ Statistic</td>
<td>637.215 ([0.0000])</td>
<td>208.389 ([0.0000])</td>
<td>497.691 ([0.0000])</td>
</tr>
</tbody>
</table>

Notes: This table shows sample statistics for the continuously compounded returns, $R_t$, for the individual stocks and the index considered. The sample period is January 2, 1976 to December 29, 1995 for a total of $N = 5055$ observations. For the skewness (Skewness) and excess kurtosis (Ex. Kurt.) statistics, the brackets below the statistics report the p-values from testing the significance of the difference between the empirical values and the theoretical values from the Normal distribution using a t-test. For the normality (Normality) statistic the p-value of a t-version of the well known Jarque-Bera test for normality is reported in brackets below the statistics. Under the heading ACF the autocorrelation functions of lags 1 through 5 are reported for the absolute returns. The asymptotic variance of the autocorrelations is $1/N$. Finally, $Q(20)$ is the Ljung-Box portmanteau test for up to 20th order serial correlation in the returns, whereas $Q^2(20)$ is for up to 20th order serial correlation in the squared returns. In square brakets below these test statistics p-values are reported.
Table 4: Estimation results for individual stocks

### Panel A: General Motors (GM)

<table>
<thead>
<tr>
<th></th>
<th>CV</th>
<th>GARCH</th>
<th>NGARCH</th>
<th>TGARCH</th>
<th>EGARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.0174</td>
<td>0.0322</td>
<td>0.0134</td>
<td>0.0186</td>
<td>0.0187</td>
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<tr>
<td></td>
<td>(0.0141)</td>
<td>(0.0140)</td>
<td>(0.0140)</td>
<td>(0.0139)</td>
<td>(0.0139)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>2.7077</td>
<td>0.0305</td>
<td>0.0191</td>
<td>0.0148</td>
<td>0.0128</td>
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<tr>
<td></td>
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<td>(0.0079)</td>
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<td>$\beta$</td>
<td>0.9330</td>
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<tr>
<td></td>
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<td>(0.0167)</td>
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<td>$\alpha$</td>
<td>0.0574</td>
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<tr>
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<td>(0.0168)</td>
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<td>$\gamma/\psi/\theta$</td>
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<td>$-0.5668$</td>
<td>$-0.4536$</td>
<td>$-0.4175$</td>
<td></td>
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<tr>
<td>log-like</td>
<td>$-9690.36$</td>
<td>$-9360.67$</td>
<td>$-9337.06$</td>
<td>$-9339.40$</td>
<td>$-9338.45$</td>
</tr>
<tr>
<td>J-B</td>
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<td>1815.68</td>
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<td>1225.06</td>
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</tr>
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<td>[0.0000]</td>
<td>[0.0000]</td>
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</tr>
<tr>
<td>$Q(20)$</td>
<td>41.62</td>
<td>34.20</td>
<td>32.80</td>
<td>30.37</td>
<td>30.90</td>
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<td>[0.0031]</td>
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<td>[0.0355]</td>
<td>[0.0649]</td>
<td>[0.0565]</td>
</tr>
<tr>
<td>$Q^2(20)$</td>
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<td>[0.0001]</td>
<td>[0.0006]</td>
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<td>SIC</td>
<td>3.8342</td>
<td>3.7041</td>
<td>(3.6949)</td>
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</table>

### Panel B: International Business Machines (IBM)

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<th>TGARCH</th>
<th>EGARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.0098</td>
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<td>0.0137</td>
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<td>(0.0171)</td>
<td>(0.0140)</td>
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<td>(0.0150)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>2.1542</td>
<td>0.0246</td>
<td>0.0270</td>
<td>0.0159</td>
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<td>(0.0143)</td>
<td>(0.0131)</td>
<td>(0.0056)</td>
<td>(0.0054)</td>
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<tr>
<td>log-like</td>
<td>$-9112.36$</td>
<td>$-8798.75$</td>
<td>$-8769.71$</td>
<td>$-8752.04$</td>
<td>$-8751.54$</td>
</tr>
<tr>
<td>J-B</td>
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<td>13673.47</td>
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<tr>
<td>$Q(20)$</td>
<td>48.30</td>
<td>24.82</td>
<td>24.90</td>
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</tr>
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<tr>
<td>$Q^2(20)$</td>
<td>207.82</td>
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<td>14.34</td>
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<td>[0.7964]</td>
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### Panel C: Merck and Company Inc. (MRK)

<table>
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<td>$-8849.72$</td>
<td>$-8846.39$</td>
<td>$-8847.33$</td>
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<td>J-B</td>
<td>2459.82</td>
<td>523.78</td>
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<td>429.68</td>
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<td>[0.0007]</td>
<td>[0.0091]</td>
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<tr>
<td>$Q^2(20)$</td>
<td>507.39</td>
<td>20.02</td>
<td>22.85</td>
<td>23.22</td>
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<td>3.5020</td>
<td>(3.5008)</td>
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</table>

**Notes:** This table reports Quasi Maximum Likelihood Estimates (QMLE) for the daily returns assuming a risk-free interest rate of 5.4% corresponding to the value on December 29, 1995. Robust standard errors are reported in parentheses. J-B is the value of the usual Jarque-Bera normality test for the standardized residuals. $Q(20)$ is the Ljung-Box portmanteau test for up to 20th order serial correlation in the standardized residuals, whereas $Q^2(20)$ is for up to 20th order serial correlation in the squared standardized residuals. In square brackets below all test statistics p-values are reported. The last row reports the Schwarz Information Criteria, with an asterix denoting the minimum value.
Table 5: Continuous time parameter estimates

Panel A: General Motors

<table>
<thead>
<tr>
<th></th>
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<th>NGARCH</th>
<th>TGARCH</th>
<th>EGARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.0322 (0.0140)</td>
<td>0.0134 (0.0140)</td>
<td>0.0186 (0.0139)</td>
<td>0.0187 (0.0139)</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>0.0305 (0.0207)</td>
<td>0.0191 (0.0102)</td>
<td>0.0297 (0.0157)</td>
<td>0.0937 (0.0327)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.0095 (0.0063)</td>
<td>0.0057 (0.0035)</td>
<td>0.0074 (0.0043)</td>
<td>-0.0107 (0.0057)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.0812 (0.0365)</td>
<td>0.0589 (0.0211)</td>
<td>0.0309 (0.0101)</td>
<td>0.0611 (0.0201)</td>
</tr>
<tr>
<td>log-like</td>
<td>-9360.67</td>
<td>-937.06</td>
<td>-9339.4</td>
<td>-9338.45</td>
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</table>

Implied $\rho$ = 0

Panel B: International Business Machines

<table>
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<tr>
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<th>NGARCH</th>
<th>TGARCH</th>
<th>EGARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.0316 (0.0171)</td>
<td>0.0100 (0.0140)</td>
<td>0.0134 (0.0151)</td>
<td>0.0137 (0.0150)</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>0.0246 (0.0143)</td>
<td>0.0270 (0.0131)</td>
<td>0.0318 (0.0152)</td>
<td>0.1012 (0.0291)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.0087 (0.0064)</td>
<td>0.0104 (0.0061)</td>
<td>0.0082 (0.0050)</td>
<td>-0.0123 (0.0054)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.0738 (0.0365)</td>
<td>0.0689 (0.0254)</td>
<td>0.0353 (0.0104)</td>
<td>0.0666 (0.0184)</td>
</tr>
<tr>
<td>log-like</td>
<td>-8798.75</td>
<td>-8769.71</td>
<td>-8752.04</td>
<td>-8751.84</td>
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Implied $\rho$ = 0

Panel C: Merck and Company Inc.

<table>
<thead>
<tr>
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<th>NGARCH</th>
<th>TGARCH</th>
<th>EGARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>0.0512 (0.0139)</td>
<td>0.0407 (0.0141)</td>
<td>0.0410 (0.0142)</td>
<td>0.0415 (0.0142)</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>0.0692 (0.0281)</td>
<td>0.0628 (0.0198)</td>
<td>0.0930 (0.0292)</td>
<td>0.1252 (0.0269)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-0.0328 (0.0131)</td>
<td>-0.0300 (0.0094)</td>
<td>-0.0304 (0.0099)</td>
<td>-0.0310 (0.0097)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.0848 (0.0239)</td>
<td>0.0756 (0.0181)</td>
<td>0.0399 (0.0084)</td>
<td>0.0756 (0.0154)</td>
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<tr>
<td>log-like</td>
<td>-8858.53</td>
<td>-8849.72</td>
<td>-8846.59</td>
<td>-8847.33</td>
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</table>

Implied $\rho$ = 0

Notes: This table reports the continuous time parameter estimates for the diffusion limits, together with the corresponding correlation coefficient.
Table 6: Parameter values used in Monte Carlo study

<table>
<thead>
<tr>
<th>Mean specification</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation:</td>
<td>$S_t = S_{t-1} \times \exp { r - \delta - \frac{1}{2} h_t + \sqrt{h_t}\varepsilon_t }$</td>
</tr>
<tr>
<td>Parameters:</td>
<td>$r$</td>
</tr>
<tr>
<td>Annualized values</td>
<td>0.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volatility specifications</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation:</td>
<td></td>
</tr>
<tr>
<td>CV</td>
<td>$h_t = \omega$</td>
</tr>
<tr>
<td>GARCH</td>
<td>$h_t = \omega + \beta h_{t-1} + \alpha h_{t-1} \varepsilon_{t-1}^2$</td>
</tr>
<tr>
<td>NGARCH</td>
<td>$h_t = \omega + \beta h_{t-1} + \alpha h_{t-1} (\varepsilon_{t-1} + \gamma)^2$</td>
</tr>
<tr>
<td>Parameters:</td>
<td>$\omega$</td>
</tr>
<tr>
<td>CV</td>
<td>$2.48 \times 10^{-4}$</td>
</tr>
<tr>
<td>GARCH</td>
<td>$4.96 \times 10^{-6}$</td>
</tr>
<tr>
<td>NGARCH</td>
<td>$4.96 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

Notes: This table reports the parameter values used in the Monte Carlo study. In all specifications $\lambda = 0.05$. Furthermore, in all GARCH specifications $h_1$ is set to $2.48 \times 10^{-4}$.
Table 7: Parameter values used in Monte Carlo study of GARCH diffusion limits

<table>
<thead>
<tr>
<th>Mean specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation: ( d\ln S_t = (r - \delta - \frac{1}{2} h_t) , dt + \sqrt{h_t} dW_t )</td>
</tr>
<tr>
<td>Parameters: ( r \quad \delta )</td>
</tr>
<tr>
<td>Annualized values: 0.05 0.03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volatility specifications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation:</td>
</tr>
<tr>
<td>C-GARCH ( dh_t = \beta_0 dt + \beta_1 h_t dt + \beta_3 h_t dW_{1t}^* )</td>
</tr>
<tr>
<td>C-NGARCH ( dh_t = \beta_0 dt + \beta_1 h_t dt + \beta_2 h_t dW_{1t}^* + \beta_3 h_t dW_{2t}^* )</td>
</tr>
<tr>
<td>Parameters: ( \beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3 )</td>
</tr>
<tr>
<td>C-GARCH ( 4.96 \times 10^{-6} \quad -0.032 \quad 0.0679 )</td>
</tr>
<tr>
<td>C-NGARCH ( 4.96 \times 10^{-6} \quad -0.02 \quad -0.048 \quad 0.0679 )</td>
</tr>
</tbody>
</table>

Notes: This table reports the parameter values used in the Monte Carlo study of the diffusion limits. In all specifications the relevant parameters are backed out from the discret time parameters.
Table 8: American put price estimates in a model with different GARCH volatility processes

<table>
<thead>
<tr>
<th>T</th>
<th>K</th>
<th>BM Price</th>
<th>GARCH Price</th>
<th>NGARCH Price</th>
<th>C-GARCH Price</th>
<th>C-NGARCH Price</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>90</td>
<td>0.210</td>
<td>0.234 (0.0074)</td>
<td>-10.19%</td>
<td>0.276 (0.0091)</td>
<td>-23.82%</td>
</tr>
<tr>
<td>21</td>
<td>95</td>
<td>0.949</td>
<td>0.944 (0.0166)</td>
<td>0.57%</td>
<td>0.984 (0.0176)</td>
<td>-3.58%</td>
</tr>
<tr>
<td>21</td>
<td>100</td>
<td>2.795</td>
<td>2.752 (0.0275)</td>
<td>1.55%</td>
<td>2.728 (0.0257)</td>
<td>2.44%</td>
</tr>
<tr>
<td>21</td>
<td>105</td>
<td>5.974</td>
<td>5.948 (0.0315)</td>
<td>0.43%</td>
<td>5.859 (0.0321)</td>
<td>1.96%</td>
</tr>
<tr>
<td>21</td>
<td>110</td>
<td>10.206</td>
<td>10.006 (0.0376)</td>
<td>0.00%</td>
<td>10.143 (0.0294)</td>
<td>0.62%</td>
</tr>
<tr>
<td>63</td>
<td>90</td>
<td>1.222</td>
<td>1.235 (0.0249)</td>
<td>-1.02%</td>
<td>1.404 (0.0315)</td>
<td>-12.96%</td>
</tr>
<tr>
<td>63</td>
<td>95</td>
<td>2.589</td>
<td>2.551 (0.0333)</td>
<td>1.50%</td>
<td>2.682 (0.0421)</td>
<td>-3.44%</td>
</tr>
<tr>
<td>63</td>
<td>100</td>
<td>4.721</td>
<td>4.642 (0.0432)</td>
<td>1.70%</td>
<td>4.679 (0.0487)</td>
<td>0.90%</td>
</tr>
<tr>
<td>63</td>
<td>105</td>
<td>7.645</td>
<td>7.567 (0.0521)</td>
<td>1.04%</td>
<td>7.487 (0.0593)</td>
<td>2.11%</td>
</tr>
<tr>
<td>63</td>
<td>110</td>
<td>11.288</td>
<td>11.232 (0.0583)</td>
<td>0.50%</td>
<td>11.078 (0.0645)</td>
<td>1.89%</td>
</tr>
<tr>
<td>126</td>
<td>90</td>
<td>2.561</td>
<td>2.552 (0.0430)</td>
<td>0.32%</td>
<td>2.853 (0.0441)</td>
<td>-10.24%</td>
</tr>
<tr>
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<td>95</td>
<td>4.248</td>
<td>4.191 (0.0477)</td>
<td>1.36%</td>
<td>4.433 (0.0546)</td>
<td>-4.18%</td>
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<tr>
<td>126</td>
<td>100</td>
<td>6.506</td>
<td>6.411 (0.0660)</td>
<td>1.47%</td>
<td>6.558 (0.0674)</td>
<td>-0.81%</td>
</tr>
<tr>
<td>126</td>
<td>105</td>
<td>9.336</td>
<td>9.221 (0.0645)</td>
<td>1.24%</td>
<td>9.245 (0.0774)</td>
<td>0.98%</td>
</tr>
<tr>
<td>126</td>
<td>110</td>
<td>12.697</td>
<td>12.584 (0.0796)</td>
<td>0.90%</td>
<td>12.507 (0.0837)</td>
<td>1.52%</td>
</tr>
</tbody>
</table>

Notes: This table shows American put prices for a set of artificial options. The interest rate is fixed at 5% with a dividend yield of 3% both of which are annualized using 252 days a year. T denotes the time to maturity in days, K denotes the strike price, and for all the options the stock price is 100. The parameter values for the different GARCH processes are the ones specified in the text and in Table 6 for the discrete time models and 7 for the diffusion models. In the cross-sectional regressions powers of and cross-products between the stock level and the level of the volatility of total order less than or equal to two were used. Exercise is considered once every trading day. Prices reported are averages of 100 calculated prices using 20,000 paths and different seeds in the random number generator, “rann”, in Ox. In parentheses standard errors of these price estimates are reported. The column headed RBIAS reports the difference between the Binomial Model (BM) and the average of the simulated values relative to the simulated values. Thus, it indicates the relative mispricing by the constant volatility model which will be observed if the true model is the corresponding GARCH specification.
Table 9: Overall performance for the individual stocks and the index using GARCH volatility specifications

Panel A: All individual options for a total of 7832 options

<table>
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<th>Model</th>
<th>RBIAS</th>
<th>RSE</th>
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<td>Call</td>
</tr>
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<td>-0.1447</td>
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</tr>
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<td>-0.0539</td>
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<td>-0.0475</td>
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<td>C-TGARCH</td>
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</tr>
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<td>C-TGARCH</td>
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<td>CV</td>
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</tbody>
</table>

Notes: This table shows the performance of the different GARCH volatility specifications. We report results for all the metrics described in the text. Thus, denoting the k'th price estimate by $\hat{P}_k$ and the k'th observed price by $P_k$ these are the relative mean bias, $RBIAS \equiv K^{-1} \sum_{k=1}^{K} \frac{(\hat{P}_k-P_k)}{P_k}$, and the relative mean squared error, $RSE \equiv K^{-1} \sum_{k=1}^{K} \frac{(P_k-\hat{P}_k)^2}{P_k^2}$. 

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Table 10: Performance for the individual stocks in terms of moneyness and maturity for the TGARCH volatility specifications

<table>
<thead>
<tr>
<th></th>
<th>Panel A: All options</th>
<th></th>
<th>Panel B: Call options</th>
<th></th>
<th>Panel C: Put options</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TGARCH RBIAS</td>
<td></td>
<td>C-TGARCH RBIAS</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>ST</td>
<td>MT</td>
<td>LT</td>
<td>All</td>
<td>ST</td>
</tr>
<tr>
<td>D-ITM</td>
<td>-0.0098</td>
<td>-0.0037</td>
<td>0.0077</td>
<td>-0.0065</td>
<td>-0.0101</td>
</tr>
<tr>
<td>ITM</td>
<td>-0.0098</td>
<td>-0.0014</td>
<td>0.0262</td>
<td>-0.0054</td>
<td>-0.0056</td>
</tr>
<tr>
<td>ATM</td>
<td>-0.0152</td>
<td>0.0009</td>
<td>0.0177</td>
<td>-0.0081</td>
<td>-0.0073</td>
</tr>
<tr>
<td>OTM</td>
<td>-0.0503</td>
<td>0.0059</td>
<td>-0.0081</td>
<td>-0.0273</td>
<td>-0.0374</td>
</tr>
<tr>
<td>D-OTM</td>
<td>-0.1752</td>
<td>-0.0483</td>
<td>-0.0121</td>
<td>-0.0833</td>
<td>-0.1862</td>
</tr>
<tr>
<td>All</td>
<td>-0.0427</td>
<td>-0.0165</td>
<td>0.0014</td>
<td>-0.0292</td>
<td>-0.0385</td>
</tr>
</tbody>
</table>

Notes: This table shows the performance of the TGARCH and C-TGARCH volatility specifications in terms of moneyness and maturity. We report results for the relative bias metric only. Thus, denoting the k’th price estimate by \( \hat{P}_k \) and the k’th observed price by \( P_k \), this is \( RBIAS \equiv K^{-1} \sum_{k=1}^{K} (\hat{P}_k - P_k) / P_k \). Results are reported for different categories or moneyness and maturity. As specified in the text ST denotes short term options with less than 42 trading days to expiration, MT denotes middle term options with between 42 and 126 trading days to expiration, and finally LT denotes the long term options with more than 126 trading days to expiration. In terms of moneyness, an option is deep in the money (D-ITM) if it is more than 6% in the money, where moneyness is calculated using the discounted strikeprice. An option is in the money if it is less than 6% but more than 2% in the money. For the out of the money (OTM) options the same applies. Finally, the at the money (ATM) options are less than 2% in or out of the money.
Table 11: Relative performance for the individual stocks in terms of moneyness and maturity for the TGARCH volatility specifications

Panel A: All options

<table>
<thead>
<tr>
<th></th>
<th>TGARCH RBIAS</th>
<th></th>
<th>C-TGARCH RBIAS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ST</td>
<td>MT</td>
<td>LT</td>
<td>All</td>
</tr>
<tr>
<td>D-ITM</td>
<td>29.47%</td>
<td>5.77%</td>
<td>-11.59%</td>
<td>13.90%</td>
</tr>
<tr>
<td>ITM</td>
<td>17.22%</td>
<td>1.48%</td>
<td>-31.37%</td>
<td>7.84%</td>
</tr>
<tr>
<td>ATM</td>
<td>23.72%</td>
<td>-0.99%</td>
<td>-20.23%</td>
<td>10.94%</td>
</tr>
<tr>
<td>OTM</td>
<td>27.25%</td>
<td>-4.35%</td>
<td>5.69%</td>
<td>16.46%</td>
</tr>
<tr>
<td>D-OTM</td>
<td>38.78%</td>
<td>15.34%</td>
<td>4.03%</td>
<td>23.47%</td>
</tr>
<tr>
<td>All</td>
<td>31.31%</td>
<td>9.47%</td>
<td>-0.80%</td>
<td>18.94%</td>
</tr>
</tbody>
</table>

Panel B: Call options

<table>
<thead>
<tr>
<th></th>
<th>TGARCH RBIAS</th>
<th></th>
<th>C-TGARCH RBIAS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ST</td>
<td>MT</td>
<td>LT</td>
<td>All</td>
</tr>
<tr>
<td>D-ITM</td>
<td>31.01%</td>
<td>6.27%</td>
<td>-13.96%</td>
<td>13.77%</td>
</tr>
<tr>
<td>ITM</td>
<td>9.31%</td>
<td>1.86%</td>
<td>-38.38%</td>
<td>2.65%</td>
</tr>
<tr>
<td>ATM</td>
<td>27.54%</td>
<td>2.91%</td>
<td>-32.80%</td>
<td>11.52%</td>
</tr>
<tr>
<td>OTM</td>
<td>36.79%</td>
<td>6.22%</td>
<td>11.38%</td>
<td>24.66%</td>
</tr>
<tr>
<td>D-OTM</td>
<td>51.60%</td>
<td>36.02%</td>
<td>20.71%</td>
<td>39.32%</td>
</tr>
<tr>
<td>All</td>
<td>38.40%</td>
<td>23.33%</td>
<td>7.63%</td>
<td>27.98%</td>
</tr>
</tbody>
</table>

Panel C: Put options

<table>
<thead>
<tr>
<th></th>
<th>TGARCH RBIAS</th>
<th></th>
<th>C-TGARCH RBIAS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ST</td>
<td>MT</td>
<td>LT</td>
<td>All</td>
</tr>
<tr>
<td>D-ITM</td>
<td>23.41%</td>
<td>1.89%</td>
<td>16.22%</td>
<td>14.71%</td>
</tr>
<tr>
<td>ITM</td>
<td>29.33%</td>
<td>0.30%</td>
<td>100.00%</td>
<td>18.95%</td>
</tr>
<tr>
<td>ATM</td>
<td>19.02%</td>
<td>-11.04%</td>
<td>27.29%</td>
<td>9.92%</td>
</tr>
<tr>
<td>OTM</td>
<td>14.84%</td>
<td>-24.12%</td>
<td>-88.83%</td>
<td>3.32%</td>
</tr>
<tr>
<td>D-OTM</td>
<td>28.32%</td>
<td>-17.77%</td>
<td>-34.29%</td>
<td>3.48%</td>
</tr>
<tr>
<td>All</td>
<td>23.46%</td>
<td>-16.97%</td>
<td>-29.70%</td>
<td>5.18%</td>
</tr>
</tbody>
</table>

Notes: This table shows the performance of the TGARCH and C-TGARCH volatility specifications relative to the CV specification in terms of moneyness and maturity. We report results for the relative bias metric only. Thus, denoting the k'th price estimate by \( \bar{P}_k \) and the k'th observed price by \( P_k \) this is \( RBIAS = \frac{1}{K} \sum_{k=1}^{K} \frac{(\bar{P}_k - P_k)}{P_k} \). Results are reported for different categories or moneyness and maturity. As specified in the text ST denotes short term options with less than 42 trading days to expiration, MT denotes middle term options with between 42 and 126 trading days to expiration, and finally LT denotes the long term options with more than 126 trading days to expiration. In terms of moneyness, an option is deep in the money (D-ITM) if it is more than 6% in the money, where moneyness is calculated using the discounted strikeprice. An option is in the money if it is less than 6% but more than 2% in the money. For the out of the money (OTM) options the same applies. Finally, the at the money (ATM) options are less than 2% in or out of the money.
Figure 1: Plot of the RBIAS surface for the CV model.
Figure 2: The left hand panels in this figure plot the log-densities of the standardized residuals from the constant volatility model for the four return series. The normal density is shown as the dotted line. The right hand panels in the figure show time series plots of the raw continuously compounded returns, $R_t$. 
Figure 3: Plot of the RBIAS surface for put options when using the CV model when the true underlying model is a GARCH specification.
Figure 4: Plot of the RBIAS surface for the TGARCH (left hand panels) and C-TGARCH (right hand panels) volatility specifications for individual stock options.