44. Class Number One Criteria for Real Quadratic Fields. II

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This paper continues the work begun in [6]. Therein we gave criteria for real quadratic fields of narrow Richaud-Degert (R-D) type to have class number one. This was a consequence of more general criteria given for real quadratic fields \(Q(\sqrt{n})\) with \(n \equiv 1 \pmod{4}\).

Herein we will deal with positive square-free integers \(n\) of wide (R-D) type; i.e., \(n = m^2 + r\) where \(r\) divides \(4m\) and \(r \in (-m, m] \) with \(|r| \neq 1, 4\). The first result generalizes results in [1, 3, 4, 9 and 11].

**Theorem 1.** Let \(n = 4^v + r > 7\) be of wide R-D type such that \(n \equiv 1 \pmod{4}\). If \(h(n) = 1\) then:

1. \(|r| = 2\).
2. \(p\) is inert in \(Q(\sqrt{n})\) for all odd primes \(p\) dividing \(l\).
3. If \(r = 2\) then \(l \equiv 0 \pmod{3}\).
4. If \(r = -2\) then \(l \equiv 0 \pmod{3}\).

**Proof.** Since \(n \equiv 1 \pmod{4}\) then \(2\) is ramified in \(Q(\sqrt{n})\). Therefore, there are integers \(x\) and \(y\) such that \(x^2 - 4ny^2 = \pm 2\). By [5, Theorem 1.1]

\[2\geq|r|\] for \(|r| = 2\) since \(|r| \neq 1\) by hypothesis. This secures (1). If \(p\) is an odd prime dividing \(l\) such that \(p\) is not inert in \(Q(\sqrt{n})\) then there are integers \(u\) and \(v\) such that \(u^2 - 4nv^2 = \pm p\). By [5, Theorem 1.2] \(n = 7\) and \(p = 3\) are forced. This secures (2).

If \(3\) is not inert in \(Q(\sqrt{n})\) then \(x^2 - 4ny^2 = \pm 3\) for some integers \(x\) and \(y\). Assume that \(x > 0\) and that \(y > 0\) is the least positive solution. Thus we may invoke [7, Theorem 108-108a, pp. 205-207] to get that if \(x^2 - 4ny^2 = 3\) then for \(x_i = (2^l + r)/|r|\) and \(y_i = 2l/|r|\) (see [2] and [8]):

\[0 < y_i \leq y_i \sqrt{3 + \sqrt{2(x_i + 1)}}\]

and if \(x^2 - 4ny^2 = -3\) then:

\[0 < y \leq y_i \sqrt{3 + \sqrt{2(x_i - 1)}}\]

A tedious check shows that \(y = 1\).

Therefore \(x^2 - n = \pm 3\); i.e., \(x^2 - l^2 = r \pm 3\). An easy check shows that the only possible solutions to the latter equation occur when either \(l = r = 2\) or \(l = 3\), and \(r = -2\). Thus, if \(n > 6\) when \(r = 2\), and \(n > 7\) when \(r = -2\) then \(3\) is inert in \(Q(\sqrt{n})\); whence \(n \equiv 2 \pmod{3}\). Therefore, \(l \equiv 0 \pmod{3}\) if \(r = 2\), and \(l \equiv 0 \pmod{3}\) if \(r = -2\). This secures (3), (4) and the theorem. Q.E.D.

**Remark 1.** The converse of Theorem 1 is false. For example, if \(n = 12^2 + 2 = 146\) then Theorem 1 (1)-(3) are satisfied, but \(h(n) = 2\).
The following Table illustrates Theorem 1.

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<th>n</th>
<th>h(n)</th>
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<td>1</td>
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<td>326</td>
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<td>-2</td>
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<td>21</td>
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</table>

All class numbers are taken from [10].

**Theorem 2.** Let \( n = l^2 + r \) be of R-D type with \( r \mid 2l \), and \( n \equiv 1 \pmod{4} \). If \( h(n) = 1 \) then:

1. If \( n \equiv 1 \pmod{8} \) then \( n = 33 \).
2. If \( n \equiv 5 \pmod{8} \) then \( r < 0 \), \( -r \) is a prime and \( p \) is inert in \( Q(\sqrt{n}) \) for all primes \( p < |r|/4 \).

**Proof.** If \( n \equiv 1 \pmod{8} \) then \( 2 \) splits in \( Q(\sqrt{n}) \). Thus there are integers \( a \) and \( b \) such that \( a^2 - nb^2 = \pm 8 \).

By [5, Theorem 1.1] \( |r| \leq 8 \). Also, using [7, Theorems 108–108a, pp. 205–207] we may achieve that \( b = 1 \) by the same reasoning as in the proof of Theorem 1. Hence \( a^2 - b = r \pm 8 \) where \( |r| \leq 8 \). However, \( n \equiv 1 \pmod{8} \) and \( |r| \neq 1, 4 \). Therefore, \( r \in \{-7, -3, 5\} \). An easy check of \( a^2 - b = r \pm 8 \) for these values of \( r \) yields that the only solution is \( l = 6 \) and \( r = -3 \); i.e., \( n = 33 \).

Suppose that \( n \equiv 5 \pmod{8} \). If \( |r| \) is not prime then there exists a prime \( p \) dividing \( |r| \) such that \( 2 < p < |r| \) and \( p \) is ramified in \( Q(\sqrt{n}) \). Therefore, there are integers \( c \) and \( d \) with \( c^2 - nd^2 = \pm 4p \); whence \( 4p \geq |r| \) by [5, Theorem 1.1]. Hence, \( |r| = 2p, 3p \) or \( 4p \). Either even case contradicts that \( n \equiv 5 \pmod{8} \). For the \( |r| = 3p \) case we note that it is well-known that if \( h(n) = 1 \) then \( n = s \) or \( pq \) where \( s, p \) and \( q \) are primes such that either \( p = 2 \) and \( q \equiv 3 \pmod{4} \) or \( p \equiv q \equiv 3 \pmod{4} \), (e.g., see [5]). Thus \( |r| = 3p \) implies that \( n \) is a product of more than two primes. Hence \( |r| \) is a prime. Moreover \( |r| \equiv 3 \pmod{4} \) and \((F+r)\mid r \equiv 3 \pmod{4} \) is prime. If \( r > 0 \) then \( F \equiv 2r \).
(mod 4) forcing \( r \) to be even, a contradiction. Thus \( r < 0 \).

If \( p < |r|/4 \) is a prime which is not inert in \( \mathbb{Q}(\sqrt{n}) \), then there are integers \( e \) and \( f \) such that \( e^2 - nf^2 = \pm 4p \) with \( |r| > 4p \). This contradicts [5, Theorem 1.1]. This secures the theorem.

Q.E.D.

Two examples which illustrate Theorem 2 (2) are \( n = 141 = 12^2 - 8 \) and \( n = 1757 = 42^2 - 7 \) for which \( h(n) = 1 \).

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References


