Infinite Families of Pellian Polynomials and Their Continued Fraction Expansions*

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Abstract

We investigate families \( \{D_k(X)\}_{k \in \mathbb{N}} \) of quadratic integral polynomials and show that, for a fixed \( k \in \mathbb{N} \) and arbitrary \( X \in \mathbb{N} \), the period length of the simple continued fraction expansion of \( \sqrt{D_k(X)} \) is constant. Furthermore, we show that the period lengths of \( \sqrt{D_k(X)} \) go to infinity with \( k \). For each member of the families involved, we show how to easily determine the fundamental unit of the underlying quadratic field. We also demonstrate how the simple continued fraction expansion of \( \sqrt{D_k(X)} \) is related to that of \( \sqrt{C} \). This continues work in [3]–[5].

1 Introduction

There is a long history to the problem of finding infinite parametric families of non-square integers \( \{D_k\}_{k \in \mathbb{N}} \) for which the fundamental unit can be easily determined (see [1, Chapter 12]). The basic idea, for quadratic polynomials \( D(X) \) is to look at the period length \( \ell(\sqrt{D(X)}) \) of the simple continued fraction expansion expansion of \( \sqrt{D(X)} \) and to provide families of such polynomials \( \{D_k(X)\}_{k \in \mathbb{N}} \) such that \( \lim_{X \to \infty} \ell(\sqrt{D(X)}) \) is constant while \( \lim_{k \to \infty} \ell(\sqrt{D_k(X)}) = \infty \). Moreover, we are able to explicitly obtain the fundamental unit \( \varepsilon_D \) of the quadratic order \( \mathcal{O}_D = \mathbb{Z} \left[ \sqrt{D(X)} \right] \) for all \( k, X \in \mathbb{N} \). For these families \( \varepsilon_{D_k} \) are particularly small, since \( \ell(\sqrt{D_k(X)}) \) is independent of \( X \), and this means that the class number \( h_{D_k} \) of \( \mathbb{Z} \left[ \sqrt{D_k(X)} \right] \) is particularly large.

The means by which we achieve the above is to employ the well-studied theory behind Extended Richaud-Degert type (ERD)-radicands, those of the form \( D = a^2 + r > 0 \) where \( r \mid 4a \) for \( a, r \in \mathbb{Z} \) (see [2]). Polynomials \( D(X) \) of ERD-type satisfy \( \ell(\sqrt{D(X)}) \leq 12 \), so are ostensibly of little interest.

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in the aforementioned scenario. Thus, in [7], for instance, they cautiously avoid such polynomials. However, in [6], we gave infinitely many counterexamples to the principal results therein and did so by observing that, for the quadratic polynomials \( D_k(X) = A_k^2 X^2 + 2B_k X + C \) investigated therein, \( A_k^2 D_k(X) = (A_k^2 X + B_k)^2 - 1 \), which is of ERD-type. From the well-studied ERD-theory, we know that \( \varepsilon_{A_k^2 D_k(X)} = A_k^2 X + B_k + \sqrt{A_k^2 D_k(X)} \). In [6], we proved that \( \varepsilon_{A_k^2 D_k(X)} = \varepsilon_{A_k D_k(X)} \), and in the process developed for so doing, were able to explicitly determine the simple continued fraction expansion of \( \sqrt{D_k(X)} \) in terms of that for \( \sqrt{C} \). These results generalized what was (intended) in [7]. In [5], we looked at ERD-types with resulting period lengths that go to infinity with \( k \), while the period length remains constant for any \( X \in \mathbb{N} \). In this paper, we provide two more infinite families which satisfy the above criteria and that employ the above techniques, in a different direction than that elucidated in [5].

2 Notation and Preliminaries

The background for the following together with proofs and details may be found in [2]. Let \( \Delta = d^2 D_0 \) (\( d \in \mathbb{N} \), \( D_0 > 1 \) squarefree) be the discriminant of a real quadratic order \( \mathcal{O}_\Delta = \mathbb{Z} + \mathbb{Z}[\sqrt{\Delta}] = [1, \sqrt{\Delta}] \) in \( \mathbb{Q}(\sqrt{\Delta}) \), \( U_\Delta \) the group of units of \( \mathcal{O}_\Delta \), and \( \varepsilon_\Delta \) the fundamental unit of \( \mathcal{O}_\Delta \).

Now we introduce the notation for continued fractions. Let \( \alpha \in \mathcal{O}_\Delta \). We denote the simple continued fraction expansion of \( \alpha \) (in terms of its partial quotients) by:

\[
\alpha = (q_0; q_1, \ldots, q_n, \ldots).
\]

If \( \alpha \) is periodic, we use the notation:

\[
\alpha = (q_0; q_1, q_2, \ldots, q_k; \overline{q_k, q_{k+1}, \ldots, q_{k+\ell-1}}),
\]

to denote the fact that \( q_n = q_{n+k} \) for all \( n \geq k \). The smallest such \( \ell = \ell(\alpha) \in \mathbb{N} \) is called the period length of \( \alpha \) and \( q_0, q_1, \ldots, q_k-1 \) is called the pre-period of \( \alpha \). If \( k = 0 \) is the least such nonnegative value, then \( \alpha \) is purely periodic, namely,

\[
\alpha = (q_0; q_1, \ldots, q_{\ell-1}).
\]

The convergents (for \( n \geq 0 \)) of \( \alpha \) are denoted by

\[
\frac{x_n}{y_n} = \left( q_0; q_1, \ldots, q_n \right) = \frac{q_n x_{n-1} + x_{n-2}}{q_n y_{n-1} + y_{n-2}}.
\]

We will need the following facts, the proofs of which can be found in most standard undergraduate number theory texts (for example see [4], and see [2] for a more advanced exposition).
\begin{align*}
x_j &= q_j x_{j-1} + x_{j-2} \quad \text{(for } j \geq 0 \text{ with } x_{-2} = 0, \text{ and } x_{-1} = 1\text{)}, \\
y_j &= q_j y_{j-1} + y_{j-2} \quad \text{(for } j \geq 0 \text{ with } y_{-2} = 1, \text{ and } y_{-1} = 0\text{)},
\end{align*}

\begin{align*}
x_j y_{j-1} - x_{j-1} y_j &= (-1)^{j-1} \quad (j \in \mathbb{N}),
\langle q_j, q_{j-1}, \ldots, q_1 \rangle &= y_j / y_{j-1} \quad (j \in \mathbb{N}),
\langle q_j, q_{j-1}, \ldots, q_1, q_0 \rangle &= x_j / x_{j-1} \quad (j \in \mathbb{N}),
\end{align*}

In particular, we will be dealing with \( \alpha = \sqrt{D} \) where \( D \) is a radicand. In this case, the **complete quotients** are given by \((P_j + \sqrt{D})/Q_j\) where the \( P_j \) and \( Q_j \) are given by the recursive formulae as follows for any \( j \geq 0 \) (with \( P_0 = 0 \) and \( Q_0 = 1 \)):

\begin{align*}
q_j &= \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor, \\
P_{j+1} &= q_j Q_j - P_j,
\end{align*}

and

\[ D = P_{j+1}^2 + Q_j Q_{j+1}. \]

Thus, we may write:

\[ \alpha = \langle q_0; q_1, \ldots, q_{\ell}, (P_{\ell+1} + \sqrt{D})/Q_{\ell+1} \rangle. \]

We will also need the following facts for \( \alpha = \sqrt{D} \). For any integer \( j \geq 0 \), and \( \ell = \ell(\sqrt{D}) \):

\[ \sqrt{D} = \langle q_0; q_1, \ldots, q_{\ell-1}, 2q_0 \rangle, \]

where \( q_j = q_{\ell-j} \) for \( j = 1, 2, \ldots, \ell - 1 \), and \( q_0 = \lfloor \sqrt{D} \rfloor \)

\begin{align*}
x_{j+1} &= q_0 y_{j+1} + y_{j+2}, \\
D y_{j+1} &= q_0 x_{j+1} + x_{j+2}.
\end{align*}

Also, for any \( j \in \mathbb{N} \)

\begin{align*}
x_j x_{j-1} - D y_j y_{j-1} &= (-1)^j P_{j+1},
\end{align*}

\begin{align*}
P_1 &= P_0 = q_0 \quad \text{and} \quad Q_0 = Q_1 = 1, \\
x_{j-1}^2 - y_{j-1}^2 D &= (-1)^j Q_j,
\end{align*}

\[ q_j < 2q_0. \]
When $\ell$ is even,

$$P_{\ell/2} = P_{\ell/2+1} = P_{(2j-1)\ell/2+1} = P_{(2j-1)\ell/2} \text{ and } Q_{\ell/2} = Q_{(2j-1)\ell/2},$$

whereas when $\ell$ is odd,

$$Q_{(t-1)/2} = Q_{(t+1)/2}.$$  

(2.20)

Lastly, the following result on Pell's equation will be quite useful in establishing results in the next section. In fact, this gives more detail to the fact exhibited in (2.17) in the case where the index is a multiple of $\ell$.

**Theorem 2.1** For $j \geq 0$, let $x_j$ and $y_j$ be as above in the simple continued fraction expansion of $\sqrt{D}$ for a nonsquare $D > 0$ and let $\ell = \ell(\sqrt{D})$. If $\ell$ is even, then all positive integer solutions of

$$x^2 - Dy^2 = 1$$

are given by $(x, y) = (x_{j(t-1)}, y_{j(t-1)})$ for $j \geq 1$, whereas there are no integer solutions of

$$x^2 - Dy^2 = -1.$$  

(2.22)

If $\ell$ is odd, then all positive solutions of (2.21) are given by

$$(x, y) = (x_{2j(t-1)}, y_{2j(t-1)})$$

for $j \geq 1$, whereas all positive solutions of (2.22) are given by

$$(x, y) = (x_{(2j-1)(t-1)}, y_{(2j-1)(t-1)}).$$

Proof. See [4, Corollary 5.3.3, p. 249].

The following lemmas were all proved in [5], so we state them here for the convenience of the reader given that we will be using them in the next section.

*For the balance of the paper, we make the following assumptions.*

We let $A, B, C, k, X \in \mathbb{N}$ with $C$ not a perfect square. Suppose that $(x, y) = (B, A)$ is the smallest positive solution of $x^2 - Cy^2 = 1$ and define, for each $k \in \mathbb{N}$,

$$B_k + A_k\sqrt{C} = (B + A\sqrt{C})^k.$$  

Also, for $n \geq 0$, set

$$\sqrt{C} = \langle c_0; c_1, \ldots, c_n, 2c_0 \rangle,$$

where this is understood to mean $\sqrt{C} = \langle c_0; 2c_0 \rangle$ in the case where $n = 0.$
Lemma 2.1 For $n \in \mathbb{N}$ odd and $k = (2m + 1)$ where $m$ is a nonnegative integer:

$$B_k = x_{(2m+1)(n+1)/2}y_{(2m+1)(n+1)/2-1} + x_{(2m+1)(n+1)/2-1}y_{(2m+1)(n+1)/2-2}$$

and

$$A_k = y_{(2m+1)(n+1)/2-1}(y_{(2m+1)(n+1)/2} + y_{(2m+1)(n+1)/2-2}).$$

Lemma 2.2 If $n$ is odd and $k = 2m + 1$ with $m$ a nonnegative integer, then

$$2x_{(2m+1)(n+1)/2-1} = Q_{(n+1)/2}(y_{(2m+1)(n+1)/2} + y_{(2m+1)(n+1)/2-1}).$$

Lemma 2.3 For $n \in \mathbb{N}$ odd, $k = 2m + 1$, and $m \geq 0$,

$$Q_{(n+1)/2}B_k = x_{(2m+1)(n+1)/2-1}^2 + y_{(2m+1)(n+1)/2-1}^2C$$

and

$$Q_{(n+1)/2}A_k = 2x_{(2m+1)(n+1)/2-1}y_{(2m+1)(n+1)/2-1}.$$

Lemma 2.4 For $n \in \mathbb{N}$ odd, $k = 2m + 1$, and $m \geq 0$,

$$A_kx_{(2m+1)(n+1)/2-1} - (B_k \pm 1)y_{(2m+1)(n+1)/2-1} = y_{(2m+1)(n+1)/2-1}(-1)^{(n+1)/2} \mp 1).$$

Lemma 2.5 If $n \in \mathbb{N}$ is odd, and $m \geq 0$, then

$$B_k + (-1)^{(n+3)/2} = y_{(2m+1)(n+1)/2-1}(x_{(2m+1)(n+1)/2} + x_{(2m+1)(n+1)/2-2})$$

3 Main Results

Assuming $m \geq 0$, in the following, we set:

$$w_m = c_1, \ldots, c_n, 2c_0, c_1, \ldots, c_n, 2c_0, \ldots, c_1, \ldots, c_n,$$

which is $m$ iterations of $c_1, \ldots, c_n, 2c_0$ followed by one iteration of $c_1, \ldots, c_n$. In the case where $n = 0$, $w_m$ is just $m$ iterations of $2c_0$, and when $m = n = 0$, $w_m$ is the empty string. Also, for $m \geq 0$ and odd $n > 1$,

$$v_m = c_1, c_2, \ldots, c_n, 2c_0, c_1, c_2, \ldots, c_n, 2c_0, \ldots, c_1, \ldots, 2c_0, c_1, c_2, \ldots, c_{(n-1)/2},$$
which means $m$ iterations of $c_1, c_2, \ldots, c_n, 2c_0$ followed by one iteration of $c_1, c_2, \ldots, c_{(n-1)/2}$, and the reverse of this is denoted by

$$
\overline{v_m} = c_{(n-1)/2}, \ldots, c_2, c_1, 2c_0, c_n, \ldots, c_1, \ldots, 2c_0, c_n, \ldots, c_1,$$

one iteration of $c_{(n-1)/2}, c_n, \ldots, c_1$ followed by $m$ iterations of $2c_0, c_n, \ldots, c_1$. Lastly, the symbols $Q_j$, defined in formulas (2.7)-(2.9), refer to the continued fraction expansion of $\sqrt{C}$.

The following results provide infinite families of polynomials $D_k(X)$ such that $\ell(\sqrt{D_k(X)})$ is independent of the variable $X$ and

$$
\lim_{k \to \infty} (\ell(\sqrt{D_k(X)})) = \infty,
$$

while at the same time having an explicit formulation of the fundamental unit of $\mathbb{Z}[\sqrt{D_k(X)}]$ which in turn, have large class numbers. We also show how the simple continued fraction expansion of $\sqrt{D_k(X)}$ is intimately related to the simple continued fraction expansion of $\sqrt{C}$.

**Theorem 3.1** Let

$$
D_k(X) = (B_k - 1)^2 A_k^2 X^2 (A_k^2 X + 2) + 2(B_k - 1)^2 X (A_k^2 X + 2) + 2(B_k - 1)X + C.
$$

Then the fundamental solution of

$$
x^2 - D_k(X)y^2 = 1
$$

is given by

$$(x, y) = ((B_k - 1)(A_k^2 X + 1)^3 + 1, A_k(A_k^2 X + 1)),$$

and for

$$
g_0 = (B_k - 1)(A_k X (A_k^2 X + 2)) + c_0 :$$

(a) If both $n, k \in \mathbb{N}$ are even, then,

$$
\sqrt{D_k(X)} = \left\langle g_0; w_{k-1}, c_0, 2(B_k - 1)A_k X/C, c_0, w_{k-1}, 2g_0 \right\rangle,
$$

with $\ell(\sqrt{D_k(X)}) = 2k(n + 1) + 2$.

(b) If $n \geq 0$ is even and $k$ is odd, then

$$
\sqrt{D_k(X)} = \left\langle g_0; w_{k-1}, 2(B_k - 1)A_k X + 2c_0, w_{k-1}, 2g_0 \right\rangle,
$$

and $\ell(\sqrt{D_k(X)}) = 2k(n + 1)$. 
(c) If $n$ is odd, then one of the following holds.

(i) If $k = 2m + 1$, $m \geq 0$, and $(n+1)/2 > 1$ is odd,

\[ \sqrt{D_k(X)} = \left\{ q_0; \overline{v_m, 2(B_k - 1)A_k X / Q_{(n+1)/2} + c_{(n+1)/2}, \overline{v_m, 2q_0}} \right\} \]

with $\ell\left(\sqrt{D_k(X)}\right) = k(n+1)$.

(ii) If $k = 2m + 1$, $m \geq 0$, and $(n+1)/2$ is even, $\sqrt{D_k(X)} = \left\{ q_0; \overline{v_m, c_{(n+1)/2}, \frac{2Q_{(n+1)/2}(B_k - 1)A_k X}{C}, \frac{c_{(n+1)/2}}{2}, \overline{v_m, 2q_0}} \right\}$

with $\ell\left(\sqrt{D_k(X)}\right) = k(n+1) + 2$.

(iii) If $k = 2m$, $m \in \mathbb{N}$, then

\[ \sqrt{D_k(X)} = \left\{ q_0; \overline{w_{m-1}, c_0, 2(B_k - 1)A_k X / C, c_0, w_{m-1}, 2q_0} \right\} \]

with $\ell(\sqrt{D_k(X)}) = k(n+1) + 2$.

Proof. Since it may be calculated that

\[ A_k^2 D_k(X) = (B_k - 1)^2 (A_k^2 X + 1)^4 + 2(B_k - 1)(A_k^2 X + 1), \]

and

\[ [(B_k - 1)(A_k^2 X + 1)^4 + 1]^2 - (A_k^2 X + A_k)^2 D_k(X) = 1, \]

then $D_k(X)$ is not a perfect square, so by [2, Theorem 3.2.1, p. 78],

\[ \varepsilon_{A_k^2 D_k(X)} = (B_k - 1)(A_k^2 X + 1)^3 + 1 + (A_k^2 X + A_k) \sqrt{D_k(X)}. \]

Let $\ell = \ell(\sqrt{D_k(X)})$ and $X_i/Y_i$ be the $i$-th convergent of $\sqrt{D_k(X)}$. Then by Theorem 2.1, there is a $j \in \mathbb{N}$ such that

\[ (B_k - 1)(A_k^2 X + 1)^3 + 1 = X_{j\ell-1} \text{ and } A_k^2 X + A_k = Y_{j\ell-1}. \]

First, we prove part (a) for which we now show that $j = 1$. Thus, we show via Theorem 2.1 that $
\varepsilon_{A_k^2 D_k(X)} = \varepsilon_{D_k(X)}$, since $\ell$ will be shown to be even. In the process of doing this, the continued fraction expansion in (a) will be shown to hold.

If $x_i/y_i$ is the $i$-th convergent of $\sqrt{C}$, then by (2.1),

\[ \left\{ q_0, w_{k-1}, c_0, 2(B_k - 1)A_k X / C, c_0, w_{k-1} \right\} = \]
\[
(q_0, w_{k-1}, c_0, 2(B_k - 1)A_kX/C + y_{k(n+1)-1}/x_{k(n+1)-1}) = \\
\left\langle q_0, w_{k-1}, \frac{2c_0(B_k - 1)A_kXx_{k(n+1)-1} + Cc_0y_{k(n+1)-1} + Cx_{k(n+1)-1}}{2(B_k - 1)A_kXx_{k(n+1)-1} + Cy_{k(n+1)-1}} \right\rangle.
\]

(3.23)

If we set,

\[
M = \frac{2c_0(B_k - 1)A_kXx_{k(n+1)-1} + Cc_0y_{k(n+1)-1} + Cx_{k(n+1)-1}}{2(B_k - 1)A_kXx_{k(n+1)-1} + Cy_{k(n+1)-1}},
\]

then (3.23) equals,

\[
(B_k - 1)(A_k^3X^2 + 2A_kX) + \frac{Mx_{k(n+1)-1} + x_{k(n+1)-2}}{My_{k(n+1)-1} + y_{k(n+1)-2}},
\]

which may be manipulated using (2.4), (2.13), and (2.17) to equal:

\[
(B_k - 1)(A_k^3X^2 + 2A_kX) + \frac{2x_{k(n+1)-1}y_{k(n+1)-1}C(B_k - 1)A_kX + C(2x_{k(n+1)-1}^2 + (-1)^{k(n+1)-1})}{2Cy_{k(n+1)-1}x_{k(n+1)-1} + 2(B_k - 1)A_kXx_{k(n+1)-1}}.
\]

(3.24)

However, since \( n + 1 \) is odd, we must have that \( B_k = x_{2k(n+1)-1} \) and \( A_k = y_{2k(n+1)-1} \) by Theorem 2.1. Also since

\[
B_k + A_k\sqrt{C} = (x_n + y_n\sqrt{C})^{2k} = (x_{k(n+1)-1} + y_{k(n+1)-1}\sqrt{C})^2 = \\
x_{k(n+1)-1}^2 + y_{k(n+1)-1}^2C + 2x_{k(n+1)-1}y_{k(n+1)-1}\sqrt{C},
\]

and in turn, by (2.17), this equals,

\[
2x_{k(n+1)-1}^2 - 1 + 2x_{k(n+1)-1}y_{k(n+1)-1}\sqrt{C},
\]

so,

\[
x_{2k(n+1)-1} = 2x_{k(n+1)-1}^2 - 1 = B_k
\]

and

\[
y_{2k(n+1)-1} = 2x_{k(n+1)-1}y_{k(n+1)-1} = A_k.
\]

Thus, (3.24) equals

\[
(B_k - 1)(A_k^3X^2 + 2A_kX) + \frac{C(B_k - 1)A_k^2X + CB_k}{CA_k + (B_k - 1)A_kX} = \\
\frac{(B_k - 1)(A_k^2X + 1)^3 + 1}{A_k(A_k^2X + 1)}.
\]

given that \((n + 1)k\) is even by hypothesis, and since \( B_k^2 - 1 = CA_k^2 \) this equals

\[
(B_k - 1)(A_k^3X^2 + 2A_kX) + \frac{C(B_k - 1)A_k^2X + CB_k}{CA_k(A_k^2X + 1)} = \\
\frac{(B_k - 1)(A_k^2X + 1)^3 + 1}{A_k(A_k^2X + 1)}.
\]
Since $2q_0 \neq c_j$ for any $0 < j < \ell$ by (2.18), and Theorem 2.1 tells us that a convergent $X_{j-1}/Y_{j-1}$ can only occur at the end of the $j$-th period, then $j = 1$. We have shown that

$$\langle q_0; w_{k-1}, c_0, 2(B_k - 1)A_k x/C, c_0, w_{k-1} \rangle = \frac{X_{k-1}}{Y_{k-1}}.$$

Since this is the $(\ell - 1)$-th convergent, then $\sqrt{D_k(x)}$ is as given in (a) and $\ell(\sqrt{D_k(x)}) = 2k(n + 1) + 2$. Also, observe that by (3.25) and Theorem 2.1,

$$B_k - 1 = 2(x_{k(n+1)-1}^2 - 1) = 2Cy_{k(n+1)-1}^2,$$

so $C \mid (B_k - 1)$, ensuring that $2(B_k - 1)A_k x/C \in \mathbb{Z}$ in the simple continued fraction expansion of $\sqrt{D_k(x)}$, thereby establishing part (a).

For part (b), we have

$$\langle q_0; w_{k-1}, 2(B_k - 1)A_k x + 2c_0, w_{k-1} \rangle =$$

$$\langle q_0; w_{k-1}, 2(B_k - 1)A_k x + c_0 + x_{k(n+1)-1}/y_{k(n+1)-1} \rangle =$$

$$\langle q_0; w_{k-1}, \frac{2(B_k - 1)A_k x y_{k(n+1)-1} + c_0 y_{k(n+1)-1} + x_{k(n+1)-1}}{y_{k(n+1)-1}} \rangle =$$

$$(B_k - 1)(A_k x (A_k^2 x + 2)) + \frac{Mx_{k(n+1)-1} + x_{k(n+1)-2}}{My_{k(n+1)-1} + y_{k(n+1)-2}},$$

(3.26)

where

$$M = \frac{2(B_k - 1)A_k x y_{k(n+1)-1} + c_0 y_{k(n+1)-1} + x_{k(n+1)-1}}{y_{k(n+1)-1}}.$$

Thus, the (3.26) equals,

$$\frac{(B_k - 1)(A_k x (A_k^2 x + 2)) + 2(B_k - 1)A_k x y_{k(n+1)-1} x_{k(n+1)-1} + c_0 y_{k(n+1)-1} x_{k(n+1)-1} + x_{k(n+1)-1}^2 + x_{k(n+1)-1}^2 - 2y_{k(n+1)-1}}{2(B_k - 1)A_k x y_{k(n+1)-1} + c_0 y_{k(n+1)-1} + x_{k(n+1)-1} - y_{k(n+1)-1} y_{k(n+1)-1} + y_{k(n+1)-1} y_{k(n+1)-2}}.$$

By the argument given to establish (3.25) in part (a), we have that $A_k = 2x_{k(n+1)-1} y_{k(n+1)-1}$ and $B_k = 2x_{k(n+1)-1} + 1 = x_{k(n+1)-1}^2 + y_{k(n+1)-1}^2$, and by (2.13), $x_{k(n+1)-1} = c_0 y_{k(n+1)-1} + y_{k(n+1)-2}$, so the above equals:

$$\frac{(B_k - 1)(A_k x (A_k^2 x + 2)) + (B_k - 1)(A_k^2 x + B_k)}{A_k^2 x + A_k} =$$

$$\frac{(B_k - 1)A_k^2 x (A_k^2 x + 2)(A_k^2 x + 1) + (B_k - 1)(A_k^2 x + B_k)}{A_k^2 x + A_k} =$$

$$\frac{(B_k - 1)(A_k^2 x + 1)^2 + 1}{A_k^2 x + A_k}.$$

Hence, we have established part (b).
For the proof of case (c), we present only sketches of the proofs. The reader may fill in the
details using the methodology presented in cases (a)–(b). We now establish case (c), part (i). Let
\( N = (2m + 1)(n + 1) \). Then,
\[
\left\langle q_0; v_m, 2(B_k - 1)A_kX/Q_{(n+1)/2} + c_{(n+1)/2}, \frac{2}{y_{n/2}} v_m \right\rangle = \\
\left\langle q_0; v_m, 2(B_k - 1)A_kX/Q_{(n+1)/2} + y_{n/2}/y_{n/2-1} \right\rangle = \\
\left\langle q_0; v_m, 2(B_k - 1)A_kX_{y_{n/2-1}} + Q_{(n+1)/2}y_{n/2-1} \right\rangle.
\]
If we set
\[
M = \frac{2(B_k - 1)A_kX_{y_{n/2-1}} + Q_{(n+1)/2}y_{n/2}}{Q_{(n+1)/2}y_{n/2-1}},
\]
then by (2.1) the above equals,
\[
(B_k - 1)(A_k^2X^2 + 2A_kX) + \frac{Mx_{n/2-1} + x_{n/2-2}}{My_{n/2-1} + y_{n/2-2}} = (B_k - 1)(A_k^3X^2 + 2A_kX) + \\
2(B_k - 1)A_kX_{y_{n/2-1}} + (y_{n/2}x_{n/2-1} + x_{n/2-2}y_{n/2-1})Q_{(n+1)/2},
\]
and by Lemmas 2.1–2.4 and (2.4) this equals,
\[
(B_k - 1)(A_k^3X^2 + 2A_kX) + \\
\frac{(B_k - 1)A_k^2Q_{(n+1)/2}X + (x_{n/2}y_{n/2-1} + y_{n/2-2}x_{n/2-1})Q_{(n+1)/2}}{2A_k^2X_{x_{n/2-1}y_{n/2-1} + A_kQ_{(n+1)/2}},
\]
so by Lemmas 2.1 and 2.3, this equals,
\[
(B_k - 1)(A_k^3X^2 + 2A_kX) + \frac{(B_k - 1)A_k^2Q_{(n+1)/2}X + B_kQ_{(n+1)/2}}{A_k^2XQ_{(n+1)/2} + A_kQ_{(n+1)/2}} = \\
(B_k - 1)(A_k^2X^2 + 2A_kX)(A_k^3X + A_k) + (B_k - 1)A_k^2X + (B_k - 1)A_k^3X + B_k = \\
(A_k^3X + A_k)
(((B_k - 1)(A_k^2X + 1)^3 + 1)/(A_k^2X + A_k).
\]
Hence, as in cases (a)–(b), the result follows for \( D_k(X) \).

We now establish c, part (ii) for \( D_k(X) \). We have,
\[
\left\langle q_0; v_m, \frac{c_{(n+1)/2}}{2}, \frac{2(B_k - 1)A_kXQ_{(n+1)/2}}{C}, \frac{c_{(n+1)/2}}{2}, v_m \right\rangle = \\
\left\langle q_0; v_m, \frac{c_{(n+1)/2}}{2}, \frac{2(B_k - 1)A_kXQ_{(n+1)/2}}{C}, -\frac{c_{(n+1)/2}}{2} + \frac{y_{n/2}}{y_{n/2-1}} \right\rangle = \\
\left\langle q_0; v_m, \frac{c_{(n+1)/2}}{2}, \frac{2(B_k - 1)A_kXQ_{(n+1)/2}}{C}, \frac{2y_{n/2} - c_{(n+1)/2}y_{n/2-1}}{2y_{n/2-1}} \right\rangle.
\]
By Lemma 2.1, (2.3) and (2.19), this equals,

\[
\left\langle \frac{c_{(n+1)/2}}{2}, \frac{2(B_k - 1)A_k X Q_{(n+1)/2}}{C} + \frac{2y_{N/2-1}^2}{A_k} \right\rangle = \\
\left\langle \frac{(B_k - 1)A_k^2 X Q_{(n+1)/2} + Cy_{N/2-1}^2 c_{(n+1)/2} + 2A_k C}{2(B_k - 1)A_k^2 X Q_{(n+1)/2} + 2Cy_{N/2-1}^2} \right\rangle = \\
\left\langle \bar{q}_0; \overline{u_m}, M \right\rangle,
\]

where \( M \) is the last term in the preceding continued fraction, and by (2.1), this equals,

\[
(B_k - 1)A_k X (A_k^2 X + 2) + \frac{M x_{N/2-1} + x_{N/2-2}}{M y_{N/2-1} + y_{N/2-2}}.
\tag{3.27}
\]

The denominator of (3.27) equals,

\[
(B_k - 1)A_k^2 X Q_{(n+1)/2} (c_{(n+1)/2} y_{N/2-1} + 2y_{N/2-2}) + A_k C y_{N/2-1} + \\
Cy_{N/2-1}^2 (c_{(n+1)/2} y_{N/2-1} + 2y_{N/2-2}),
\]

so by (2.3) and (2.19), this equals,

\[
(B_k - 1)A_k^2 X Q_{(n+1)/2} (y_{N/2} + y_{N/2-2}) + A_k C y_{N/2-1} + Cy_{N/2-1} (y_{N/2} + y_{N/2-2}),
\]

and by Lemma 2.1, this equals,

\[
(B_k - 1)A_k^2 X Q_{(n+1)/2} y_{N/2-1} + A_k C y_{N/2-1} + Cy_{N/2-1} A_k = \\
\frac{A_k C}{y_{N/2-1}} ((B_k - 1)A_k^2 X Q_{(n+1)/2}/C + 2y_{N/2-1}^2).
\tag{3.28}
\]

However, we may employ Lemmas 2.3–2.4, (2.17), and (2.19) to verify that \((B_k - 1)Q_{(n+1)/2}/C = 2y_{N/2-1}^2\), so (3.28) equals,

\[
2A_k C y_{N/2-1} (A_k^2 X + 1).
\tag{3.29}
\]

Given the calculated denominator (3.29) of (3.27), we may now use it to calculate the numerator, which is,

\[
2(B_k - 1)A_k^2 X (A_k^2 X + 2)(A_k^2 + 1) C y_{N/2-1} + \\
(B_k - 1)A_k^2 X (2Cy_{N/2-1} + Q_{(n+1)/2} (c_{(n+1)/2} x_{N/2-1} + 2x_{N/2-2}))+ \\
A_k C x_{N/2-1} + Cy_{N/2-1}^2 (c_{(n+1)/2} x_{N/2-1} + 2x_{N/2-2}),
\]

and by using Lemma 2.5, (2.2), and (2.19), this equals,

\[
2(B_k - 1)A_k^2 X (A_k^2 X + 2)(A_k^2 + 1) C y_{N/2-1} + \\
A_k C x_{N/2-1} + Cy_{N/2-1}^2 (c_{(n+1)/2} x_{N/2-1} + 2x_{N/2-2}),
\]
\[(B_k - 1)A_k^2X(2Cy_{n/2-1} + Q_{(n+1)/2}(x_{n/2} + x_{n/2-2})) + A_kCx_{n/2-1} + Cy_{n/2-1}^2(x_{n/2} + x_{n/2-2})) = 2(B_k - 1)A_k^2X(A_k^2X + 2)(A_k^2 + 1)Cy_{n/2-1} + (B_k - 1)A_k^2X(2Cy_{n/2-1} + Q_{(n+1)/2}(B_k - 1)/y_{n/2-1}) + A_kCx_{n/2-1} + C(B_k - 1)y_{n/2-1}.

However, it may be verified using Lemma 2.3 and (2.17) that

\[2Cy_{n/2-1} = Q_{(n+1)/2}(B_k - 1)/y_{n/2-1}\]

so the above equals

\[2(B_k - 1)A_k^2X(A_k^2X + 2)(A_k^2 + 1)Cy_{n/2-1} + 4(B_k - 1)A_k^2XCy_{n/2-1} + A_kCx_{n/2-1} + C(B_k - 1)y_{n/2-1} - 2Cy_{n/2-1}\]

and since Lemma 2.4 tells us that \(A_kx_{n/2-1} = (B_k - 1)y_{n/2-1} + 2y_{n/2-1}\), then the above equals,

\[2(B_k - 1)A_k^2X(A_k^2X + 2)(A_k^2 + 1)Cy_{n/2-1} + 4(B_k - 1)A_k^2XCy_{n/2-1} + 2C(B_k - 1)y_{n/2-1} + 2Cy_{n/2-1} = 2(B_k - 1)Cy_{n/2-1}(A_k^2X + 2)(A_k^2X + 1) + 2A_k^2X + 1) + 2Cy_{n/2-1} = 2Cy_{n/2-1}(B_k - 1)(A_k^2X + 1)^3 + 1).

Hence, we have shown that (3.27) equals

\[\frac{2Cy_{n/2-1}(B_k - 1)(A_k^2X + 1)^3 + 1)}{2A_kCy_{n/2-1}(A_k^2X + 1)} = \frac{(B_k - 1)(A_k^2X + 1)^3 + 1}{A_k(A_k^2X + 1)}.
\]

The balance of the proof now follows as in cases (a)-(b). One final note is in order for this case. Since Lemmas 2.3–2.4 and (2.17) allow us to deduce that when \(n + 1\)/2 is even, we must have, \(Q_{(n+1)/2}(B_k - 1) = 2Cy_{n/2-1}\), then \(C \mid Q_{(n+1)/2}(B_k - 1)\), so \(2(B_k - 1)AkQ_{(n+1)/2}X/C \in \mathbb{N}\) in the simple continued fraction expansion of \(\sqrt{D_k(X)}\).

It remains to verify case (c), part (iii) for \(D_k(X)\). By (2.1) and (2.5) we have,

\[\left\langle q_0; w_{m-1}, c_0, \frac{2(B_k - 1)A_kX}{C}, c_0, w_{m-1} \right\rangle = \left\langle q_0; w_{m-1}, c_0, \frac{2(B_k - 1)A_kX}{C} + \frac{y_{m(n+1)-1}}{x_{m(n+1)-1}} \right\rangle = \]
\[
\left\langle q_0; w_{m-1}, \frac{2(B_k - 1)A_k X c_0 x_{m(n+1)-1} + C c_0 y_{m(n+1)-1} + C x_{m(n+1)-1}}{2(B_k - 1)A_k X x_{m(n+1)-1} + y_{m(n+1)-1} C} \right\rangle = \nonumber
\]
\[
\left\langle q_0; w_{m-1}, M \right\rangle = (B_k - 1)A_k X (A_k^2 X + 2) + \frac{M x_{m(n+1)-1} + x_{m(n+1)-2}}{M y_{m(n+1)-1} + y_{m(n+1)-2}},
\]  
(3.30)

where \( M \) is the last term in the above continued fraction expansion. Now we calculate the denominator of (3.30). It is
\[
2(B_k - 1)A_k X x_{m(n+1)-1}(c_0 y_{m(n+1)-1} + y_{m(n+1)-2}) + 
\]
\[
C y_{m(n+1)-1}(c_0 y_{m(n+1)-1} + y_{m(n+1)-2}) + C x_{m(n+1)-1}y_{m(n+1)-1},
\]

and by (2.13), this equals
\[
2(B_k - 1)A_k X x_{m(n+1)-1}^2 + 2C x_{m(n+1)-1}y_{m(n+1)-1}.
\]  
(3.31)

However, by Theorem 2.1,
\[
B_k + A_k \sqrt{C} = (x_n + y_n \sqrt{C})^k = (x_n + y_n \sqrt{C})^m = (x_{m(n+1)-1} + y_{m(n+1)-1} \sqrt{C})^2 = 
\]
\[
x_{m(n+1)-1}^2 + y_{m(n+1)-1}^2 C + 2x_{m(n+1)-1}y_{m(n+1)-1} \sqrt{C}.
\]

Hence,
\[
B_k = x_{m(n+1)-1}^2 + y_{m(n+1)-1}^2 C \quad \text{and} \quad A_k = 2x_{m(n+1)-1}y_{m(n+1)-1}.
\]  
(3.32)

Thus, using (3.32) in conjunction with (2.17), we get that (3.31) equals,
\[
4A_k X x_{m(n+1)-1}^2 y_{m(n+1)-1}^2 C + A_k C = C(A_k^2 X + A_k),
\]
which is the denominator of (3.30), so we can now calculate its numerator:
\[
(B_k - 1)A_k X (A_k^2 X + 2)(A_k^3 X + A_k) C + 2(B_k - 1)A_k X x_{m(n+1)-1}(c_0 x_{m(n+1)-1} + x_{m(n+1)-2}) + 
\]
\[
C(c_0 x_{m(n+1)-1} + x_{m(n+1)-2})y_{m(n+1)-1} + C x_{m(n+1)-1}^2
\]
which (2.14) tells us is equal to,
\[
(B_k - 1)A_k X (A_k^2 X + 2)(A_k^3 X + A_k) C + 
\]
\[
2(B_k - 1)A_k X C x_{m(n+1)-1}y_{m(n+1)-1} + C^2 y_{m(n+1)-1}^2 + C x_{m(n+1)-1}^2
\]

and (3.32) allows us to rewrite this as
\[
(B_k - 1)A_k X (A_k^2 X + 2)(A_k^3 X + A_k) C + (B_k - 1)A_k^2 X C + CB_k =
\]
\[ C[(B_k - 1)(A_k^2 X + 1)^3 + 1] \]

Hence, we have shown that (3.30) equals,
\[
\frac{C[(B_k - 1)(A_k^2 X + 1)^3 + 1]}{C(A_k^3 X + A_k)} = \frac{(B_k - 1)(A_k^2 X + 1)^3 + 1}{A_k^3 X + A_k},
\]
and the balance of the proof follows as in cases (a)–(b).

We now provide an illustration of each case in Theorem 3.1

**Example 3.1** Let \( C = 13 \) for which \( \sqrt{C} = (3; 1, 1, 1, 1, 5), \) so \( n = 4 \) and for \( k = 2, \) \( B_2 = 842401, \) and \( A_2 = 233640. \) Thus, for \( X = 1, \) \( \sqrt{D_2(1)} = \)
\[
\left\langle q_0; 1, 1, 1, 1, 1, 6, 1, 1, 1, 1, 3, 30279744000, 3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 1, 1, 2q_0 \right\rangle =
\left\langle q_0; w_1, c_0, 2(B_2 - 1)A_2X/13, c_0, w_1, 2q_0 \right\rangle,
\]
where \( q_0 = 10743850360816702272003 = (B_2 - 1)(A_2X(A_2^2X + 2)) + c_0, \) and \( \ell(\sqrt{D_2(1)}) = 22 = 2k(n + 1) + 2. \) Thus, this illustrates Theorem 3.1, part (a).

Also, since \( B_1 = 649 \) and \( A_1 = 180, \) then for \( X = 1, \)
\[
\sqrt{D_1(1)} = \sqrt{1428363218181978525} = \left\langle q_0; 1, 1, 1, 1, 1, 233286, 1, 1, 1, 1, 2q_0 \right\rangle =
\left\langle q_0; w_0, 2(B_1 - 1)A_1X + 2c_0, w_0, 2q_0 \right\rangle,
\]
where \( q_0 = 3779369283 = (B_1 - 1)(A_1X(A_1X + 2)) + c_0, \) and \( \ell(\sqrt{D_1(1)}) = 10 = 2k(n + 1). \) Lastly, the fundamental solution of \( x^2 \) \( - D_1(X)y^2 = 1 \) is given by
\[
(x, y) = (22041961948426249, 5832180) =
((B_1 - 1)(A_1^2 + 1)^3 + 1, A_1(A_1^3 + 1)),
\]
which illustrates Theorem 3.1, part (b).

**Example 3.2** Let \( C = 19 \) for which \( \sqrt{C} = (4; 2, 1, 3, 1, 2, 8), \) so \( n = 5 \) and for \( k = 1, \) \( B_k = 170, \)
and \( A_2 = 39. \) Thus, for \( X = 1, \) \( \sqrt{D_2(1)} = \)
\[
\left\langle q_0; 1, 1, 6594, 1, 2, 2q_0 \right\rangle = \left\langle q_0; v_0, 2(B_1 - 1)A_1X/Q_2 + c_2, v_0, 2q_0 \right\rangle,
\]
where \( q_0 = 10038097 = (B_1 - 1)(A_1X(A_2^2X + 2)) + c_0, \) and \( \ell(\sqrt{D_2(1)}) = 6 = k(n + 1). \) Also, the fundamental solution of \( x^2 \) \( - D_1(1)y^2 = 1 \) is
\[
(x, y) = (595841381513, 59358) = ((B_1 - 1)(A_1^2 + 1)^3 + 1, A_1^3 + A_1),
\]
Thus, this illustrates Theorem 3.1, part (c)–(i).
Example 3.3 Let $C = 14$ for which $\sqrt{C} = (3; 1, 2, 1, 6)$, so $n = 3$ and for $k = 1$, $B_k = 15$, and $A_k = 4$. Thus, for $X = 1$,

$$\sqrt{D_1(1)} = \langle 10111; 1, 1, 16, 1, 1, 2022 \rangle = \left\langle q_0; \overline{v_0, c_{n+1}/2, 2Q_2(B_1 - 1)A_1X/14, c_{n+1}/2, \overline{v_0, 2q_0}} \right\rangle,$$

where $\ell(\sqrt{D_1(1)}) = 6 = k(n + 1) + 2$. Also, the fundamental solution of $x^2 - D_1(1)y^2 = 1$ is

$$(x, y) = (68783, 68) = ((B_1 - 1)(A_k^2 + 1)^3 + 1, A_k^3 + A_1).$$

Note as well, that $h_{4D(1)} = 64$, while $\log(x_{4D(1)}) = 11.83185 \ldots$, and $\log(4D(1)) = 7.61235$, all of which illustrate Theorem 3.1, part (c)-(i).

Example 3.4 Let $C = 19$ for which $\sqrt{C} = (4; 2, 1, 3, 1, 2, 8)$, so $n = 5$ and for $k = 4$, $B_k = 6681448801$, and $A_k = 1532829480$. Thus, for $X = 1$, $\sqrt{D_1(1)} =

$$\langle q_0; 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 4, 1078054914710592000, 4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, q_0 \rangle = \left\langle q_0; \overline{v_0, c_{n+1}/2, 2Q_2(B_1 - 1)A_1X/14, c_{n+1}/2, \overline{v_0, 2q_0}} \right\rangle,$$

where $q_0 = 24063133349939809178568760191 = (B_1 - 1)(A_1X(A_k^2X + 2)) + c_0$, and $\ell(\sqrt{D_4(1)}) = 26 = k(n + 1) + 2$. This illustrate Theorem 3.1, part (c)-(iii).

In order to provide our last infinite families, we need the following. In [5], we proved a result of which the next theorem is a small subset.

Theorem 3.2 Let

$$D_k(X) = (B_k - 1)^2A_k^2X^2 + 2(B_k - 1)^2X + C.$$  

Then the fundamental solution of

$$x^2 - D_k(X)y^2 = 1$$

is

$$(x, y) = ((B_k - 1)(A_k^2X + 1)^2 + 1, A_k^3X + A_k),$$

and, for

$$q_0 = (B_k - 1)A_kX + c_0.$$
(a) If \( n \geq 0 \) is even and \( k \) is odd, then
\[
\sqrt{D_k(X)} = \langle q_0; \overline{w_{k-1}}, 2q_0 \rangle
\]
and \( \ell \left( \sqrt{D_k(X)} \right) = k(n + 1) \).

(b) If \( n \) is odd, \( k = 2m + 1, m \geq 0, \) and \( (n + 1)/2 > 1 \) is odd, then
\[
\sqrt{D_k(X)} = \langle q_0; \overline{v_m}, 2(B_k - 1)A_kX/Q_{(n+1)/2} + c_{(n+1)/2}, \overline{v_m}, 2q_0 \rangle
\]
with \( \ell \left( \sqrt{D_k(X)} \right) = k(n + 1) \).

We will need this for the following result.

**Theorem 3.3** Let
\[
D_k(Y) = (B_k + 1)^2 A_k^2 Y^2 + 2(B_k^2 - 1)Y + C.
\]
Then the fundamental solution of
\[
x^2 - D_k(Y)y^2 = 1
\]
is
\[
(x, y) = \left( (B_k - 1) \left( A_k^2 \left( \frac{(B_k + 1)Y}{B_k - 1} \right) + 1 \right)^2 + 1, \frac{B_k + 1}{B_k - 1} A_k^3 Y + A_k \right),
\]
and, for
\[
q_0 = (B_k + 1)A_kY + c_0:
\]

(a) If \( n \geq 0 \) is even and \( k \) is odd, then
\[
\sqrt{D_k(Y)} = \langle q_0; \overline{w_{k-1}}, 2q_0 \rangle
\]
and \( \ell \left( \sqrt{D_k(Y)} \right) = k(n + 1) \).

(b) If \( n \) is odd, \( k = 2m + 1, m \geq 0, \) and \( (n + 1)/2 > 1 \) is odd, then
\[
\sqrt{D_k(Y)} = \left\langle q_0; \overline{v_m}, 2(B_k + 1)A_kY/Q_{(n+1)/2} + c_{(n+1)/2}, \overline{v_m}, 2q_0 \right\rangle
\]
with \( \ell \left( \sqrt{D_k(Y)} \right) = k(n + 1) \).

**Proof.** Merely set \( X = (B_k + 1)Y/(B_k - 1) \) in Theorem 3.2. \( \square \)

We conclude with two illustrations of Theorem 3.3.
Example 3.5 We revisit Example 3.1, where $C = 13$, $n = 4$, $k = 1 = Y$, $B_1 = 649$ and $A_1 = 180$. Then in the case of Theorem 3.3,

$$\sqrt{D_1(1)} = \sqrt{13689842413} = \langle 117003; 1, 1, 1, 1, 234006 \rangle = \langle q_0; w_0, 2q_0 \rangle,$$

where $\ell(\sqrt{D_1(1)}) = 5 = k(n + 1)$.

Also, the fundamental solution of $x^2 - D_1(1)y^2 = 1$ is given by

$$(x, y) = (684492120649, 5850180). \tag{3.33}$$

Note that this is not the fundamental unit of $\mathbb{Z}[\sqrt{D_1(1)}]$, which is

$$\epsilon_{4D_1(1)} = 585018 + 5\sqrt{D_1(1)},$$

of norm $-1$. The value given in (3.33) arises from $\epsilon_{2D_1(1)}^2$. This illustrates Theorem 3.3, part (a).

Example 3.6 We revisit Example 3.2, where $C = 19$, $n = 5$, and for $k = Y = 1$, $B_1 = 170$ and $A_1 = 39$. Thus, in the case of Theorem 3.3,

$$\sqrt{D_1(1)} = \sqrt{44533378} = \langle 6673; 2, 1, 6672, 1, 2, 13346 \rangle = \langle q_0; v_0, 2(B_1 + 1)/Q_2 + c_2, v_0, 2q_0 \rangle.$$

The fundamental solution of $x^2 - D_1(1)y^2 = 1$ is given by

$$(x, y) = (400800401, 60060),$$

and

$$\epsilon_{4D_1(1)} = x + y\sqrt{D_1(1)}.$$

References


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