Prime producing quadratic polynomials and real quadratic fields of class number one

R.A. Mollin and H.C. Williams

Abstract

The purpose of this paper is to determine (modulo the generalized Riemann hypothesis (GRH)) all real quadratic fields of Richaud-Degert (R-D) type which have class number one; and to establish a connection with certain prime producing quadratic polynomials. This completes our previous work in which all real quadratic fields of narrow R-D type with class number one were determined modulo GRH.

1. Introduction

In [6] we established, for any square-free positive integer $d$, three sufficient conditions for the class number $h(d)$ of $\mathbb{Q}(\sqrt{d})$ to equal 1. One of these conditions is that if $f_d(x) = -x^2 + x + (d-1)/4$ when $d \equiv 1 \pmod{4}$ and $f_d(x) = d - x^2$ when $d \not\equiv 1 \pmod{4}$ then $f_d(x)$ is prime for all integers $x$ such that $1 < x < \alpha$ where $\alpha = (\sqrt{d} - 1)/2$ when $d \equiv 1 \pmod{4}$ and $\alpha = \sqrt{d}$ if $d \not\equiv 1 \pmod{4}$. A remarkable property which we obtained is that if $f_d(x)$ is prime for all integers $x$ with $1 < x < \alpha$ and $d > 13$ then $d \equiv 1 \pmod{4}$ is of narrow R-D type; (i.e., $d = \ell^2 + r$ where $|r| \in \{1, 4\}$).

Then assuming GRH we established that for $d$ of narrow R-D type $h(d) = 1$ if and only if $d$ is one of 14 values. Thus (modulo GRH) we proved that $f_d(x)$ is prime for all integers $x$ with $1 < x < \alpha$ if and only if

$$d \in \{2, 3, 5, 6, 7, 11, 13, 17, 21, 29, 37, 53, 77, 101, 173, 197, 293, 437, 677\}.$$ 

It is natural to consider the remaining wide R-D types; (i.e., those $d = t^2 + r$ where $r$ divides $4t$, $-t < r \leq t$ and $|r| \notin \{1, 4\}$). It is the purpose of this paper to complete the task begun in [5] [6] and [2] by determining all real quadratic fields of R-D type which have class number 1. In the process we forge an intimate link with certain prime producing quadratic polynomials most of which have heretofore not appeared in the literature.
2. Results

Let \( f_d(x) = -x^2 + x + (d - 1)/4 \) and \( \alpha = (\sqrt{d} - 1)/2 \) when \( d \equiv 1 \pmod{4} \); and \( f_d(x) = -x^2 + d \) and \( \alpha = \sqrt{d} \) when \( d \not\equiv 1 \pmod{4} \). Among other equivalences we proved (modulo GRH) in [6] that \( f_d(x) \) is prime for all integers \( x \) with \( 1 < x < \alpha \) if and only if

\[
d \in \{2, 3, 5, 6, 7, 11, 13, 17, 21, 29, 37, 53, 77, 101, 173, 197, 293, 437, 677\}.
\]

As a consequence we have that all real quadratic fields of narrow R-D type with \( h(d) = 1 \) are contained in this set. They are specifically:

\[
\{2, 3, 17, 21, 29, 37, 53, 77, 101, 173, 197, 293, 437, 677\}.
\]

(Note that 5 is not generally considered to be an R-D type and 6, 7, 11 are wide R-D types; whereas 13 = \( 3^2 + 4 \) is not an R-D type since \( r \) must be less than \( t \)). We are now able to handle all other R-D types.

First we deal with wide R-D types \( d \not\equiv 1 \pmod{4} \). In [4] Mollin proved that if \( h(d) = 1 \) for such \( d \) then \( d = t^2 + r \) with \( |r| = 2 \) (see also [3]). We consider two cases separately, that when \( t \) is even and that when \( t \) is odd. We deal with the even case first.

**Theorem 1.** Let \( d = 4\ell^2 \pm 2 > 2 \). If \( f_d(x) = -2x^2 + d/2 \) is prime or 1 for all integers \( x \) with \( 0 \leq x < \sqrt{d}/2 \) then \( h(d) = 1 \).

**Proof.** If \( h(d) > 1 \) then by Kutsuna [1, Theorem 2, p. 126] there exists an integer \( x \) and a prime \( p \) with \( 0 \leq x < p < \sqrt{d} \) such that

(a) \( N(x - \sqrt{d}) \equiv 0 \pmod{p} \), and

(b) \( |N(x + kp - \sqrt{d})| \geq p^2 \) for all integers \( k \).

By (a), \( d \equiv x^2 \pmod{p} \). Without loss of generality we may assume that \( x \) is even, (since if it is odd we may replace it by \( p - x \)). Thus, \(-2(x/2)^2 + d/2 = p^\alpha \) for \( \alpha \in \{0, 1\} \), by hypothesis. Setting \( k = 0 \) in (b) we get \( 2p^\alpha = |N(x + kp + \sqrt{d})| \geq p^2 \), whence \( p = 2 \) and \( x \in \{0, 1\} \). From (a) \( x = 0 \) is forced. Setting \( k = \ell \) in (b) yields that \( 2 = |4\ell^2 - (4\ell^2 \pm 2)| \geq 4 \), which is absurd. \( \blacksquare \)
Table 1

<table>
<thead>
<tr>
<th>$d$</th>
<th>$f_d(x) = -2x^2 + d/2$ for $0 \leq x &lt; \sqrt{d}/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6 3</td>
</tr>
<tr>
<td>4</td>
<td>14 7, 5</td>
</tr>
<tr>
<td>2</td>
<td>38 19, 17, 11, 1</td>
</tr>
<tr>
<td>4</td>
<td>62 31, 29, 23, 13</td>
</tr>
<tr>
<td>y</td>
<td>398 199, 197, 191, 181, 167, 149, 127, 101, 71, 37</td>
</tr>
</tbody>
</table>

Now we deal with the case where $t$ is odd.

**Theorem 2.** If $d = (2\ell+1)^2 \pm 2$ with $\ell > 0$ and $f_d(x) = -2x^2 + 2x + (d-1)/2$ is prime or 1 for all integers $x$ with $0 < x < (\sqrt{d} + 1)/2$ then $h(d) = 1$.

**Proof.** If $h(d) > 1$ then as in the proof of Theorem 1 there exists an integer $y$ and a prime $p$ with $0 \leq y < p < \sqrt{d}$ such that

(a) $N(y - \sqrt{d}) \equiv 0 \pmod{p}$,

(b) $|N(y + kp - \sqrt{d})| \geq p^2$ for all integers $k$.

By (a), $d \equiv y^2 \pmod{p}$. Without loss of generality we may assume that $y$ is odd, (since we may replace it by $p - y$ if it is even). Set $y = 2x - 1$ and we have $d \equiv (2x - 1)^2 \pmod{p}$ for $0 < x \leq (p+1)/2$. By hypothesis $p^\alpha = -2x^2 + 2x + (d-1)/2$ for $\alpha \in \{0, 1\}$. Putting $k = 0$ in (b) yields that $2p^\alpha = |(2x - 1)^2 - d| \geq p^2$, whence $p = 2$, $x = 1$ and $y = 1$. Therefore $d = 5$, a contradiction. 

Table 2

<table>
<thead>
<tr>
<th>$d$</th>
<th>$f_d(x) = -2x^2 + 2x + (d-1)/2$ for $0 &lt; x &lt; (\sqrt{d} + 1)/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7 3</td>
</tr>
<tr>
<td>2</td>
<td>11, 5, 1</td>
</tr>
<tr>
<td>4</td>
<td>23, 11, 7</td>
</tr>
<tr>
<td>2</td>
<td>47, 23, 19, 11</td>
</tr>
<tr>
<td>4</td>
<td>83, 41, 37, 29, 17</td>
</tr>
<tr>
<td>4</td>
<td>167, 83, 79, 71, 59, 43, 23</td>
</tr>
<tr>
<td>2</td>
<td>227, 113, 109, 101, 89, 73, 73, 53, 29, 1</td>
</tr>
</tbody>
</table>

**Conjecture 1.** If $d = t^2 \pm 2 > 3$ then $h(d) = 1$ if and only if $d$ is an entry on either Table 1 or Table 2.
We now establish this conjecture modulo GRH.

**Theorem 3.** If the GRH holds then Conjecture 1 holds.

**Proof.** Let

\[
t_1(x) = (3/\pi) + (15.9/(2 \log \log x)) + (2/(\pi \log \log x)) + (5.3/(\log \log x)^2) + (8/(\log x \log \log x)) + (6 \log \log x/(\pi \log x)) + (12/\log x) + (4/\pi \log x) + 1/(\log x)^2;
\]

\[
t_2(x) = 3(\log 4x/(\pi \log x)) + (15.9 \log 4x/(2 \log \log x \log x)) + (2 \log 4x/(\pi \log x \log \log x)) + 5.3(\log 4x/(\log x(\log \log x))) + (8/(\log x \log \log x)) + 6(\log \log x/(\pi \log x)) + (12/\log x) + (4/(\pi \log x)) + 1/(\log x)^2
\]

and

\[
B(d) = \Pi(q/(q + 1))
\]

where the product runs over all primes \( q < (\log d)^2 \).

By using the reasoning which we employed in [5] and [6], we have for any positive square-free \( d \) that:

\[
h(d) > i\sqrt{d}e^{-t_1(d)}B(d)/2R,
\]

when the GRH is assumed. Here \( i = 1 \) when \( d \equiv 1 \pmod{4} \), \( i = 2 \) when \( d \equiv 2, 3 \pmod{4} \) and \( R \) is the regulator of \( \mathbb{Q}(\sqrt{d}) \). Furthermore \( B(d) > 1/(\log d)^2 \).

For \( d = t^2 + r \) with \( |r| = 2 \) a unit \( \varepsilon > 1 \) in \( \mathbb{Q}(\sqrt{d}) \) is given by

\[
\varepsilon = (t + \sqrt{d})^2/2.
\]

Hence \( R < 2\log(2\sqrt{d} + 1) - \log 2 \). Since in this case \( d \equiv 2, 3 \pmod{4} \) and both \( e^{-t_2(d)} \) and \( \sqrt{d}/((2\log(2\sqrt{d} + 1) - \log 2)(\log d)^2) \) are increasing functions of \( d \) then for \( d > e^6 \) we see that if \( t > 3.6 \times 10^6 \) then \( d > 1.295 \times 10^{13} \) and

\[
h(d) > 1.004861647 > 1.
\]

We must now find all those values of \( d = t^2 \pm 2 \) with \( t < 3.6 \times 10^6 \) for which \( h(d) = 1 \). To do this we make use of our Lemma 2.2 of [6]. By using this result we are able to say that if there is an odd prime \( q \) such that \( q < \sqrt{d} \) and \( (d/q) = 1 \) then \( h(d) > 1 \). By employing the same type of sieve
technique as that which we described in [5] and applying it to polynomials $g(s) = 4s^2 - 2, 4s^2 + 2, 4s^2 + 4s - 1$ and $4s^2 + 4s + 3$ we were able to use a computer (about 20 seconds per $g(s)$) to show that if $s < 1.8 \times 10^6$ then the only values of $d = t^2 \pm 2$ such that $h(d) = 1$ are given in Tables 1 and 2. 

This completes all wide R-D types $d \not\equiv 1 \pmod{4}$. We now turn to wide R-D types $d \equiv 1 \pmod{4}$. First we consider the case where $r$ divides $2\ell$. In [4] Mollin proved that if $h(d) = 1$ for such $d$ then $d = 33$ if $d \equiv 1 \pmod{8}$ and $d = t^2 - p$ where $p$ is an odd prime dividing $t$ if $d \equiv 5 \pmod{8}$. Therefore $q = d/p$ must be an odd prime and $p \equiv q \equiv 3 \pmod{4}$. The following conjecture is made on the basis of the techniques used in the proofs of Theorems 1 and 2 which just barely fail in this case.

**Conjecture 2.** Suppose $d = t^2 - p$ where $p$ is an odd prime dividing $t$ and $d \equiv 5 \pmod{8}$. If $|px^2 + px - \frac{d-2}{4}|$ is prime for all integers $x$ with $0 \leq x < (\sqrt{d-1}/4 - 1/2$ then $h(d) = 1$.

To see more clearly why the bounds on $x$ are conjectured we develop the case which we are able to prove using the techniques cited above. We assume $r$ is inert in $Q(\sqrt{d})$ for all primes $r < p$.

If $h(d) > 1$ then by Kutsuna (op. cit.) there exist an integer $x$ and a prime $r$ with $0 \leq x < r < (\sqrt{d-1}/2$ such that both:

(a) $N(x - (1 + \sqrt{d})/2) \equiv 0 \pmod{r}$ and

(b) $|N(x + k\ell - (1 + \sqrt{d})/2/2| \geq r^2$ for all integers $k$.

From (a) we get that $d \equiv (2x - 1)^2 \pmod{4r}$, whence $r > 2$. The problem occurs when $p = r$, so we assume $p \not\equiv r$. Let $p^{-1}$ denote the multiplicative inverse of $p$ modulo $4r$. Hence $d \equiv p^2(p^{-1}(2x - 1))^2 \pmod{4r}$ by (a). Let $z \equiv p^{-1}(2x - 1) \pmod{r}$ with $0 \leq z < r$; whence, $d \equiv (pz)^2 \pmod{r}$. We may assume that $z$ is odd since we may replace it by $r - z$ if necessary. Set $z = 2w + 1$, whence $d \equiv p^2(2w + 1)^2 \pmod{r}$, i.e., $pw^2 + pw - (d - p^2)/4p \equiv 0 \pmod{r}$ with $0 \leq w < (\sqrt{d-1}/4 - 1/2$, (since $0 \leq z < (\sqrt{d-1}/2)$. By hypothesis $|pw^2 + pw - (d - p^2)/4p| = r$; i.e., $4pr = |(2pw + p)^2 - d|$. Set $k = (pz - 2x + 1)/2r$ in (b), and observe that $k$ is an integer since $pz \equiv (2x - 1) \pmod{r}$. Thus $4pr = |(2pw + p)^2 - d| \geq 4r^2$, whence $p \geq r$. By assumption $r \neq p$, and so $r$ is inert in $Q(\sqrt{d})$, contradicting (a).

It turns out, using GRH that when $h(d) = 1$ for $d = pq$ then all primes $r < p$ are inert in $Q(\sqrt{d})$. The following table provides illustrations. It will be shown, using GRH, that as with Tables 1 and 2, this tells the whole story.
### Table 3

| d    | p  | q  | t   | $|px^2 + px - (d - p^2)/4p|$ for $0 \leq x < (\sqrt{d - 1})/2 - 1/2$ |
|------|----|----|-----|---------------------------------------------|
| 141  | 3  | 47 | 12  | 11, 5, 7                                    |
| 573  | 3  | 191| 24  | 47, 41, 29, 11, 13, 43                     |
| 1293 | 3  | 431| 36  | 107, 101, 89, 71, 47, 27, 19, 61, 109        |
| 1757 | 7  | 251| 42  | 61, 47, 19, 23, 79, 149, 233, 331, 443, 569 |

Conjecture 3. Suppose $d = t^2 + r \equiv 1 \pmod{4}$ where $d$ is a wide R-D type. If $h(d) = 1$ then $d$ is an entry in Table 3 or $d = 33$.

Later we will establish Conjecture 3 (modulo GRH). First we consider the remaining R-D types; i.e., those of the form $d = t^2 \pm 4m \equiv 1 \pmod{4}$ with $m > 1$, $m$ divides $t$, $t$ odd and $m$ odd. Therefore if $h(d) = 1$ then $p = m$ must be prime, as must $q = d/m$ and $p \equiv q \equiv 3 \pmod{4}$.

For similar reasons as those given for Conjecture 2 we pose the following.

Conjecture 4. Let $d = t^2 \pm 4p$ where $p$ is an odd prime dividing $t$. If $|px^2 + px - (d - p^2)/4p|$ is 1 or prime for all integers $x$ with $0 \leq x < (\sqrt{d - 1})/2 + 1/2$ then $h(d) = 1$.

### Table 4

| d    | p  | q  | t   | $|px^2 + px - (d - p^2)/4p|$ for $0 \leq x < (\sqrt{d - 1})/2 + 1/2$ |
|------|----|----|-----|---------------------------------------------|
| 213  | 3  | 71 | 15  | 17, 11, 1, 19, 43, 73, 109, 151            |
| 237  | 3  | 79 | 15  | 19, 13, 1, 17, 41, 71, 107, 149, 197        |
| 413  | 7  | 59 | 21  | 13, 1, 13, 71, 127, 197, 281, 379, 491, 617, 757 |
| 453  | 3  | 151| 21  | 37, 31, 19, 1, 23, 31, 37, 53, 89, 131, 173, 233, 293, 359 |
| 1133 | 11 | 103| 33  | 23, 1, 43, 109, 197, 307, 439, 599, 769, 967, 1187, 1429, 1693, 1979, 2287, 2617, 2969, 3343 |
| 1253 | 7  | 179| 35  | 43, 29, 1, 41, 97, 167, 251, 349, 461, 587, 727, 881, 1049, 1231, 1427, 1637, 1861, 2079, 2351 |

Conjecture 5. Suppose $d = t^2 + 4p \equiv 1 \pmod{4}$ where $p$ divides $t$, $p$ an odd prime. If $h(d) = 1$ then $d$ is an entry in Table 4.
Theorem 4. If the GRH holds then Conjectures 3 and 5 hold.

Proof. The problem of finding all \( d \equiv 1 \pmod{4} \) of the form \( m^2p^2 + \kappa p \) where \( p \equiv 3 \pmod{4} \) is prime and \( \kappa \in \{-1, 4, -4\} \) such that \( h(d) = 1 \) is somewhat more difficult than the previous problem in Theorem 3. The reason is that each of the forms of \( d \) involves two variables instead of just one. To facilitate our search in this case we must first lower the search bound on \( d \) by obtaining a better bound on \( B(d) \) than the simple one employed in Theorem 3; viz., \( B(d) > 1/(\log d)^2 \). (Of course, we could have done the same thing in the previous case; but since we must employ the GRH, it is reasonable to attempt to push the search somewhat beyond the limit needed.)

We note that if \( t(x) = \prod(q/(q + 1)) \) where the product runs over all primes \( q < x \), then:

\[
-\log t(x) = \log(t(x))^{-1} = \sum_{q < x} \log(1 + 1/q) < \sum_{q < x} 1/q.
\]

By using (3.20) of [7] we get:

\[
\sum_{q < x} 1/q < \log \log x + B + 1/(\log x)^2; \ x > 1
\]

where \( B = 0.2614972129 \). It follows that:

\[
t(x) > (e^{-B}/\log x)(1 - 1/(\log x)^2).
\]

If \( x > 530 \) then \( t(x) > 0.75/\log x \); hence \( B(d) > 0.75/(2 \log \log d) \), when \( (\log d)^2 > 530 \); i.e., \( d > 9.96 \times 10^9 \).

Thus, under the GRH, if \( D > 9.96 \times 10^9 \) then:

\[
h(d) > 0.75\sqrt{d}e^{-t_1(d)}/(4R \log \log d).
\]

Now if \( d = p^2m^2 + \kappa p \equiv 1 \pmod{4} \) and \( \kappa \in \{-1, 4, -4\} \) then a unit \( \epsilon > 1 \) of \( \mathbb{Q}(\sqrt{d}) \) is given by

\[
\epsilon = (pm + \sqrt{d})^2/|\kappa|p < 3d/2.
\]

It follows that if

\[
F(d) = 0.75\sqrt{d}e^{-t_1(d)}/(4 \log(3d/2) \log \log d),
\]

then \( h(d) > F(d) \) and \( F(d) \) is an increasing function of \( d \) for \( d > 9.96 \times 10^9 \). Since \( F(9.96 \times 10^9) > 1.432095699 > 1 \) we may assume under the GRH that \( h(d) > 1 \) for \( d > 9.96 \times 10^9 \).
As in the previous cases in Theorem 3 it now remains to consider those values of $d \leq 9.96 \times 10^9$. Once again, by appealing to Lemma 2.2 of [6] we know that $h(d) > 1$ if we can find a prime $q$ such that $(d/q) = 1$ and $q < \sqrt{d - 1}/2$. Since $d \leq 9.96 \times 10^9$, we have $p = 4r - 1 < 10^5 - 1$ and $r < 25,000$. Putting $\lambda = 2$ when $|\kappa| = 4$ and $\lambda = 1$ when $\kappa = -1$ we see that $t = (m + \lambda - 1)/2$ must be an integer, $d \equiv 1 \pmod{4}$ and $t \leq \lfloor 10^5/(8r - 2) \rfloor + 1$. Furthermore,

$$4rt - 2r\lambda + 2r - t - 1 = (pm - \lambda - 1)/2 < (\sqrt{d - 1})/2.$$ 

For a fixed value $\alpha$ of $\kappa$ put:

$$f(r, t) = (4r - 1)^2(2t - \lambda + 1)^2 + \kappa(4r - 1).$$

For each element $p_i$ in the set of the first $k$ odd primes (we used $k = 20$) \{p_1, p_2, \ldots, p_k\}, and each value of $j (0 \leq j < p_i)$ we compute those pairs $(j, t_{ij})$ such that $0 \leq t_{ij} < p_i$ and $(f(j, t_{ij})/p_i) = 1$. For each value of $r$ from 1 to 25,000 we calculate $B_r = (10^5/2(4r - 1)) + 1$. If $j \equiv r \pmod{p_i}$, $t \equiv t_{ij} \pmod{p_i}$, $t \leq B_r$ and $p_i < 4rt - 2r\lambda + 2r - t - 1$ we can eliminate $f(r, t)$ as a possible value of $d$ such that $h(d) = 1$. This sieving technique was programmed in Fortran and run in the Amdahl 5870 computer at the University of Manitoba. In a total of 15 seconds, it was determined that the only possible values of $d$ of the above considered form such that $d < 9.96 \times 10^9$ and $h(d) = 1$ are those given in Tables 3 and 4.

Therefore we have proved:

**Theorem 5.** If the GRH holds then all real quadratic fields of R-D type $\mathbb{Q}(\sqrt{d})$ with $h(d) = 1$ are one of the 39 values of $d$ given in the set \{2, 3, 6, 7, 11, 14, 17, 21, 23, 29, 33, 37, 38, 47, 53, 62, 77, 83, 101, 141, 167, 173, 197, 219, 227, 293, 398, 413, 457, 458, 573, 667, 717, 1077, 1103, 1253, 1293, 1757\}. Of these values 14 are of narrow R-D type, and the remaining 25 are of wide R-D type. (Note that 5 is not included since it is not generally considered to be of R-D type, whereas 13, 69 and 93 are excluded since they are of the form $d = \ell^2 + r$ with $|r| > \ell$.)

On the basis of an ongoing investigation into the link between prime producing quadratic polynomials and the class number one problem for real quadratic fields we pose the following:

**Conjecture 6.** Let $d > 0$ be any square-free integer. $h(d) = 1$ if and only if there exists some polynomial $ax^2 + bx + c = f_d(x)$ whose discriminant
$b^2 - 4ac = v^2 d$ for some integer $v \geq 1$ and $f_d(x)$ is prime (or has a prescribed factorization over the integers) for all integers $x$ in an interval depending on $a, b$ and $c$.

Conjecture 6 does not specify the bounds on $x$ since they are not yet fully understood. Moreover, a glance at the various bounds achieved for R-D types in this paper shows that the final answer may be quite complicated in terms of being "case-bound" in some fashion. The consequences of an investigation of Conjecture 6 will be completed in a later paper. An affirmative answer to Conjecture 6 would be a complete Rabinovitch result for real quadratic fields.

Another investigation which is currently underway is to look at the density of primes in prime producing polynomials, such as those in this paper, and the connection with class number 1. It may be of interest to give, at this juncture, some preliminary data in this regard. Restricting our attention to certain prime producing polynomials in this paper we have found the following.

First we need a definition. Let $\nu(f, N)$ be the cardinality of the set \( \{ x \in \mathbb{Z} : 0 \leq x \leq N; |f(x)| = 1 \text{ or prime} \} \) where $f$ is a polynomial.

### Table 5

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\nu(f, 10000)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x^2 + 3x - 11$</td>
<td>1216</td>
</tr>
<tr>
<td>$3x^2 + 3x - 19$</td>
<td>2174</td>
</tr>
<tr>
<td>$3x^2 + 3x - 47$</td>
<td>2192</td>
</tr>
<tr>
<td>$7x^2 + 7x - 13$</td>
<td>2476</td>
</tr>
<tr>
<td>$3x^2 + 3x - 107$</td>
<td>3035</td>
</tr>
<tr>
<td>$3x^2 + 3x - 59$</td>
<td>3036</td>
</tr>
<tr>
<td>$7x^2 + 7x - 61$</td>
<td>3099</td>
</tr>
<tr>
<td>$7x^2 + 7x - 43$</td>
<td>3512</td>
</tr>
<tr>
<td>$3x^2 + 3x - 89$</td>
<td>3515</td>
</tr>
<tr>
<td>$11x^2 + 11x - 23$</td>
<td>3516</td>
</tr>
<tr>
<td>$-2x^2 + 2x + 83$</td>
<td>3216</td>
</tr>
<tr>
<td>$-2x^2 + 2x + 113$</td>
<td>3585</td>
</tr>
<tr>
<td>$2x^2 - 199$</td>
<td>4373</td>
</tr>
</tbody>
</table>
We note that the Euler-Rabinovitch polynomial $f(x) = x^2 + x + 41$ has $\nu(f, 10000) = 4149$. Further results in this direction will be published at a later date.

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REFERENCES


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