38. Solution of a Problem of Yokoi

By R. A. MOLLIN* and H. C. WILLIAMS **

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In [12]–[16] Yokoi studied what he called $p$-invariants for a real quadratic field $Q(\sqrt{p})$ where $p \equiv 1 \pmod{4}$ is prime. In [9] we generalized this concept to an arbitrary real quadratic field $Q(\sqrt{d})$ where $d$ is positive and square-free. We provided numerous applications including bounds for fundamental units and an investigation of the class number one problem related to non-zero $n_d$, (defined below). It is the purpose of this paper to give a complete list and a proof that the list is valid (with one possible value remaining) of all $Q(\sqrt{d})$ having class number $h(d)=1$ when $n_d \neq 0$. Moreover we show that if the exceptional value of $d$ exists then it is a counterexample to the Generalized Riemann Hypothesis. This completes the task of Yokoi begun in [15]–[16].

In what follows the fundamental unit $\varepsilon_d(>1)$ of $Q(\sqrt{d})$ is denoted $(t_d + u_d \sqrt{d})/\sigma$ where $\sigma = \begin{cases} 2 & \text{if } d \equiv 1 \pmod{4} \\ 1 & \text{if } d \equiv 2, 3 \pmod{4} \end{cases}$. Now set:

$$B = (2t_d)/\sigma - N(\varepsilon_d) - 1)u_d^2$$

where $N$ is norm from $Q(\sqrt{d})$. This boundary $B$ was studied in [4], [5] and [14].

The following generalizes Yokoi’s notion of a $p$-invariant $n_p$ where $p \equiv 1 \pmod{4}$ is prime (see [12]–[16]).

Let $n_d$ be the nearest integer to $B$; i.e.,

$$n_d = \begin{cases} [B] & \text{if } B - [B] < \frac{1}{2} \\ [B] + 1 & \text{if } B - [B] \geq \frac{1}{2} \end{cases}$$

(where $[x]$ is the greatest integer less than or equal to $x$).

In [9] we proved the following:

Theorem 1. Let $d > 0$ be square-free and let $u_d > 2$. Then the following are equivalent:

1. $n_d = 0$
2. $t_d > 4d/\sigma$
3. $u_d^2 > 16d/\sigma^2$.

The above generalizes the main result of Yokoi in [12].

We also proved in [9] the following consequences of Theorem 1.

Corollary 1. If $n_d \neq 0$ then $\varepsilon_d < 8d/\sigma^2$.

Corollary 2. If $n_d \neq 0$ then there are only finitely many $d$ with $h(d)=1$.

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Corollary 3. Let $d_0$ be a fixed positive square-free integer. Then there are only finitely many $d$ with $n_d = n_0$ and $h(d) = 1$.

The above generalize results of Yokoi in [13]–[16]. Moreover this has consequences for the Gauss conjecture as follows.

Let:

$(G_1)$: There exist infinitely many real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with $h(d) = 1$; (Gauss's conjecture).

$(G_2)$: There exist infinitely many $d$ with $n_d = 0$ and $h(d) = 1$.

$(G_3)$: For a given natural number $n$, there exists at least one real quadratic field with $h(d) = 1$ and $n_d \geq n$.

In fact it is easily seen that:

Theorem 2. $(G_1) \iff (G_2) \iff (G_3)$.

Moreover there are applications for the Artin-Ankeny-Chowla conjecture; that $u_p \not\equiv 0 \pmod{p}$ if $p \equiv 1 \pmod{4}$ is prime; as well as the Mollin-Walsh conjecture [6], that if $d \equiv 7 \pmod{8}$ is positive square-free then $u_d \not\equiv 0 \pmod{d}$. In fact we proved the following in [9].

Theorem 3. If $d > 0$ is square-free and $n_d = 0$ then $u_d \not\equiv 0 \pmod{d}$.

Thus the aforementioned two conjectures hold when $n_d = 0$.

Now we turn to the main function of this paper which is to use the above results to actually determine all $d$ with $h(d) = 1$ and $n_d = 0$.

First we provide a table of such values, and then prove that we have all of them, (except possibly one which we show would be a counter-example to the Generalized Riemann Hypothesis).

Theorem 4. If $h(d) = 1$ and $n_d = 0$ then (with possibly one more value remaining) $d$ is an entry in the following Table.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\log (\sqrt{d})$</th>
<th>$d$</th>
<th>$\log (\sqrt{d})$</th>
<th>$d$</th>
<th>$\log (\sqrt{d})$</th>
<th>$d$</th>
<th>$\log (\sqrt{d})$</th>
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<td>53</td>
<td>1.965720471</td>
<td>237</td>
<td>4.346361767</td>
<td>917</td>
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<td>269</td>
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<td>7.087867062</td>
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<td>0.48121218521</td>
<td>62</td>
<td>4.486323128</td>
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<td>1013</td>
<td>6.627804083</td>
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<td>6</td>
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<td>69</td>
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<td>4.488762502</td>
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<td>5.989703264</td>
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<td>77</td>
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<td>4.610624728</td>
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<td>5.098292345</td>
<td>398</td>
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<td>1253</td>
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<td>4.585615389</td>
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<td>7.755150289</td>
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<td>5.183281804</td>
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<td>778</td>
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<tr>
<td>47</td>
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<td>6.113677285</td>
<td>797</td>
<td>5.906362725</td>
<td>3533</td>
<td>7.098523222</td>
</tr>
</tbody>
</table>
Proof. By Corollary 1 we have that \( \varepsilon_d < 8d/\sigma^2 \). Thus our task is to find all positive square-free \( d \) such that \( h(d) = 1 \) and \( 1 < \varepsilon_d < 8d/\sigma^2 \). Let \( \Delta = 4d/\sigma^2 \). A classical class number formula is:

\[ 2h(d) \log(\varepsilon_d) = \sqrt{\Delta} L(1, \chi). \]

Moreover a result of Tatzuwa [11] says:

If \( \frac{1}{2} > \alpha > 0 \) and \( \Delta \geq \max(e^{1/4}, e^{0.5}) \) then with one possible exception \( L(1, \chi) > 0.655 \alpha / d \) where \( \chi \) is a real, non-principal, primitive character modulo \( \Delta \).

We now use the above to complete our task.

Choose \( \alpha = 0.0885 \) and \( \Delta > 80,775.9 \). Then, since \( \log \varepsilon_d < \log 2\Delta \) we have:

(with one possible exception):

\[ h(d) > (\sqrt{\Delta})(0.0885)(0.655)/(2 \log 2\Delta)(\Delta^{0.085}). \]

Hence \( h(d) > 1 \) if \( \Delta > 5 \times 10^4 \); (in fact \( h(d) > 1.026755418 \)).

Now we proceed to show that below this bound the only \( h(d) = 1 \) with \( \varepsilon_d < 8d/\sigma^2 \) are those in the Table. First we need some notation and facts from the theory of continued fractions.

Let \( w_d = (\sigma - 1 + \sqrt{d})/\sigma \) and denote the continued fraction of \( w_d \) by \( w_d = (a, a_1, a_2, \ldots, a_k) \); whence having period \( k \); and \( a_0 = 1 = \lfloor w_d \rfloor \) while:

\[ a_i = \lfloor (P_i + \sqrt{d})/Q_i \rfloor \text{ for } i \geq 1, \]

where:

\[ (P_0, Q_0) = (\sigma - 1, \sigma); \quad P_{i+1} = a_i Q_i - P_i \quad \text{for } i \geq 0 \]

\[ Q_{i+1} = d - P_{i+1}, \quad \text{for } i \geq 0. \]

Now we return to our task.

Case 1. \( d \equiv 2, 3 \pmod{4} \); whence \( \Delta = 4d \).

Since \( \Delta \) is even then 2 ramifies. Thus by [3, Theorem 2.1], \( Q_i \equiv 2 \), with \( k \) even whenever \( h(d) = 1 \), provided \( \Delta > 20 \). (If \( \Delta \leq 20 \) then we get our values \( d = 2, 3 \) of the Table).

From [7] we also have:

\[ \varepsilon_d = \prod_{i=1}^{k} \left( (P_i + \sqrt{d})/Q_{i-1} \right) \quad (P_i \geq 1). \]

Thus:

\[ \varepsilon_d > (\sqrt{d})(\sqrt{d}/2) \prod_{i=1}^{k} \left( (P_i + \sqrt{d})/Q_{i-1} \right) \]

where the product runs from \( i = 2 \) to \( i = k \), excluding \( i = k/2 + 1 \).

Now:

\[ (P_i + \sqrt{d})/Q_{i-1} = (P_{i+1} + \sqrt{d})/Q_i \]

\[ = (P_{i+1} + \sqrt{d})/(\sqrt{d} - P_i) = a_i Q_i / (\sqrt{d} - P_i) + 1 > 2. \]

If \( k \geq 10 \) then \( \varepsilon_d > (\sqrt{d})(\sqrt{d}/2) 2^{(k/2) - 1} \geq 8d \), a contradiction. Since \( k \leq 10 \) then by computation we arrive at \( d \leq 7653 \). Our computation shows that of those values only the following satisfy our criteria and appear in the Table:

\[ d \in \{ 2, 3, 6, 7, 11, 14, 23, 38, 47, 62, 83, 167, 227, 398 \}. \]

Case 2. \( d = 4 \equiv 1 \pmod{8} \).

Thus 2 splits and so since \( h(d) = 1 \) we get \( Q_i / 2 = Q_i / (\sigma - 2) / 2 \) for some \( j \neq 0 \) (provided \( d > 20 \)). (If \( d \leq 20 \) then we get only the value \( d = 17 \) which is on our Table).
Therefore:
\[ \varepsilon_d > (\sqrt{d}/2)(\sqrt{d}/4)^x \prod (P_i + \sqrt{d})/Q_{i-1} \geq d\sqrt{d}/32 > 2d \]
whenever \( d > 64 \), a contradiction. (Here the product runs from \( i = 2 \) to \( i = k \) excluding \( i = j + 1 \) and \( i = k - j + 1 \).)

Hence \( \sqrt{d} \leq 64 \); i.e., \( d \leq 4096 \). In this range our computation gives us only the following values satisfying our criteria: \( d \in \{17, 38, 41\} \).

Case 3. \( d = 6 \equiv 5 \pmod{8} \).

By [2], since \( d < 5 \times 10^6 \) there exists a prime \( p < 67 \) such that \( (d/p) = 1 \), where \( (\cdot) \) is the Kronecker symbol. Suppose \( \sqrt{d}/2 > 67 \). Then \( p \) splits in \( Q(\sqrt{d}) \) and so \( Q_j = Q_{k-j} = 2p \) for some \( j \neq 0 \) (provided \( d > 20 \). If \( d > 20 \) then we get only \( d = 5, 13 \).

Now let \( \varepsilon = (1 + \sqrt{5})/2 \) and \( \psi_i = (P_i + \sqrt{d})/Q_{i-1} \). By [10, Corollary 1, p. 873] \( \prod_{i=a} b \psi_i > \tau^{a+b} \) for \( b \geq a \). Thus:
\[ \prod_{i=3}^k \psi_i \prod_{i=j+2}^k \psi_i \psi_i > \tau^{j-k-j-(j+2)k-(k-j+2)} = \tau^{k-j} \]
(where the initial product ranges over \( i = 2 \) to \( i = k \) excluding \( i = j + 1 \) and \( i = k - j + 1 \)).

Hence \( \varepsilon_d > (\sqrt{d}/2)(\sqrt{d}/2p)^x \prod \psi_i > 2d(\sqrt{d}\tau^{k-j}/16p^x) \)
where the product ranges as in the previous one. Since \( p \leq 67 \) we get that if \( \tau^{k-j} > 536 \) then \( \sqrt{d}\tau^{k-j} > 71824 > 16p^2 \). But \( \tau^{k-j} > 536 \) implies \( k - 6 > (\log 536)/\log \tau = 13.06 \) so \( k > 19.06 \). Thus: If \( d > 17956 \) and \( k \geq 20 \) then \( \varepsilon_d > 2d \), a contradiction. If \( d > 17956 \) and \( k < 20 \) then \( h(d) = 1 \) by computation that \( d \leq 30917 \). In this case there exists a prime \( p \leq 29 \) such that \( (d/p) = 1 \). Hence if \( \sqrt{d}/2 > 2 \cdot 29 = 58 \) we get \( Q_j = Q_{k-j} = 2p \) for some \( p \leq 29 \). Thus \( \varepsilon_d > 2d(\sqrt{d}\tau^{k-j}/16 \cdot 29^x) \) as above. Hence, if \( d > 13456 \) and \( k \geq 16 \) then \( \varepsilon_d > 2d \), a contradiction. If \( d > 13456 \) and \( k \leq 15 \) then \( d < 23117 \).

Our computation on this bound now yields the remaining values in the Table.

Remark 1. In [15] Yokoi found the 30 primes \( p \equiv 1 \pmod{4} \) with \( h(p) = 1 \) and \( n_p \neq 0 \) (with one possible exception). We have completed the task by adding another 35 values to the list for a total of 68. As seen by the above proof there are 14 values of \( d \equiv 2 \pmod{4} \) of which 9 are primes. For \( d \equiv 1 \pmod{8} \) we got only 17, 33 and 41. The remainder are \( d \equiv 5 \pmod{8} \). Of these 51 remaining values 28 are primes, those found by Yokoi along with 17 and 41.

The composite values which we added are the 23 values:

\[ \{21, 69, 77, 93, 133, 141, 213, 237, 341, 413, 437, 453, 573, 717, 917, 1077, 1133, 1253, 1293, 1757, 2453, 3053, 3817\} \]

We also have a list, too long to include here, of all values of squarefree \( d \) with \( n_d = 0 \), up to 39,999 with their class numbers and regulators.

Remark 2. Kim [1] has shown that if the Generalized Riemann Hypothesis (GRH) holds then Tatzuwara's theorem is true without exception. Hence if the exceptional value exists then it is a counterexample to the GRH.
Remark 3. Observe that the Table contains all the ERD-types with $h(d)=1$ (i.e., all types $h(d)=1$ where $d=F+r$ with $4l\equiv 0 \pmod{r}$). These were found by the authors in [8]. Thus there are 25 non-ERD type and they are

\[ 41, 61, 133, 149, 157, 269, 317, 341, 461, 509, 557, 773, 797, \\
917, 941, 1013, 1493, 1613, 1877, 2453, 2477, 2693, 3053, 3317, \\
3533. \]

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References

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