Characterization of $D=P^2+Q^2$ when $\gcd(P,Q)=1$ and $x^2-Dy^2=-1$ has no integer solutions.


The author considers a problem posed by J. P. Robertson and K. R. Mathews [Am. Math. Mon. 115, No. 4, 346–349 (2008; Zbl 1144.11008)]. Let $F$ be a quadratic field, $\mathcal{O}_F$ its maximal order $\mathcal{I}_F$ the group of fractional ideals, $\mathcal{P}_F$ the group of principal ideals and $\mathcal{P}_F^+$ the group of fractional ideals $I=(a)$ such that $N_F(a)>0$. Let $\mathcal{C}_F=\mathcal{I}_F/\mathcal{P}_F$ be the wide ideal class group and $\mathcal{C}_F^+=\mathcal{I}_F/\mathcal{P}_F^+$ the narrow ideal class group. Moreover, let $\Delta_F$ be the discriminant of $F$ and put $D_F=\Delta_F$ if $\Delta_F \equiv 1 \mod 4$ and $D_F=\Delta_F/4$ otherwise. Then the author shows the following equivalence: $D_F$ is a sum of two relative prime squares and there is no unit $u \in \mathcal{O}_F^*$ with $N_F(u)=-1$ if and only if there exists an ideal $I \in \mathcal{C}_F$ of order 2 that is not an image of an ambiguous class of the canonical map $\mathcal{C}_F^+ \to \mathcal{C}_F$. A similar result is also obtained for non-maximal orders $\mathcal{O}$ of $F$.

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MSC:
11D09 Quadratic and bilinear diophantine equations
11D85 Representation problems of integers
11R11 Quadratic extensions
11R29 Class numbers, class groups, discriminants

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