Class numbers, quadratics, and exponential Diophantine equations.

In a previous paper, the author has observed, under certain conditions, an ideal class of order \( n \) in an order of a real quadratic field yields an ideal class of order \( n \) in the maximal order of this quadratic field [Glasg. Math. J. 41, No. 2, 197–206 (1999; Zbl 0990.11066)]. In this paper, the author generalizes several existing results obtained by other authors such as M. R. Murty, M. J. Cowles, B. H. Gross, D. E. Rohrlich. Also the genesis of this paper is an earlier paper [Acta Math. Acad. Paedagog. Nyházi. (N.S.) 21, No. 1, 21–24 (2005; Zbl 1102.11019)], which corrected errors in [Y. Bugeaud and T. N. Shorey, J. Reine Angew. Math. 539, 55–74 (2001; Zbl 0995.11027)].

In fact, let \( D \) be a squarefree negative integer. First, the author supposes that \( D=r^2-r<0 \) with \( r, r \geq 1 \) and \( D \equiv 1 \pmod{4} \). He proves that if there exist \( m \geq 0, N > 1 \), and \( n > 1 \), such that
\[
N^2 = 4m^2 + 4m + r,
\]
and if \( n \) is odd, \( 2m - r \neq \left\lfloor \frac{N^{n/2}}{2} \right\rfloor, \left\lfloor \frac{N^{n/2}-1}{2} \right\rfloor \), then the class group \( C_{4D} \) has a cyclic subgroup of order \( n \). Secondly, he considers the case \( D=r^2-4r \equiv 1 \pmod{4} \) with \( r, r \geq 1 \). He shows that if there exist \( m \geq 0, N > 1 \), and \( n > 1 \) such that
\[
N^2 = m^2 + mt + r,
\]
and if \( 2m + r \neq \left\lfloor 2N^{n/2} \right\rfloor \) when \( n \) is odd, and \( 2m + r \neq 2N^{n/2} - 1 \) when \( n \) is even, then the class group \( C_D \) has a cyclic subgroup of order \( n \). Finally, necessary and sufficient conditions for a class group to have an element of order \( n \) are given for each of the above cases. The paper is illustrated by many numerical examples.

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MSC:
- 11R11 Quadratic extensions
- 11R29 Class numbers, class groups, discriminants
- 11R65 Class groups and Picard groups of orders
- 11D61 Exponential diophantine equations

Keywords:
quadratic extensions; class number; quadratic order; exponential equations