These problems can be solved using mathematical induction. You might also be able to solve some of these problems in a different way, without using mathematical induction at all! However, the purpose of this exercise is to give you practice using mathematical induction, so you should look for a way to use mathematical induction when you answer these questions.

1. Prove that $n^2 - n$ is even, for every integer $n \geq 0$.

   **Proof.** Induction on $n$. The base case is $n = 0$, and $0^2 - 0 = 0$ is certainly even. For the induction step, you need to show that $(k + 1)^2 - (k + 1)$ is even whenever $k^2 - k$ is even, for every integer $k \geq 0$. So let $k$ be any non-negative integer, and assume that $k^2 - k$ is even. Now

   $$(k + 1)^2 - (k + 1) = (k^2 + 2k + 1) - (k + 1) = k^2 + 2k + 1 - k - 1 = k^2 + k ,$$

   which by induction hypothesis is even. This proves out claim.

   An alternative way to do the algebraic manipulation above is to pull out the factor $k + 1$ that is common to both the terms $(k + 1)^2$ and $k + 1$, to obtain

   $$(k + 1)^2 - (k + 1) = (k + 1)((k + 1) - 1) = (k + 1)k = k^2 - k ,$$

   which again by induction hypothesis is even.

2. Prove that $n^3 - n$ is divisible by 6, for every integer $n \geq 0$. You may use the result of Problem 1.

   **Proof.** Induction on $n$. The base case is $n = 0$, and $0^2 - 0 = 0$ is certainly divisible by 6. For the induction step, you need to show that $(k + 1)^3 - (k + 1)$ is divisible by 6 whenever $k^3 - k$ is divisible by 6, for every integer $k \geq 0$. So let $k$ be any non-negative integer, and assume that $k^3 - k$ is divisible by 6. Now

   $$(k + 1)^3 - (k + 1) = (k^3 + 3k^2 + 3k + 1) - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 + 3k^2 + 2k .$$

   We want to use the induction hypothesis, so we need to “isolate” an expression of the form $k^3 - k$ in the right hand side above. To that end, consider

   $$k^3 + 3k^2 + 2k = (k^3 - k) + 3k^2 + 3k = (k^3 - k) + 3(k^2 + k).$$
By induction hypothesis, the first term $k^3 - k$ is divisible by 6. The second term, $3(k^2 + k)$, is certainly divisible by 3, but that is not enough; we also need it to be divisible by 2 to obtain that it is divisible by 6. We’d like to use the result of Problem 1, so we need to isolate an expression of the form $k^2 - k$:

$$3(k^2 + k) = 3k^2 + 3k = 3k^2 - 3k + 6k = 3(k^2 - k) + 6k.$$  

The first term $3(k^2 - k)$ is divisible by 6, since it is obviously divisible by 3 and $k^2 - k$ is divisible by 2 by Problem 1. The second term $6k$ is obviously divisible by 6. Hence, $3(k^2 + k)$ is divisible by 6, and from our initial expression, we obtain that $(k + 1)^3 - (k + 1)$ is divisible by 6. This concludes our induction proof.

3. Prove that $5^n + 2 \cdot 3^{n-1} + 1$ is divisible by 8, for every integer $n \geq 1$.

**Proof.** For $n = 1$, we have $5^1 + 2 \cdot 3^0 + 1 = 5 + 1 = 8$ which is divisible by 8. Now let $k$ be any positive integer and assume that $5^k + 2 \cdot 3^{k-1} + 1$ is divisible by 8; we need to prove that $5^{k+1} + 2 \cdot 3^k + 1$ is divisible by 8. Once again, in order to use our induction hypothesis, we need to manipulate the expression $5^{k+1} + 2 \cdot 3^k + 1$ in such a way that we “isolate” the expression $5^k + 2 \cdot 3^{k-1} + 1$.

So let’s try to “reduce” the exponent $k + 1$ in $5^k$ down to $k$ (to get $5^k$), and the exponent in $3^k$ down to $k - 1$ (to get $3^{k-1}$):

$$5^{k+1} + 2 \cdot 3^k + 1 = 5 \cdot 5^k + 2 \cdot 3 \cdot 3^{k-1} + 1 = 5 \cdot 5^k + 6 \cdot 3^{k-1} + 1 = (1 + 4)5^k + (2 + 4)3^{k-1} + 1 = (5^k + 2 \cdot 3^{k-1} + 1) + (4 \cdot 5^k + 4 \cdot 3^{k-1}).$$  

The first term on the right hand side above is divisible by 8 by induction hypothesis. The second term is $4(5^k + 3^{k-1})$ which is certainly divisible by 4. Now $5^k$ is the power of an odd number (5), so it is odd. Similarly, $3^{k-1}$ is a power of the odd number 3 and is hence odd. The sum of two odd numbers is even, so $5^k + 3^{k-1}$ is even. Hence, $4(5^k + 3^{k-1})$ is divisible by 8. If follows that $5^{k+1} + 2 \cdot 3^k + 1$ is divisible by 8, and our claim now follows from induction.

4. Prove that

$$\sum_{i=0}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$$

for every integer $n \geq 0$.

**Proof.** The base case is $n = 0$. We have $\sum_{i=0}^{0} i^2 = 0^2 = 0$ and $\frac{0(0 + 1)(2 \cdot 0 + 1)}{6} = 0$. So the claim holds for $n = 0$.

Now let $k \geq 0$ be any integer, and assume that

$$\sum_{i=0}^{k} i^2 = \frac{k(k + 1)(2k + 1)}{6}.$$  

We need to prove that

$$\sum_{i=0}^{k+1} i^2 = \frac{(k + 1)((k + 1) + 1)(2(k + 1) + 1)}{6}.$$
So let us calculate both sides of this expression. For the left hand side, we just separate out the last term in the sum:

\[ \sum_{i=0}^{k+1} i^2 = \sum_{i=0}^{k} i^2 + (k + 1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2, \]

where the last equality follows from the induction hypothesis. Pulling out a common factor \((k + 1)/6\), we obtain

\[ \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 = \frac{k + 1}{6} \left( 2k + 7k + 6 \right). \]

On the other hand,

\[ \frac{(k + 1)((k + 1) + 1)(2(k + 1) + 1)}{6} = \frac{(k + 1)(k + 2)(2k + 3)}{6} \]
\[ = \frac{k + 1}{6} \left( 2k + 7k + 6 \right). \]

Since both sides of the equation in the induction claim are equal, the assertion is proved.

6. Prove that every binary tree with \(n\) edges has exactly \(n + 1\) nodes, for every integer \(n \geq 0\).

**Proof.** A binary tree with 0 edges has one node, so the base case holds. Let \(k \geq 0\) be any integer, and assume that every binary tree with \(k\) edges has \(k + 1\) nodes. Let \(T\) be any binary tree with \(k + 1\) edges; we need to prove that \(T\) has \(k + 2\) nodes.

Since \(k \geq 0\), \(T\) has at least one edge, and hence a leaf node \(v\). Remove the (unique) edge \(e\) that is incident with this leaf node \(v\) from \(T\); this of course also removes the end point \(v\) of \(e\) from \(T\). We claim that the resulting graph \(G = T \setminus e\) is again a binary tree:

- \(G\) is connected since removing a leaf node and its corresponding edge from a tree does not disconnect the resulting graph;
- \(G\) has no cycles since removing an edge from a graph does not introduce cycles;
- every node in \(G\) has at most two children, since removing an edge does not increase the number of children of any node in \(T\).

Hence \(G\) is a binary tree. Since \(G\) has one less edge than \(T\), it has \(k\) edges, and hence \(k + 1\) nodes by the induction hypothesis. Now \(T\) also has one more node than \(G\) because \(G\) was obtained from \(T\) by removing the edge \(e\) and its end point \(v\) of \(e\). Thus, \(T\) has \(k + 2\) nodes as claimed.