§1. Philosophical background: iteration, ineffability, reflection. There are at least two heuristic motivations for the axioms of standard set theory, by which we mean, as usual, first-order Zermelo-Fraenkel set theory with the axiom of choice (ZFC): the iterative conception and limitation of size (see [6]). Each strand provides a rather hospitable environment for the hypothesis that the set-theoretic universe is ineffable, which is our target in this paper, although the motivation is different in each case.

Contemporary ZFC does not countenance urelements. It is a theorem that every set is a member of some $V_\alpha$, where, as usual:

\[ V_0 = \emptyset; V_{\alpha+1} = \mathcal{P}(V_\alpha); \text{ and } V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \text{ for limit ordinals } \lambda. \]

The picture of the set-theoretic universe as divided into stages of a cumulative hierarchy has become a powerful heuristic for standard set theory: the iterative conception of set. On the iterative conception, sets are usually described with a temporal metaphor, or at least we take it to be a metaphor. In the beginning we have non-sets (or urelements as they are sometimes known), or else in the case of pure set theory we begin with nothing—creation ex nihilo. We proceed in stages. To move from one stage to the next, we form all sets corresponding to arbitrary collections of members from previous stages. The temporal metaphor is stretched, beyond recognition, when the iteration continues into the transfinite. At limit stages, we form sets corresponding to arbitrary collections of items from earlier stages. Then we continue forming sets corresponding to collections of those sets (see, for example, [5] and [6]). It follows that the membership relation is well-founded.

How far is this iteration to continue? How many sets are there? The stock, and unhelpful, answer is that the iteration goes all the way, through all of the ordinals, i.e., the order-types of well-orderings. But how far is that? When interpreted against the backdrop of the iterative conception, the axioms of ZFC reflect intuitions supporting some partial, but more substantial answers to the question of how far to proceed. In effect, the axiom of infinity amounts to the existence of at least one transfinite stage, $V_\omega$. The next stage after that gives us an uncountable set, the power set of $V_\omega$. Another stock heuristic is that the iterative is as “high” as it is “wide”. The axiom of choice gives us a well-ordering of the
size of the power set of $V_\omega$. Replacement yields a von Neumann ordinal of that order-type. So the iteration goes at least that far. The usual axioms take the iteration to what many find to be dizzying heights. As usual, let $\beth_0 = \aleph_0$, the smallest infinite ordinal; for each ordinal $\alpha$, let $\beth_{\alpha+1}$ be the smallest ordinal equinumerous with the power set of $\beth_\alpha$; and if $\lambda$ is a limit ordinal, then $\beth_\lambda$ is the union of all $\beth_\alpha$, for $\alpha < \lambda$. The axioms entail that there is a fixed point in this Beth series: a cardinal $\kappa$ such that $\kappa = \beth_\kappa$ (see [8]). Each fixed point in the Beth series has the curious property that it is larger than the power set of any smaller cardinal.

When it comes to recapturing virtually all of ordinary mathematics (excluding foundational theories such as set theory, category theory and the like), this is massive overkill. However, working set theorists do not generally take the axioms of ZFC to set a limit on the length of the ordinal sequence.$^1$ A cardinal $\kappa$ is strongly inaccessible if it is regular and larger than the power set of any smaller cardinal (and so every strong inaccessible is a fixed point in the Beth-series). The axioms of ZFC do not entail the existence of any strongly inaccessible cardinals. Indeed, if $\kappa$ is a strong inaccessible, then the result of iterating the hierarchy to stage $\kappa$ will result in a model of ZFC. Nevertheless, the existence of (lots of) strong inaccessibles is a staple of set theory. But strong inaccessibles are the very smallest of the so-called small large cardinals. In this context, reflection gives us an attractive answer to the question of how far to proceed. In particular, it can be argued to partially articulate a widely held view among working set theorists according to which the set-theoretic hierarchy is ineffable, or indescribable. Suppose that we find that a certain description $\Phi$ applies to the iterative hierarchy. Since, it is presumed, the set-theoretic universe is ineffable, $\Phi$ cannot characterize the iterative hierarchy uniquely. So, the argument concludes, there is a set in the hierarchy that satisfies $\Phi$.

For example, it follows from the axioms of ZFC, that the hierarchy is inaccessible, in the sense that it is larger than the power set of any of its members (thanks to the power set axiom) and it is not the union of a set sized collection of sets (replacement). So we conclude, via reflection, that there is a set that is strongly inaccessible. The iteration goes on at least that far. And, of course, once we get that far, we go on past that. There is a power set of the first strong inaccessible, . . .

Another motivation for reflection comes from the other heuristic motivation for some the axioms of set theory: limitation of size. This is roughly the thought that some objects form a set just in case they are not too many. What is too many? Are uncountably many sets too many? Are strongly inaccessibly many sets too many? There is the Cantorian, albeit vague, answer according to which some sets are too many if they are indeterminately or indefinitely many. Reflection can be motivated as a way of articulating this answer. This

---

$^1$An exception is Abraham Fraenkel’s short lived axiom of restriction which asserts, in effect, that there are no sets other than those whose existence follows from directly from the other axioms. This is an analogue of the principle of induction in arithmetic. See Fraenkel [1922] and [1], 113-116.
is made explicit, for example, in [9] and [10] (see ch. 3), where the following heuristic train of inter-connections is proposed to take us from the hypothesis that there are too many sets, i.e., indeterminately or indefinitely many, to reflection:

1. . . . the sets are indeterminately or indefinitely many.
2. . . . the sets are indefinably or indescribably many.
3. . . . any statement \( \Phi \) that holds of them fails to describe how many they are.
4. . . . any statement \( \Phi \) that holds of them continues to hold if reinterpreted to be not about all of them but just about some of them, fewer than all of them.
5. . . . any statement \( \Phi \) that holds of them continues to hold if reinterpreted to be not about all of them but just some of them, few enough to form a set.

Burgess is, of course, aware that these transitions are not immune to challenge. The steps are not meant to be anything like a deductive argument supporting reflection. Still, reflection seems to be in line with the working hypothesis of practicing set theorists according to which the universe of set theory is ineffable or indescribable. 2

Even when motivated by limitation of size, this hypothesis promises to shed some light on the length of the cumulative hierarchy against the background of the iterative conception. In an early articulation of the iterative conception, [36] (see p. 1233), Ernst Zermelo proposed the existence of an unbounded sequence of [inaccessible cardinals] as a new axiom of meta-set theory.” According to this principle, for each ordinal \( \alpha \), there is a unique inaccessible cardinal \( \kappa_\alpha \). But this is not the end of the process of reflection. Reflection on the new axiom would entail that there is a set \( x \) such that for every ordinal \( \alpha \in x \), \( \kappa_\alpha \in x \). This, in turn, entails the existence of a fixed point in the \( \kappa \)-series: an ordinal \( \kappa \) such that \( \kappa = \kappa_\kappa \). This is called a “hyper-inaccessible”. Reflection goes on, yielding Mahlo cardinals, hyper-Mahlo cardinals, and the like, up to indescribable cardinals. As noted, these are sometimes called small, large cardinals.

In [36] (see p. 1232), Zermelo wrote:

If we now put forward the general hypothesis that every categorically determined domain can also be interpreted as a set in some way, i.e., can appear as an element of a normal domain [a set-theoretic model of set theory], it follows that to each normal domain there is a higher domain with the same basis.

In [35] (see p. 555), Hao Wang captured the idea succinctly:

---

2The present, vague articulation of ineffability/reflection seems to be related to Michael Dummett’s notion of “indefinite extensibility” (see [11], p. 317). According to Dummett, the error that led Frege to Basic Law V did not lie “in his supposing every definite concept to have an extension”. Rather the “mistake lay in supposing there to be a totality containing the extension of every concept defined over it; more generally it lay in his not having the glimmering of a suspicion of the existence of indefinitely extensible concepts.” See [28]. [9] and [10] follow [7] in having “no use” for the notion of indefinite extensibility. The underlying issues relating to absolute generality cannot be pursued here (see [24]).
Any time we try to capture the universe from what we positively possess (or can express) we fail the task and the characterization is satisfied by certain (large) sets.

In the remainder of this paper, we set out to isolate a reasonably pure version of ineffability in the context of a Fregean theory of extensions. So, in formulating our interpretation of ineffability, we make sure not to rely on any set-theoretic assumptions concerning the structure of the set-theoretic universe. We are interested in unpacking the very idea of ineffability with a view to answering the question of what set-theoretic principles can be motivated with the help of a reasonably pure version of ineffability alone and what set-theoretic principles require further assumptions motivated perhaps by a different heuristic. Our results will be mixed. While the very idea of ineffability cannot be used to motivate unrestricted versions of all of the existence axioms of set theory (union and power set are the two exceptions), it nevertheless encodes a remarkable amount of set theory, enough, for example, to show the consistency of Zermelo set theory.

§2. Formulating reflection, and the background theory. We work in a second-(or higher) order language. Although we presuppose what has been called standard semantics, we want to remain as neutral as possible concerning what the second-order variables range over: Fregean concepts, (proper) classes, logical sets, whatever. This is not to say that we can in fact remain completely neutral. It is important to us to make free use of quantification over polyadic relations, which puts some strain on the prospects of a plural interpretation of second-order quantification. At all events, since we are largely concerned with modifications of Basic Law V in this paper, we settled on Fregean concepts. However, the reader is free to substitute a different locution if she so wishes.

We define the relativization of a sentence Φ to a concept F to be the result of relativizing the quantifiers in Φ to F, by which we mean, as usual, the result of replacing:

\[ \exists x(\ldots) \quad \text{with} \quad \exists x(Fx \& \ldots) \]
\[ \exists X(\ldots) \quad \text{with} \quad \exists X(\forall x(Xx \rightarrow Fx) \& \ldots) \]
\[ \forall x(\ldots) \quad \text{with} \quad \forall x(Fx \rightarrow \ldots) \]
\[ \forall X(\ldots) \quad \text{with} \quad \forall X(\forall x(Xx \rightarrow Fx) \rightarrow \ldots). \]

We will write \( \Phi^F \) for the relativization of \( \Phi \) to \( F \). If the language in question has a relation for membership, and if \( t \) is a term, then \( \Phi^t \) is the relativization of \( \Phi \) to the concept of being a member of \( t \). That is, \( \Phi^t \) is the result of replacing:

\[ \exists x(\ldots) \quad \text{with} \quad \exists x(x \in t \& \ldots) \]
\[ \exists X(\ldots) \quad \text{with} \quad \exists X(\forall x(Xx \rightarrow x \in t) \& \ldots) \]
\[ \forall x(\ldots) \quad \text{with} \quad \forall x(x \in t \rightarrow \ldots) \]
\[ \forall X(\ldots) \quad \text{with} \quad \forall X(\forall x(Xx \rightarrow x \in t) \rightarrow \ldots). \]
Reflection principles make at least implicit reference to structural properties of the iterative hierarchy expressed by various statements of the background language. This raises the question of what resources should be available in order to express the features and statements in question. One reflection principle is in fact a theorem of first-order set theory. If a first-order sentence $\Phi$ does not contain the variable $x$, then $(\Phi \rightarrow \exists x \Phi^x)$ is already a theorem of first-order ZFC. The extra power from reflection comes either from explicitly invoking stronger statements, or moving to a higher-order language.

In [20] and [19], Azriel Lévy provided a study of some reflection principles, in the context of first-order set theory. One of them is a schema he calls the principle of sentential reflection ([19], 1):

If $\Phi$ holds then there exists a standard model of [set theory] in which $\Phi$ holds too, one instance for each sentence $\Phi$ in the language of first-order set theory. In terms of the iterative conception, a “standard model” is the result of taking the iteration through a strongly inaccessible cardinal $\kappa$. So the schema says that any truth of set theory is satisfied in some strongly inaccessible rank.

[25] collects together results concerning reflecting sentences (and open formulas) in the language of second-order set theory (and the language of pure second- (and higher-) order logic; see also [26], Chapter 6). Let $\Phi$ be any sentence in the language of second-order set theory (or the language of pure logic, for that matter). Then

\[ \Phi \rightarrow \exists x \Phi^x \]

is an instance of one of the schemata.

In this case, we do not need Levy’s clause about standard models in order to secure the existence of small large cardinals. This is thanks to a brilliant move used, over and over, by Paul Bernays in [3] and [4]. The simple observation is that if a sentence $\Phi$ holds, then so does $\Phi \& \Psi$, where $\Psi$ is any theorem in the system. So we can “reflect” on the conjunction $\Phi \& \Psi$. To illustrate, let $Z_2$ be the conjunction of the axioms of second-order ZFC. Then, of course, second-order ZFC entails $Z_2$. So the reflection principle gives us a set that satisfies $Z_2$. Every such set is the size of a strong inaccessible.

In the context of second-order set theory, the principle of sentential reflection is a natural formulation of what [25] calls “Kreisel’s principle, a thesis that if a sentence is satisfiable at all, then it can be satisfied on a set. Kreisel’s principle is a presupposition of the use of the iterative hierarchy to give the model theory for higher-order languages.

The background for [20], [19], and [25] is, of course, ordinary set theory. That is, those papers explore the results of adding reflection principles to (first- or second-order) ZFC. So they start with some fairly substantial axioms concerning the existence of sets, and they look to establish even stronger existence principles. In contrast, [3], [4] showed how most of the axioms of second-order ZFC are themselves consequences of a strengthened reflection
principle formulated in a language with variables ranging over sets and (proper) classes. For present purposes, one can think of that as a second-order language. In particular, Bernays shows that pairs, union, power set, infinity and replacement of ZFC (and the existence of inaccessibles, hyperinaccessibles, and other small large cardinals) are derivable from the axioms of extensionality, separation, and a reflection principle formulated as a scheme, one instance for each formula $\Phi$ which does not contain $x$ free:

$$\forall y_1...\forall y_n(\Phi \rightarrow \exists x (\text{trans}(x) \& \Phi^x)),$$

where $y_1...y_n$ are all of the free variables in $\Phi$, and $\text{trans}(x)$ is the usual statement that $x$ is transitive: $\forall y \forall z ((y \in x \& z \in y) \rightarrow z \in y)$.

In other words, most of the existence axioms of ZFC and the existence of various small large cardinals follows from extensionality, separation, and a principle that for any $\Phi$, if $\Phi$, then $\Phi$ is satisfied by a transitive set. Who would have thought that so much is derivable from so little? [9] shows how to transpose the results to a language with singular and plural quantification (see also [10], pp. 190-201). With the relevant primitives, he is able to weaken the reflection scheme to a simpler schema, whose instances are the universal closures of

$$\Phi \rightarrow \exists x \Phi^x,$$

where $\Phi$ is a formula in the chosen language. In particular, with the help of an axiom of Heredity, which governs the predicate $\beta$, for “is a set”, Burgess is able to derive Bernays’s reflection principle from the weaker form of reflection. However, the argument is highly sensitive to the choice of primitives. Heredity has the appearance of an explicit definition of $\beta$ in terms of the plural predicate “form a set”. If Heredity is treated as such, then, of course, $\beta$ is no longer a primitive. Remarkably, in that case, we can no longer justify the transition from the weaker principle to Bernays’s reflection principle. So what theory one obtains in this framework depends crucially on the choice of primitives, as Burgess himself notes. It is a commonplace in mathematics that the choice of primitives is, to a certain extent, arbitrary. Quite often, one can think of some terms as primitive and define others in terms of those, or vice versa. In such cases, one would think, it does not matter where one starts. There is a marked trend in contemporary mathematics away from relying on what can be defined by this or that resources when developing our basic theories.

That trend is violated here. Since the above reflection principles are formulated as schemata, one instance for each sentence, or formula, in the respective language, the exact theory one obtains is tied directly to the expressive resources in the given language.

Not only that, [22] has recently argued that the very consistency of the framework is sensitive to the choice of primitive vocabulary. He shows, in particular, that, modulo reasonable assumptions on the behavior of plurals, if we add another rather natural primitive

\footnote{Compare also [29], [30], [13] and [18].}
to the system, a sign $\approx$ for plural identity (which would correspond to an identity relation on the items in the range of second-order variables), together with the obvious axiom of extensionality, the system is rendered inconsistent. In particular, it follows that every plurality forms a set. There is, in particular, a set of non-self-membered sets, which shows the system to be vulnerable to Russell’s paradox. Of course, there is no problem with introducing $\approx$ as a defined term, governed by the same principle of extensionality, now regarded as a definition. The trouble comes when we regard it as primitive.

§3. Framework. [22] concludes:

The only satisfactory solution would be to provide a better motivation for the reflection principle that explains what expressive resources may figure in formulas to which the reflection principle is applied. But it is completely unclear what such a motivation would look like.

We do not contribute to such a satisfactory solution here. Indeed, we are not even out to formulate a theory as rich as ZFC, let alone motivate or justify small large cardinal principles. Our more modest aim is to explore the consequences of reflection on minimal expressive resources, namely those of pure (higher-order) logic. We want to see what can be justified on the basis of ineffability-cum-reflection alone, reflecting on sentences with no non-logical terminology, primitive or otherwise. Our theories have models within the iterative hierarchy, below the first strong inaccessible. However, they are more resourceful than one might have first expected, and they are rich enough to prove the consistency of Zermelo set theory.

In order to keep our framework free from assumptions concerning the iterative hierarchy, we set out to develop a theory of extensions. Fregean extensions are ideal for this purpose because we cannot assume that the universe of extensions is well-founded or that they have any specific structural feature akin to the cumulative hierarchy. (See [17], especially sections 6 and 7, for some illustration of the flexibility afforded by extensions.) We want to develop a modification of Freges Basic Law V:

$$\forall x \left( Fx \equiv Gx \right)$$

Of course, as stated, this is inconsistent. To remove the inconsistency, we need to restrict the range of concepts to which an extension is assigned. Our goal will be to assign extensions only to concepts that are describable by using logical resources alone. Introduce a word \textit{GOOD} for those concepts, and let \textit{BAD} stand for its complement.

There are several ways to restrict Basic Law V, with the choice between them only a matter of convenience. One option is to think of the extension function as a partial function from \textit{GOOD} concepts to objects, in which case $\forall x \left( Fx \equiv Gx \right)$ becomes a non-denoting singular term when $F$ is \textit{BAD}. This is the option followed in [17] and [2]. However, the presence of non-denoting singular terms requires a free logic in the background.
Another option is to take the extension function to be total (i.e., defined on all concepts), and to assign a dummy object to all concepts without Fregean extensions. The dummy object is officially an extension but in practice it differs from Fregean extensions by not respecting extensionality. This is the route followed by [6] and [27]. The relevant principle would be:

\[ \exists F x = \exists G x \equiv ((BAD(F) \& BAD(G)) \lor \forall x (Fx \equiv Gx)). \]

We adopt a third option here, which has the advantages of free logic without the inconvenience of non-denoting singular terms. Instead of a term for a function from concepts to objects, we introduce a higher-order predicate \( EXT(x, F) \), which is satisfied by an object \( x \) and a concept \( F \) just in case \( x \) is an (or the) extension of \( F \). This is in fact the procedure used in [10], where he develops a Fregeanized version of Bernays-Boolos set theory (BB), which he calls FB. So, a further reason for our choice is to facilitate comparison between our system and his. Our axiom is thus:

\[ (RV) \quad EXT(x, X) \& EXT(y, Y) \rightarrow (x = y \equiv \forall z (Xz \equiv Yz)) \]

The \( EXT \) predicate is the only non-logical primitive in our system. An extension predicate and a symbol for membership are explicitly defined in terms of \( EXT \) as follows:

\[
\begin{align*}
\text{ext}(x) & \equiv df \exists X(EXT(x, X)) \\
x \in y & \equiv df \exists X(EXT(y, X) \& Xx).
\end{align*}
\]

We can define \( \text{GOOD} \) and \( \text{BAD} \) as abbreviations:

\[
\begin{align*}
\text{GOOD}(F) & \equiv df \exists x EXT(x, F) \\
\text{BAD}(F) & \equiv df \neg \text{GOOD}(F).
\end{align*}
\]

From (RV) and our definitions, we can quickly derive a principle of extensionality for extensions:

\[ (\text{ext}(x) \& \text{ext}(y)) \rightarrow (x = y \equiv \forall z (z \in x \equiv z \in y)) \]

Another consequence of (RV) is that if a concept has an extension, then it has at most one. So we will speak, informally, of “the extension of” a concept \( F \) to refer to the unique \( x \) such that \( EXT(x, F) \).

Since we are on the matter of terminology, we want to insist on a distinction between “extension” and “set” for some of the reasons given in [17]. While we take sets to lie in ranks of an iterative hierarchy and therefore to be well-founded, the membership relation on extensions need not be well-founded at all (see §6 below). Moreover, it will be convenient to reserve the word “set” for the later discussion of models of our theory (e.g., §7), since the meta-theory is ordinary first-order ZFC. So we will only use the word “extension” here.

\[ ^{\text{5}}\text{Notice that we do not invoke reflection on sentences containing this (or any other) non-logical primitive.} \]
There is a potential ambiguity with the membership relation, $\in$, as well, but that will not cause trouble.

Thus far (RV) is just a principle of extensionality for extensions. If we think of $\exists x \text{EXT}(x, F)$ as short for “$F$ has an extension”, then (RV) is clearly true on just about any conception of extension. However, (RV) remains silent on the question of what concepts have extensions assigned to them. This is of course the question we would like to answer. The plan for the following two sections is to rehearse a partial response in terms of ineffability and reflection. We show that the very nature of our answer prevents us from extending it to a complete answer (see §8 below).

§4. Characterizing indefinability, first attempt. If $F$ and $G$ are concepts, then let $F \sim G$ be the usual second-order statement that $F$ is equinumerous with $G$. That is, $F \sim G$ if and only if there is a one-to-one relation from $F$ to $G$. Similarly, let $F \preceq G$ say that there is a one-to-one relation from $F$ to $G$, and let $F \prec G$ say that $F \preceq G$ but not $G \preceq F$. If a sentence $\Phi$ with no non-logical terminology is true in a structure $M$, then $\Phi$ is true in any structure whose universe is equinumerous with that of $M$. In effect, the only distinctions among models that can be made with such “pure” sentences are differences of cardinality. So, since we are concerned with the relative sizes of various concepts and extensions, it makes sense to invoke purely logical sentences in reflection. As an added bonus, there are no primitive terms to consider or eliminate.

For our first attempt, we take a cue from Zermelo’s hypothesis that “every categorically determined domain can also be interpreted as a set” ([36], 1232). We take this to be the suggestion that if a structure can be characterized up to isomorphism, then it is a set. In the present framework, the corresponding suggestion is that the concept of being a member of the domain of that structure has an extension. To speak of categoricity, of course, one must delimit the expressive resources available, the languages in which one attempts to characterize various domains and structures. Here, we propose to interpret “categorically determined” to be “fixed up to isomorphism by a sentence of pure second-order logic”.\footnote{Zermelo himself worked on the possibility of developing set theory by invoking infinitary languages. See, for example, [23], [32], and [12].}

Say that a concept $F$ is characterized by a sentence $\Phi$ of pure second-order logic if $\Phi^F$, and, for all concepts $X$, if $\Phi^X$, then $F \sim X$. In these terms, Zermelo’s proposal is that if $F$ is characterized by a sentence of pure second-order logic, then $F$ has an extension.

The contrapositive of this hypothesis is that if a concept $F$ lacks an extension, then $F$ is not characterized by any sentence of pure second-order logic, which corresponds to (3) in Burgess’s suggested route from limitation of size to reflection:

(3) . . . any statement $\Phi$ that holds of them fails to describe how many they are.
A cursory look at Zermelo’s proposal will now shed light on the content of (3) against the background of minimal assumptions, and, in particular, it will give us a better sense of the distance between (3) and each of the further steps in the route towards reflection.

Zermelo’s suggestion would initially seem to deliver large cardinals. Let $B$ be the conjunction of the axioms of second order ZFC together with a statement that there are no strongly inaccessible cardinals. Let $B'$ be the result of replacing the membership symbol in $B$ with a new relation variable $R$. Then the first strong inaccessible is characterized by $\exists R B'$ (see [26], 154-155). We are not even limited to small large cardinals: if $\kappa$ is the first measurable cardinal, then the power set of the power set of $\kappa$ is characterized by a sentence of pure second-order logic.

There are two problems with this first proposal. One is that since there are only countably many sentences of pure second-order logic, we would have only countably many extensions. We have not yet justified the counterpart of separation, a principle that if a concept $F$ has an extension and every $G$ is an $F$, then $G$ has one too. But perhaps that could be overcome, by motivating another principle, say one based on the limitation of size (which, arguably, fits well with the thought that the universe is ineffable). A more troubling problem is that the proposal does not, by itself, justify the existence of any extensions at all. We can show that any two models of, say, $\exists R B'$ are equinumerous. So Zermelo’s proposal would yield an extension the size of a strongly inaccessible cardinal, provided that there is such a concept. That is, we cannot assert the existence of a strong inaccessible, unless we show first that there are at least that many objects. But that is the very thing we have not yet shown. At this stage, it would beg the question to just assume that the universe is at least that size. The proposal is not (yet) motivated.

What we have to do, it seems, is bootstrap. In informal language, [9] (see p. 192), indicates how to do this when we take the step from (3) to (4):

(4) . . . any statement $\Phi$ that holds of them continues to hold if reinterpreted to be not about all of them but just about some of them, fewer than all of them.

He writes:

. . . though it is possible to make a true statement about how many objects there are, there are too many objects for it to be possible for such a statement to be definitive of how many there are: there will necessarily be not merely as many as is said, but more also . . . To begin with, there is at least one object. By [(4)], that would still be true if one were speaking not of all objects, but just some of the objects, fewer than all; in other words, it is an understatement, which means that there must be at least two objects. Then, by [(4)] again, that is an understatement, so there are at least three objects. Continuing in this way, there are infinitely many objects. But by [(4)], even that is an understatement,
so there must be uncountably many objects. But by [(4)] again, even that is an
understatement, so . . .

This procedure would take us as far as we can express cardinalities, from below so to speak.
And that, of course, depends on the expressive resources used. We propose to recapitulate
Burgess reasoning formally, using a reflection principle and exploiting the aforementioned
insight that purely logical statements can correspond to statements about the size of the
universe. The procedure described above will thus extend as far as the expressive resources
of the pure second-order language allow.

§5. Characterizing indefinability, a theory of extensions. We extend Zermelo’s
proposal in an obvious direction. Let Φ be a sentence in a formal language. The size a
concept F must have, in order for Φ^F to hold, is a lower bound that Φ imposes on the
universe. For example, the sentence ∃x∃y(x ≠ y) imposes a lower bound of 2 on the size
of the universe, since (∃x∃y(x ≠ y))^F holds only if F applies to at least two objects. And
if PA2 is the conjunction of the axioms of second-order Peano arithmetic, then PA2 sets a
lower bound of \( \aleph_0 \) on the universe. Similarly, Z2, the conjunction of the axioms of second-
order ZFC, sets a lower bound at the first strong inaccessible. So do the sentences B and
∃RB′ from the previous section.

Of course, some sentences, like ∃x(x ≠ x), fail to set a lower bound just because they
are not satisfiable at all. But even if we restrict attention to satisfiable sentences, one may
wonder whether they set a unique lower bound on the size of a concept. The answer depends
on the scope of our background assumptions. We can only show that every satisfiable
sentence sets a unique lower bound on the universe if we have the resources to show that
any two concepts are comparable in size. But this requires a choice principle.\(^7\)

More formally, say that a concept F is fixed by a sentence Φ just in case Φ^F and for
every concept X, if Φ^X then F ⊲ X. This is at least one formal analogue of the notion
of Φ setting a bound (on the universe) at the “size” of F. Say that F is bounded if there is
a sentence Φ of pure second-order logic (i.e., Φ has no non-logical terminology) and a
concept G such that F ⊲ G and G is fixed by Φ.\(^8\)

\(^7\)We return to the issue of choice principles several times below.

\(^8\)This notion of boundedness is a counterpart, in pure logic, of the model-theoretic notion of an ordinal
being “pinned down” by a sentence. An alternate definition of a sentence Φ fixing a concept F would be
that Φ^F and for every concept X, if X ≺ F then ¬Φ^X. This is entailed by the definition given above,
and is equivalent to it if there is a strong choice principle in the logic—the choice principle being needed
to show that for every X, Y, either X ⊲ Y or Y ⊲ X. The connection with Zermelo’s proposal is that if
we have a strong choice principle, and if Φ fixes F, then there is a sentence Ψ that characterizes F up to
equinumerosity, in the sense that Ψ^F and for every X if Ψ^X then X ∼ F. So we have that every satisfiable,
categorical concept has an extension. Since there is no non-logical terminology, equinumerous is the same
as isomorphic.
One obvious thought to have at this point is that all and only bounded concepts are GOOD. Unfortunately, we cannot be expected to provide necessary and sufficient conditions for a concept to be bounded in the language of pure second-order logic. For if we could, we could thereby give necessary and sufficient conditions for a concept to be GOOD. But this, in turn, would enable us to characterize the size of the least BAD concept, which would therefore be bounded and, consequently, GOOD, which is absurd. Otherwise put, a successful characterization of bounded would allow us to describe the indescribable, or, to coin an expression, to eff the ineffable.\footnote{We assume that the reader understands what we mean by eff. We were surprised to learn that the word has made it into the language, with (we presume) a different meaning. According to the Oxford English Dictionary, the word is used “as an expletive on its own account, as a milder alternative to the full form of the word fuck, or else as a euphemistic report of an actual use of the full word.” More seriously, if we could give necessary and sufficient conditions for a concept to be bounded, using only the resources of pure logic (or using abstraction principles previously introduced), then we could turn the above principle (RV) into a neo-logicist abstraction principle. See our concluding §8 below. Our skepticism concerning an explicit definition for GOOD, sketched here and developed a bit below, is the main reason for the tension between neo-logicism and contemporary set theory.}

We will have to make do with a sufficient condition for GOOD. So the thesis to be developed here, finally, is that all bounded concepts are GOOD. For each sentence $\Phi$ of pure second-order logic, we want the following:

\[(\exists G (F \preceq G \& F \preceq G \& \forall H (H \preceq G \rightarrow G \preceq H))) \rightarrow \exists x (\text{EXT}(x, F)).\]

Our theory is hereby dubbed FZBB, for Frege-Zermelo-Bernays-Burgess. It is axiomatized by the schema (RF) and the aforementioned principle of extensionality:

\[(\text{RF}) \quad \text{EXT}(x, X) \& \text{EXT}(y, Y) \rightarrow (x = y \equiv \forall z (Xz = Yz)).\]

We submit (RF) gives partial expression to (4) above. But when we presuppose a principle of choice strong enough to guarantee that for any concepts $X, Y$, either $X \preceq Y$ or $Y \preceq X$, we can derive a formal counterpart of (5) as well:

\[(5) \ldots \text{any statement } \Phi \text{ that holds of them continues to hold if reinterpreted to be not about all of them but just some of them, few enough to form a set.}\]

Here is an informal argument. Let $\Phi$ be any sentence of pure second-order logic and suppose that $\Phi^F$ holds for some concept $F$. “Choose” a concept $G$, of minimal size, such that $\Phi^G$. That is, pick $G$ such that $\Phi^G$ and for all $H$, if $H \preceq G$ then $\neg \Phi^H$. It follows that $G$ is bounded. Thus, by the relevant instance of (RF), $G$ has an extension. Since this holds for any concept $F$, it holds for the “universal concept” $[x : x = x]$. So:

\[\Phi \rightarrow \exists X (\exists x (\text{EXT}(x, X) \& \Phi^X)).\]

From (RV) and the above definitions,

\[\Phi \rightarrow \exists x (\text{ext}(x) \& \Phi^x),\]
where, as usual, $\Phi^x$ is the relativization of $\Phi$ to the members of $x$. This has the form of
Burgess's reflection principle and it is close to Bernays's. But it differs in several ways.

First, and most obviously, we work in a different framework. While [10] is set against
the background of monadic second-order logic, we make free use of dyadic second-order
quantification. However, we have restricted attention to sentences of pure second-order
logic. In particular, we only allow reflection on sentences, not on open formulas. Both
Bernays and Burgess reflect on open formulas with first-order variables for the existence
axioms of ZFC and a formula with free second-order variables for replacement (see [22]).
Furthermore, we do not have a clause, or general consequence, that the given extension
is transitive. Bernays and Burgess make crucial use of this clause in the derivation, for
example, of union and power set. Finally, we make no set-theoretic assumption other than
(RV), which amounts to extensionality. In particular, unlike Bernays and Burgess, we do
not take the axiom of separation for granted. Instead, we will make do with a restricted
version of separation that will fall out of our theory.

We might add that our official object-language theory does not include the choice princ-
iples needed to make the connection to the other reflection principles. In particular, we
cannot show, in general, that if a sentence $\Phi$ holds, then there is a “minimal” $I$ such that
$\Phi^I$. Nevertheless, our theory does have some interesting consequences, to which we now
turn.

§6. Consequences. FZBB entails the following:

**Empty extension:** $\exists x (\text{ext}(x) \& \forall y (\neg y \in x))$

Let $F$ be the empty concept $[x : x \neq x]$. Then $F$ is fixed by the sentence
$\forall x (x \neq x)$ (which, of course, is vacuously satisfied by every concept). That is,
we have $\forall x (x \neq x)^F$ and $\forall G (\forall x (x \neq x)^G \rightarrow F \preceq G)$. So $F$ is bounded, and, by
(RF), has an extension, which we dub $\emptyset$.

As a bonus, we have that the universe is not empty, without invoking it in the usual way
as a stipulation on the logic or on the class of interpretations we are considering. That is,
we could have started with a free (second-order) logic.

By beginning with the concept non-self-identical, we are taking a page from Frege’s proof
of the infinity of the universe from Hume’s Principle, which has subsequently become a
staple of neo-logicism. In order to generate proofs of various axioms, we need a guarantee
that certain sentences of pure second-order are satisfiable. This satisfiability, in turn,
requires the existence of concepts of various sizes. So we have to exercise some caution and
make sure the requisite concepts exist before we take a sentence to set a lower bound on
the size of a concept.

Because we have the existence of at least one object, we have a concept under which
at least one object falls. And this observation in turn guarantees the satisfiability of the
sentence $\exists x(x = x)$, which does then set a lower bound on the size of the universe. This observation enables us to prove:

**Singletons**: $\forall x \exists y (\text{ext}(y) & \forall z(z \in y \equiv z = x))$

If $a$ is an object, let $F$ be the concept $[x : x = a]$. The requisite sentence is $\exists x(x = x)$. We have $\exists x(x = x)^F$ and $\forall G(\exists x(x = x)^G \rightarrow F \preceq G)$. So $F$ is bounded and, by (RF), has an extension $\{a\}$ whose only member is $a$.

We now have the existence of the empty extension and its singleton, which, by (RV), are distinct. It follows that $\exists x \exists y(x \neq y)$ is satisfiable and does indeed set a lower bound on the size of the universe. So we have:

**Pairs**: $\forall x \forall y \exists z (\text{ext}(z) & \forall w(w \in z \equiv (w = x \lor w = y)))$

This is similar to what we did for singletons. If $a$ and $b$ are objects, let $F$ be the concept $[x : x = a \lor x = b]$. The requisite sentence is $\exists x \exists y(x \neq y)$, and, as usual, $\{a, b\}$ is the unordered pair of $a$ and $b$.

Note that we cannot skip the proof of Singletons in our development, for without Singletons, we have no guarantee that $\exists x \exists y(x \neq y)$ is satisfiable and our proof of Pairs is thrown into jeopardy.

As usual, we define the ordered pair $\langle a, b \rangle$ of objects $a$ and $b$, to be $\{\{a\}, \{a, b\}\}$. It is straightforward that $\langle a, b \rangle = \langle c, d \rangle$ if and only if $a = c$ and $b = d$.

**Infinity**: There is a Dedekind infinite extension.

The existence of the empty extension and pairs together entail the existence of $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, etc. By (RV) these are all different. So the universe is Dedekind infinite. Recall that Boolos’s [1989] New V is in the form of (RV), where $\text{BAD}$ is explicitly characterized as “equinumerous with the universe”. It, too, entails the existence of infinitely many extensions. However, New V does not entail the existence of an infinite extension (i.e., an extension with infinitely many members). Our theory FZBB does.

Let $F$ be the minimal closure of the empty extension under the singleton operation:

$$\forall x(Fx \equiv \forall X((X\emptyset & \forall z(Xz \rightarrow X\{z\})) \rightarrow Xx)).$$

Let $\Phi$ be the following sentence, stating that the universe is Dedekind infinite:

$$\exists f(\forall x \forall y(fx = fy \rightarrow x = y) & \exists x \forall y(x \neq fy)).$$

Then we have $\Phi^F & \forall G(\Phi^G \rightarrow F \preceq G)$. So $F$ has an extension whose members are precisely $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, etc.

We do not have unrestricted versions of the other axioms of ZFC. In most cases, the reason is that we only have a sufficient condition for a concept to have an extension. That is, (RF) is that if $F$ is bounded, then $F$ has an extension. We do not have (and, we submit,
do not want) the converse of this. So we cannot derive the power set, union, separation, and replacement axioms/schemata. But restricted versions of these principles do give the theory considerable power, or at least they do if “considerable” is assessed from the perspective of someone not jaded by contemporary set theory.

**Bounded replacement:** If $F$ is bounded and $G \subseteq F$ then $G$ is has an extension.

**Bounded separation:** If $F$ is bounded and $\forall x (Gx \rightarrow Fx)$ then $G$ has an extension.

Both of these are immediate. If $F$ is bounded and $G \subseteq F$ then $G$ is bounded and, by (RV), has an extension. A fortiori, if $F$ is bounded and $\forall x (Gx \rightarrow Fx)$ then $G \subseteq F$.

An unrestricted principle of replacement would be that if $F$ has an extension and $G \subseteq F$, then $G$ has one. This, admittedly, is part and parcel of the “limitation of size” conception of set. The thought that a concept lacks an extension just in case it is ineffable seems to be of a piece with that, at least in part. As we will see, however, even if this were added to the system, the other axioms would still be restricted.

Define the *power-concept* of a $F$, written $\mathcal{P}(F)$, to be the concept of being the extension of a subconcept of $F$. That is, $\mathcal{P}(F)$ is $[x : \exists X (\text{EXT}(x, X) \& \forall x (Xx \rightarrow Fx)]$. It follows from bounded separation that if a concept $F$ is bounded, then every subconcept of $F$ is bounded and thus, by (RF), has an extension. So by (RV), every such subconcept has an extension. In the present context, the power set principle would be that if a concept is $GOOD$, then so is its power-concept. That does not follow from FZBB, even if we add a general replacement principle. However, we do have:

**Bounded power extension:** If $F$ is bounded, then the power-concept of $F$ has an extension.

*Proof sketch:* Suppose that $F \subseteq G$ and that $G$ is fixed by a sentence $\Phi$ of pure second-order logic. Then $G$ is bounded and thus $GOOD$. We show that there is a sentence $\Phi'$ that fixes the power-concept of $G$. The construction is modified from a proof of an analogue of Cantor’s theorem in pure second-order logic (see [26] 103-104), also pp. 105-6, on the generalized continuum hypothesis). Let $R$ be a binary relation and $x$ an object. Say that $x$ *represents* $Y$ in $R$ just in case for all $y$, $Rxy$ if and only if $Yy$. And say that $R$ *represents* $Y$ if there is an $x$ such that $x$ represents $Y$ in $R$. The sentence $\Phi'$ says that there is a concept $X$ such that $\Phi^X$ and there is a relation $R$ that represents every sub-concept of $X$:

$$\exists X [\Phi^X \& \exists R (\forall Y (\forall z (Yz \rightarrow Xz) \rightarrow \exists x \forall y (Rxy \equiv Yy)))]$$

By now FZBB has the resources to generate every member of $V_\omega$, and, by Bounded replacement, the existence of $V_\omega$ itself. Moreover, we have the existence of the power

---

10 See the next section, on models of our theory, for proofs of these claims.
extension of $V_\omega$, the power extension of the power extension of $V_\omega$, the power extension of that, the power extension of that, etc. And that is not the end.

So FZBB has the resources to recapture virtually all of classical mathematics, except for foundational theories like ZFC, category theory, and the like. It is thus a reasonably powerful—if awkward—theory. By bounded replacement, we also have the existence of an extension that contains $V_\omega$ and is closed under power extensions. We will later make use of this fact in order to show that FZBB has the resources to prove the existence of a model for Zermelo set theory.

We cannot do even this well with unions. Define the union-concept of a concept $F$, written $\bigcup F$, to be the concept of being a member of an extension of which $F$ holds: $\bigcup F$ is $[x : \exists y \exists X ((\text{EXT}(y, X) \& F y \& x \in y))].$ We realize that this is a mouthful. Recall that $x \in y$ only if $y$ is an extension. So the union-concept of $F$ is just $[x : \exists y (F y \& x \in y].$

An unrestricted principle of unions would be that if $F$ has an extension, then so does the union-concept of $F$. As we will see in the next section, this does not hold in full generality here, alas.\(^{11}\) The best we can do is the following:

**Doubly bounded union:** Suppose that $F$ is bounded. Suppose further that there is a bounded concept $H$ such that for every $a$ such that $Fa$, the concept $[x : x \in a]$ is smaller than or equinumerous with $H$. Then the union-concept of $F$ has an extension.

*Proof sketch:* Suppose that $F \preceq G$ and $G$ is fixed by $\Phi$, and that $H \preceq J$ and $J$ is fixed by $\Psi$. Let $K$ be the concept of being an ordered pair $\langle x, y \rangle$ such that $Gx$ and $Jy$. Now consider the sentence $\Xi$ that says that there is a concept $X$, a concept $Y$ and a three place relation $R$ such that $\Phi X$, $\Psi Y$, and for each $x$ such that $Xx$ and each $y$ such that $Yy$ there is a unique $z$ such that $Rxyz$. Then $\Xi^K$ and for all $Z$, if $\Xi Z$, then $K \preceq Z$. So $K$ is fixed by $\Xi$. But $F \preceq K$: the union-concept of $F$ is smaller than or equinumerous with $K$. So the union-concept of $F$ has an extension.

As we will soon see, for $\bigcup F$ to have an extension, it is not sufficient for $F$ to be bounded and for every $a$ such that $Fa$, the concept $[x : x \in a]$ to be bounded. In general, to conclude that the union-concept of $F$ has an extension, we need a single sentence that fixes a bound for each of those concepts. This severely hampers the theory.

Recall that Boolos’s New V entails a strong version of the axiom of choice, namely that there is a well-ordering of the universe. But this depends on the specific characterization of

\(^{11}\)[21] provides an interesting proof that a relevant form of the union axiom follows from the other axioms of [33] and [34] axiomatization of set theory. [6] shows that the result carries over, fairly directly, to his New V, which, again, is a principle in the form of (RV) in which $\text{BAD}$ is characterized as “equinumerous with the universe”. In other words, according to New V, concepts are GOOD unless they are equinumerous with the universe.
BAD as “equinumerous with the universe”\textsuperscript{12}. Here, no principle of choice is forthcoming. The best option is to add a version of the axiom of choice to the underlying higher-order logic, as a sort of general logical principle. One plausible candidate for such an axiom is the one in \textsuperscript{[16]}:

\[
\forall R(\forall x \exists y Rxy \rightarrow \exists f Rxfx).
\]

(AC)

The logical relationship between this and the more usual axiom of choice in set theory further highlights the fact that we do not have a non-trivial, necessary condition for a concept to be GOOD\textsuperscript{13}.

In effect, (AC) is a global choice principle. Suppose, for example, that \( F \) is a concept such that for every \( x \), if \( Fx \) then \( x \) is a non-empty extension and if \( Fx, Fy \) and \( x \neq y \), then \( x \) and \( y \) are disjoint. Then (AC) entails that there is a “choice concept”, a concept \( G \) such that for each \( x \) such that \( Fx \) there is exactly one \( y \) such that \( y \in x \) and \( Gy \). We would like a local choice principle which says that if, in addition to the above, \( F \) is GOOD then it has a GOOD choice concept. But, surprisingly, we do not have that. What we get from FZBB and (AC) is:

**Bounded choice:** Suppose \( F \) is a bounded concept such that for every \( x \), if \( Fx \) then \( x \) is a non-empty extension and if \( Fx, Fy \) and \( x \neq y \), then \( x \) and \( y \) are disjoint. Then there is a GOOD concept \( G \) such that for each \( x \) such that \( Fx \) there is exactly one \( y \) such that \( y \in x \) and \( Gy \).

As above, (AC) entails the existence of a choice-concept \( G \) for \( F \). But \( G \preceq F \) and so \( G \) is bounded and thus has an extension.

The final item here is the axiom of foundation. For essentially the reasons provided by \textsuperscript{[17]} FZBB, even with (AC) and replacement, does not entail that the membership relation is well-founded. This contrasts sharply with the iterative conception of set in which all sets are well-founded. The usual move, in contexts like this one, is to restrict attention to extensions generated by a certain process. One begins with the empty extension, and, at any given stage, one takes extensions of concepts under which only extensions generated thus far fall. The resulting extensions are hereditary well-founded extensions. They are

\textsuperscript{12}To show this, one defines the notion of a von Neumann ordinal, in the usual manner. The reasoning behind the Burali-Forti paradox entails that the concept of being a von Neumann ordinal must be BAD. From New V, it follows that the von Neumann ordinals are equinumerous with the universe. This amounts to a global well-ordering. In contrast, the only conclusion available here is that the concept of being a von Neumann ordinal is not bounded.

\textsuperscript{13}In several places above, we referred to strong principles of choice in the background logic that entail that, for every \( X, Y \), either \( X \preceq Y \) or \( Y \preceq X \), and that if there is an \( X \) such that \( \Phi^X \), then there is a “smallest” such \( X \). As far as we know, neither of these follows from (AC). The existence of a global well-ordering is sufficient to get these results, but we do not know if one can call that a general logical principle. See \textsuperscript{[26]}, 106-108, and the references cited there, for more on choice principles in the context of pure higher-order languages.
hereditary because they have only extensions in their transitive closure; and they are well-founded because the membership relation on them is well-founded.

We define this notion of hereditary well-founded extension, abbreviated \textit{hwf}, by invoking a technique from \cite{6}, following \cite{21}.

Define a concept \( F \) to be \textit{closed} if:

\[
\forall y \left( \exists X \left( \text{EXT}(y, X) & \forall z \in y \rightarrow Fz \right) \rightarrow Fy \right)
\]

In words, \( F \) is closed if, whenever it holds of the members of an extension, then it holds of that extension:

\[
\text{hwf}(x) \equiv \forall F \left( \text{closed}(F) \rightarrowFx \right).
\]

In words, an object is hereditarily well-founded if every closed property holds of it.

It is straightforward to show that the membership relation, restricted to \textit{hwf} extensions, is well-founded. In the context of New V, Boolos showed that \( x \) is \textit{hwf} if and only if \( x \) is an extension and every member of \( x \) is \textit{hwf}. The proof carries over to the general context here (see \cite{27} 76-78). So the empty extension is hereditary well-founded, and if \( x \) and \( y \) are \textit{hwf}, then so is their pair, \( \{x, y\} \). The von Neumann ordinals are all \textit{hwf}, and, in particular, \( \omega \) is \textit{hwf}. If \( x \) is \textit{hwf} and if the power extension of \( x \) exists, then it is \textit{hwf}, etc. So the power extension of \( \omega \), the power extension of that, the power extension of that, etc., are all \textit{hwf}. And if \( x \) is \textit{hwf} and the union of \( x \) exists, then it, too, is \textit{hwf}.

For all its faults, FZBB is a reasonably strong theory. In fact it provides us resources to construct a model of Zermelo set theory and thus proves the consistency of \( Z \).

\textbf{Proposition 1. (FZBB)} \textit{There is a model of Zermelo set theory (Z).}

\textbf{Proof Sketch.} We work in FZBB. We begin with the concept:

\[
HF = \{ x : x \text{ is a hereditarily finite hwf} \}.
\]

Being countable, \( HF \) is bounded, and, by (RF), has an extension, which we call \( h \), i.e., \( \text{EXT}(h, HF) \). Repeated applications of bounded power extension yields the existence of every extension in the sequence \( \mathcal{P}h, \mathcal{P}\mathcal{P}h, \mathcal{P}\mathcal{P}\mathcal{P}h \), etc. Now: let \( F \) be the minimal closure of \( h \) under the power extension operation:

\[
\forall x (Fx \equiv \forall X ((Xh \& \forall z (Xz \rightarrow X\mathcal{P}z)) \rightarrow Xx))
\]

We are interested in the union-concept of \( F \), which is \( \bigcup F = \{ x : \exists y (Fy \& x \in y) \} \). In the absence of unrestricted union, we cannot just assume that if \( F \) has an extension, so does

\footnote{\cite{6} uses the term “pure” for what we have called “hereditary well-founded”. As \cite{17} note, this can be misleading. Our theory has models in which there are extensions \( e \) such that membership on \( e \) is not well-founded, but the members of \( e \), the members of the members of \( e \), etc., are all extensions. Indeed, there can be an extension \( e \) such that \( e = \{e\} \).}
∪ \mathcal{F}. However, we have independent reason to think it does, namely, \( \bigcup \mathcal{F} \) is bounded by a pure second-order sentence of the form:

\[
\neg \text{COUNT}([x : x = x]) \land \forall X((X \prec [x : x = x] \rightarrow \mathcal{P}X \prec [x : x = x])
\]

where \( \neg \text{COUNT}([x : x = x]) \) abbreviates a pure second-order sentence stating that the universe is uncountable and \( \forall X((X \prec [x : x = x] \rightarrow \mathcal{P}X \prec [x : x = x]) \) abbreviates another pure second-order sentence stating that the universe is strictly larger than the power-concept of any smaller subconcept. Being bounded, \( \bigcup \mathcal{F} \) has an extension, by (RF), which we call \( \mathcal{M} \). By construction, \( \mathcal{M} \) contains the empty extension and the extension of the concept finite \( \text{hwf} \) and is closed under pairs, union and power set. Furthermore, when restricted to \( \mathcal{M} \), the membership relation satisfies the axioms of extensionality, separation and, since every member of \( \mathcal{M} \) is, by construction, \( \text{hwf} \), foundation. □

Moreover, we have a model of Zermelo set theory plus choice when we assume (AC). Since \( \mathcal{M} \) is the extension of a bounded concept, so is any subconcept of it, and, by bounded choice, we have that any subconcept of the domain has a choice concept:

**Corollary 2.** (FZBB + AC) There is a model of Zermelo set theory plus choice (ZC).

Let us summarize the results of this section. We begin with the widely held view that the universe of set theory is ineffable, but all we take from it that is that the extent of the universe cannot be described with the resources of pure second-order logic. Our theory FZBB, axiomatized by (RV) and the instances of the reflection scheme (RF), plus the straightforward definitions, entail the usual axioms of the empty extension, pairs, infinity, bounded replacement, and bounded separation. We also have bounded power extension and doubly bounded union (whether or not unrestricted replacement is added). If we add (AC), then bounded choice follows. Finally, the axiom of foundation holds if the quantifiers are restricted to hereditarily well-founded extensions. The \( \text{hwf} \) extensions make for a substantial universe, one sufficient to recapture almost all of contemporary mathematics. Indeed, FZBB allows us to explicitly construct such a universe, which itself is a model of Zermelo set theory. And we have a model of Zermelo set theory plus choice if we assume global choice in the theory of the object-language. All in all, it is a fairly powerful set theory, at least by non-set-theoretic standards.

The time has come to briefly compare FZBB with the Fregeanized variant of Bernays-Boolos set theory developed in Burgess [2005]. Fregeanized Bernays set theory (FB) is formulated in a monadic second-order language whose primitive vocabulary contains each of the symbols \( \text{EXT}, \text{ext} \) and \( \in \). In addition to (RV), FB includes two subordination axioms that govern the predicates \( \text{ext} \) and \( \in \):

\[
\text{ext}(x) \equiv \exists X(\text{EXT}(x, X)) \\
x \in y \equiv \exists X(\text{EXT}(y, X) \land Xx).
\]
There is finally an axiom of separation and a simple reflection principle:

\[(\text{SEP}) \quad \forall x (Xx \rightarrow Yx) \rightarrow (\exists y (\text{EXT}(y, Y) \rightarrow \exists x (\text{EXT}(x, X))), \quad \Phi \rightarrow \exists x (\text{ext}(x) \& \Phi^x).\]

Note, however, that the reflection principle is not restricted to sentences of pure second-order logic but rather includes open formulas of the language of FB. With the help of the subordination axioms, Burgess derives a version of Bernays’s reflection principle:

\[\Phi \rightarrow \exists x (\text{ext}(x) \& \text{trans}(x) \& \Phi^x).\]

where \(\text{trans}(x)\), as usual, is the statement that \(x\) is transitive: \(\forall y \forall z ((y \in x \& z \in y) \rightarrow z \in y)\). All the existence axioms of second-order ZF as well as inaccessibles, hyperinaccessibles... are derivable in FB. So, FB is much stronger than FZBB.

We cannot even hope to match the strength of FB by adding the choice principles required to derive a restricted form of Burgess’s reflection principle from (RF). The most we can hope for are instances of the principle:

\[\Phi \rightarrow \exists x (\text{ext}(x) \& \Phi^x).\]

where \(\Phi\) is a sentence of pure dyadic second-order logic. But even then, by treating the subordination axioms of FB as explicit definitions of \(\text{ext}\) and \(\in\) in terms of \(\text{EXT}\), we deprive ourselves of Burgess’s derivation of Bernays’s principle. Since the clause that \(x\) is transitive plays a crucial role in the derivation of union and power set FB, we have every reason to think they will not be derivable in the weaker system. Unfortunately, the mere fact that we can substitute \(\Phi\) with a sentence involving quantification over dyadic second-order predicates does not help very much, as we will see in the next section.

Call \(\text{FB}^-\) the second-order theory of extensions that results from FB by (i) treating the subordination axioms as explicit definitions of \(\text{ext}\) and \(\in\) in terms of \(\text{EXT}\) and (ii) restricting attention to instances of reflection in which \(\Phi\) is substituted with a sentence of pure dyadic second-order logic. We will see that \(\text{FB}^-\) is considerably weaker than FB (which allows only quantification over monadic predicates) by showing that there are models of this theory in which both unrelativized forms of union and power set fail. These independence results cast some light upon the work done by the choice of primitives in FB itself. But all will come in due course.

Let us now consider the models of FZBB.

§7. Models, and consistency. As noted above, the meta-theory here is ordinary first-order ZFC. The only primitive, non-logical term in the object language is \(\text{EXT}\), which is a higher-order predicate, which relates objects to concepts. So we look at structures of the form \(\mathcal{M} = \langle d, E \rangle\), where \(d\) is a set, \(E\) is a set of ordered pairs \(\langle b, a \rangle\), where \(b \in d\) and \(a \subseteq d\).
The idea is that if \( \langle b, a \rangle \in E \), then, in \( M \), \( b \) is the extension of the “concept” \( a \). Define \( GD^M \) to be the set of concepts of \( M \) that have extensions and define \( ext^M \) to be the set of extensions of \( M \):

\[
GD^M = \{ a : \exists b \in d \ (b, a) \in E \} \\
ext^M = \{ b : \exists a \subseteq d \ (b, a) \in E \}
\]

We noted above, several times, that our theory FZBB gives only a sufficient condition for a concept to have an extension. Our first meta-theorem is that we can take any subset of the domain of a model and make it \( GOOD \).

**Proposition 3.** Let \( M = \langle d, E \rangle \) be a model of FZBB, and suppose that \( a \subseteq d \). Then there is a model \( M' = \langle d, E' \rangle \) of FZBB, in which \( GD^{M'} \) is \( GD^M \cup \{ a \} \).

**Proof Sketch.** If \( a \in GD^M \), then there is nothing to prove; \( M' \) just is \( M \). So assume that \( a \) is not in \( GD^M \). Suppose, first, that there is an object \( m \in d \) that is not in \( ext^M \). That is, according to \( M \), \( m \) is not the extension of any concept. Then let \( E' = E \cup \{ \langle m, a \rangle \} \). That is, we just make \( m \) the extension of the “new” \( GOOD \) concept \( a \). It is straightforward to verify that \( M' \) satisfies (RF) and (RV). Now suppose that \( ext^M \), the collection of the extensions of \( M \), is all of \( d \). As above, every model of FZBB is infinite. So let \( f \) be a one-to-one function from \( d \) to a proper subset of \( d \). Consider the structure \( N = \langle d, E_1 \rangle \), where \( E_1 = \{ \langle fx, y \rangle : \langle x, y \rangle \in E \} \). In words, we use \( f \) to change the extensions, without changing which concepts have extensions. It is straightforward to verify that \( N \) is a model of FZBB, with the same domain as \( M \) and \( GD^M = GD^N \). But since \( ext^N \) is not all of \( d \), we can proceed as above. \( \square \)

To illustrate, let \( M = \langle d, E \rangle \) be any model of FZBB. Define \( m \) to be a “Russell-object” of \( M \) if there is a subset \( a \) of \( d \) such that \( \langle m, d \rangle \in E \) and \( m \notin d \). Let \( r \) be the set of Russell-objects of \( M \). Then \( r \) cannot be in \( GD^M \), for the usual reasons. But, by Proposition 3, there is a model \( M' = \langle d, E' \rangle \) of FZBB, in which \( r \in GD^{M'} \). That is, in \( M' \), \( r \) has an extension. However, \( r \) is not the set of Russell-objects of \( M' \). Suppose that, in \( M' \), \( n \) is the extension of \( r \) (i.e., \( \langle n, r \rangle \in E' \)). If \( n \in r \), then \( r \) contains a non-Russell-object (of \( M' \)), namely \( n \), and if \( n \notin r \), then \( r \) fails to contain a Russell-object, namely \( n \).

In any model of FZBB, most of the subsets of the domain lack extensions, but for any such domain, there is no subset that must lack an extension (so to speak). In effect, this is why the unrestricted versions of the separation, replacement, power set, union, and choice axioms fail. Consider, for example, a model \( M \) in which the entire domain has an extension (in that model). Then replacement and separation fail in \( M \). Again, in every model, there is a subset of the domain such that that subset and its choice set both lack extensions. It follows from Proposition 3 that there is a model in which the given set has an extension and its choice set lacks one. So local choice fails—even if global choice holds. This gives us
a reason to add another axiom, which is part and parcel of the limitation of size conception of set theory:

(REP) \[ \exists x (\text{EXT}(x, X) \& Y \preceq X) \rightarrow \exists y (\text{EXT}(y, Y)) \]

In effect (REP) just is unrestricted replacement. Unrestricted separation follows, and, FZBB, (REP), and our global choice principle (AC) does entail local choice. But, as we will see, we still do not have unrestricted power extension and unrestricted union, or even what we may call “bounded union”.\(^\text{15}\)

We now show how to construct models of the theory axiomatized by FZBB, (AC), (REP), and foundation. They will all be in the form \( M = \langle d, E \rangle \) in which the domain \( d \) is a transitive set and, for each GOOD concept \( b \), the extension of \( b \) is \( b \) itself. That is, if \( \langle b, a \rangle \in E \) then \( b = a \). It follows, of course, that all of the GOOD concepts of the model are members of the domain: if \( b \in GD^M \) then \( b \in d \). Recall that we introduced a symbol for membership, in the object language, as an abbreviation:

\[ x \in y \equiv df \exists X (\text{EXT}(y, X) \& Xx) \]

It follows, happily, that in the models we construct below, this object language membership relation coincides with the membership relation in the meta-theory, at least on the objects for which the former is defined.

In line with (REP), we will look at models in which a subset of the domain is assigned an extension just in case it is smaller than a certain, fixed cardinality. To specify one of our models, then, we give its domain \( d \) and a cardinal number \( \lambda \). Then \( GD^M \) will be the set of all subsets of \( d \) that are smaller than \( \lambda \): \( \{ x \in d : |x| < \lambda \} \), and \( E \) will be \( \{ \langle x, x \rangle : x \in GD^M \} \). These correspond to what [17] call \((\kappa, \lambda)\) models (of the relevant version of (RV)).

For convenience, introduce a “dummy” symbol \( \Omega \) for the concept of being a cardinal. If \( \alpha \) is a cardinal, then just read \( \alpha \in \Omega \) and \( \alpha < \Omega \) as short for \( \alpha = \alpha \). Let \( \kappa \) be any cardinal or \( \Omega \). For each sentence \( \Phi \) of our pure second-order language, let \( f(\Phi) \) be the smallest cardinal \( \delta < \kappa \) such that \( \Phi^\delta \) is true, if there is such a cardinal \( \delta \); otherwise, let \( f(\Phi) = 0 \). Thus

\[ f(\Phi) = \begin{cases} \delta & \text{if } \Phi^\delta \text{ and for every } \gamma < \delta, \neg \Phi^\gamma \\ 0 & \text{if no such } \delta \text{ exists.} \end{cases} \]

In effect, \( f(\Phi) \) is the cardinality of the smallest model of \( \Phi \) that is itself smaller than \( \kappa \), if there is such a model. Define the \( \kappa \)-limit, \( l_\kappa \), to be the union of the set of all \( f(\Phi) \). So if a sentence \( \Phi \) is satisfiable on a set smaller than \( \kappa \), then it is satisfiable on a set smaller than the \( \kappa \)-limit.

\(^{15}\)Another possible axiom would be that a given concept \( X \) is \textit{GOOD} unless it is too big to be. That is, \( X \) is \textit{GOOD} if the sub-concepts of \( X \) can be mapped, one to one, onto the universe. This can be formulated in the language of FZBB, but it does not yield power extension and union.
The \( \Omega \)-limit is called the “Löwenheim number” for the pure second-order language (see [26], 147-157, and the references cited there). It is rather large, even by set-theoretic standards.\(^{16}\) We deal with the general case of \( \kappa \)-limits here in order to show that FZBB has some (relatively) small models.

Notice that since there are only countably many sentences in the language, if the cofinality of \( \kappa \) is uncountable, then \( l_\kappa < \kappa \). The \( \kappa \)-limits of strong inaccessibles, and \( \Omega \), give us a handle on which concepts can be GOOD.

**Proposition 4.** (ZFC). Let \( \kappa \) be any strong inaccessible or \( \Omega \), and let \( \lambda \) be any cardinal such that \( l_\kappa \leq \lambda < \kappa \). Then there is a standard model \( M = \langle d, E \rangle \) of FZBB, (AC), foundation, and (REP), such that a subset \( a \subseteq d \) is GOOD (i.e., \( a \in GD^M \)) if and only if \( |a| < \lambda \).

**Proof Sketch.** We need a set \( d \) such that for any \( a \subseteq d \), if \( |a| < \lambda \), then \( a \in d \). In other words, \( d \) contains all of its subsets that are smaller than \( \lambda \). [17] (see section 4), provide a technique for finding the least such \( d \). If \( \lambda \) is an infinite cardinal, let \( \lambda^* \) be the least cardinal of cofinality \( \geq \lambda \). Thus

\[
\lambda^* = \begin{cases} 
\lambda & \text{if } \lambda \text{ is regular} \\
\lambda^+ & \text{if } \lambda \text{ is singular.}
\end{cases}
\]

Define a sequence of sets, by transfinite recursion, as follows:

\[
\begin{align*}
d_0 &= \lambda \\
d_{\alpha+1} &= d_\alpha \cup \{x : x \subseteq d_\alpha \text{ and } |x| < \lambda\} \\
d_\lambda &= \bigcup_{\alpha < \lambda} d_\alpha, \text{ for } \lambda \text{ a limit ordinal.}
\end{align*}
\]

If \( a \subseteq d \) and \( |a| < \lambda \), then, since \( \lambda^* \) is regular, there is an ordinal \( \beta < \lambda^* \) such that \( a \subseteq d_\beta \). So \( a \in d_{\beta+1} \) and thus \( a \in d \). By the construction \( |d| < \kappa \), since \( \kappa \) is a strong inaccessible (or \( \Omega \)). Notice, incidentally, that \( \lambda^* \) is a fixed point for the construction, in that for any ordinal \( \alpha \), if \( \lambda^* \leq \alpha \) then \( d_\alpha = d_{\lambda^*} = d \).

The structure \( M \) we are looking for is \( \langle d, E \rangle \), where \( E \) is \( \{ \langle x, x \rangle : x \subseteq d \text{ and } |x| < \lambda \} \). As noted above, the defined membership relation, on the extensions of \( M \), coincides with the membership relation of the background meta-theory. So \( M \) satisfies foundation.

\(^{16}\)A property \( P(x) \) of sets is said to be “local” if there is a formula \( \Psi(x) \) in the language of first-order set theory, such that for each \( x \), \( P(x) \) holds if and only if \( \exists \delta(V_\delta \models \Psi(x)) \) (where \( V_\delta \) is the \( \delta \)th rank). The idea is that local properties are those with a characterization that only refers to the sets below a fixed rank. One does not need to refer to “arbitrarily large” sets in order to state whether a given set has the property. Inaccessible, Mahlo, hyper-Mahlo, and measurable are all local properties. Define a cardinal \( \lambda \) to be “minimal-local” if there is a local property \( P(x) \) such that \( \lambda \) is the smallest cardinal with property \( P(x) \). The \( \Omega \)-limit is the union of all minimal-local cardinals. So, for example, if there is a measurable cardinal, then the \( \Omega \)-limit is greater than the smallest one.
The satisfaction of (RV), (REP), and (AC) are immediate. That leaves only:

\[(RF) \quad (\exists G(F \leq G & \Phi^G & \forall H(\Phi^H \rightarrow G \leq H))) \rightarrow \exists x(\text{EXT}(x, F)).\]

So let \( \Phi \) be a sentence in the language of pure second-order logic. Suppose that there is a set \( F \subseteq d \) such that \( \mathcal{M} \) satisfies \( (\exists G(F \leq G & \Phi^G & \forall H(\Phi^H \rightarrow G \leq H))) \). We have to show that \( F \) is \( \text{GOOD} \), which amounts to \( |F| < \lambda \). There is a subset \( G \) of \( d \) such that \( F \leq G \), \( \mathcal{M} \) satisfies \( \Phi^G \), and \( \mathcal{M} \) satisfies \( \forall H(\Phi^H \rightarrow G \leq H)) \). Because the quantifiers in \( \Phi^G \) are all restricted, it is absolute in the structure. Since \( \mathcal{M} \) is standard, so is the “\( \leq \)” relation. So \( \Phi^G \) is true, and for all subsets \( H \subseteq d \), if \( \Phi^H \) then \( G \leq H \). Since \( |d| < \kappa \), there is a cardinal \( \delta < \kappa \) such that \( \Phi^\delta \) is true. Recall that \( f(\Phi) \) is the smallest such cardinal. Let \( X \) be any subset of \( d \) of cardinality \( f(\Phi) \). Then \( \Phi^X \) is true, and so \( \mathcal{M} \) satisfies \( \Phi^X \). So \( G \leq X \) (indeed, \( G \sim X \)). Since \( F \leq G \), we have that \( F \leq X \), and so \( |F| \leq f(\Phi) \). But \( f(\Phi) < l_\kappa \) and \( l_\kappa \leq \lambda \). So \( |F| < \lambda \), and so, in \( \mathcal{M} \), \( F \) is \( \text{GOOD} \). That is, \( \mathcal{M} \) satisfies \( \text{EXT}(F,F) \). \( \square \)

A fortiori, ZFC entails that our theory FZBB, together with (AC), foundation, and replacement (REP), is consistent. Two spinoffs of Proposition 4 are that FZBB (together with (AC), foundation, and (REP)) does not entail the unrestricted union and power extensions principles:

**Corollary 5.** (ZFC) There is a model \( \mathcal{M}_1 = (d_1, E_1) \) of FZBB, (AC), foundation, and (REP), and a subset \( a \subseteq d_1 \) such that, in \( \mathcal{M}_1 \), \( a \) has an extension, but the union-concept of \( a \) does not.

**Proof Sketch.** Let \( \kappa \) be any strong inaccessible cardinal (or \( \Omega \)), and let \( \lambda \) be the \( \kappa \)-limit \( l_\kappa \). From Proposition 4, there is a model \( \mathcal{M}_1 = (d_1, E_1) \) of FZBB, (AC), foundation, and (REP), such that a subset \( a \subseteq d_1 \) is \( \text{GOOD} \) if and only if \( |a| < \lambda \). It is straightforward to verify that \( \lambda \) has cofinality \( \omega \) (see [26] Theorem 6.17, p. 150). So there is a countable set of cardinals \( \{\lambda_1, \lambda_2, \ldots\} \) such that for each \( i \in \omega \), \( \lambda_i < \lambda \), and \( \{\lambda_i : i \in \omega\} = \lambda \). For each \( i \in \omega \), let \( a_i \) be a subset of \( d \) such that \( |a_i| = \lambda_i \). So we have that, in \( \mathcal{M} \), each \( a_i \) is \( \text{GOOD} \), and it is its own extension. Let \( a = \{a_i : i \in \omega\} \). Then \( a \) is countable, and thus has an extension in \( \mathcal{M} \). But the union-concept of \( a \) has cardinality \( \lambda \), and so lacks one. \( \square \)

We have just shown that not even countable union follows from our theory. Notice also that every member of the defined \( a \) is itself bounded. So the principle of double-bounded union, established in the previous section, is the best we can do. It is a consequence of Proposition 5 of [17] that a model constructed in Proposition 4 satisfies the union axiom if and only if the indicated cardinal \( \lambda \) is regular.

Since our model is also a model of the weakening of Fregeanized Bernays set theory we have called \( \text{FB}^- \), we have another corollary:
Corollary 6. (ZFC) There is a model $\mathcal{M}_1 = \langle d_1, E_1 \rangle$ of $\text{FB}^-$ and a subset $a \subseteq d_1$ such that, in $\mathcal{M}_1$, $a$ has an extension, but the union-concept of $a$ does not.

What is remarkable is the fact that we can rule such a model out merely by expanding our primitive vocabulary to include $\text{ext}$ and $\in$ and conceiving of our explicit definitions of them in terms of $\text{EXT}$ as axioms. This, it seems to us, gives us a sense of how much work the choice of primitives can make in the development of a theory of extensions based on reflection.

Corollary 7. (ZFC) There is a model $\mathcal{M}_2 = \langle d_2, E_2 \rangle$ of $\text{FZBB}$, (AC), foundation, and (REP), and a subset $a \subseteq d_2$ such that, in $\mathcal{M}_2$, $a$ has an extension, but the power-concept of $a$ does not.

Proof Sketch. Let $\kappa$ be any strong inaccessible cardinal (or $\Omega$), and let $\eta$ be any cardinal smaller than $\kappa$ but greater than or equal to the $\kappa$-limit $l_\kappa$. Let $\lambda$ be $\eta^+$, the smallest cardinal greater than $\eta$. So $\lambda < \kappa$. From Proposition 4, there is a model $\mathcal{M}_2 = \langle d_2, E_2 \rangle$ of $\text{FZBB}$, (AC), foundation, and (REP), such that a subset $a \subseteq d_2$ is $\text{GOOD}$ if and only if $|a| < \lambda$. Let $a$ be any subset of $d$ whose cardinality is $\eta$. Then, in $\mathcal{M}_2$, $a$ is $\text{GOOD}$, and is its own extension. By Cantor’s theorem, the cardinality of the power-concept of $a$ is at least $\lambda$, and so, in $\mathcal{M}_2$, this power-concept is $\text{BAD}$. □

A cardinal $\lambda$ is a strong-limit if $\lambda$ is a limit cardinal and if, for every cardinal $\eta < \lambda$, the power set of $\lambda$ is smaller than $\lambda$. It follows from Proposition 5 of [17] that a model constructed in Proposition 3 satisfies the power set axiom if and only if the indicated cardinal $\lambda$ is a strong limit.

It follows from our last corollary that $\text{FB}^-$ cannot prove the unrestricted version of the power set axiom:

Corollary 8. (ZFC) There is a model $\mathcal{M}_2 = \langle d_2, E_2 \rangle$ of $\text{FB}^-$ and a subset $a \subseteq d_2$ such that, in $\mathcal{M}_2$, $a$ has an extension, but the power-concept of $a$ does not.

So, $\text{FB}^-$ is a relatively weak theory. Note that we cannot blame this only on the restriction to pure second-order sentences. Many of the constructions given above carry over to the case where we assume the language to contain a primitive non-logical symbol, since, after all, the crucial observation we have been exploiting is that there are at most countably many sentences to set bounds on our concepts.

At all events, recall that a cardinal is strongly inaccessible if and only if it is regular and a strong limit. So a model in the form of the conclusion of Proposition 4 satisfies the axioms of ZFC if and only if the indicated cardinal $\lambda$ is a strong inaccessible (in which case, of course, the only relevant “$\kappa$” is $\Omega$). In light of the conditions for the proposition, $\lambda$ would also have to be larger than the $\Omega$-limit—the Löwenheim number—for pure second-order logic. It is straightforward to verify that every standard model for $\text{FZBB} + \text{ZFC}$ is a strong inaccessible that is greater than the Löwenheim number.
§8. Whereof one cannot speak. We have noted, early and often, that FZBB only gives a sufficient condition for a concept to have an extension. This bug, or feature, is attenuated a bit by the replacement principle (REP). Still, in the previous section, we saw that it is pretty easy to come up with models of FZBB and (REP) (plus foundation and (AC)). Here we briefly explore attempts to turn our sufficient condition into a definition.

Recall that a concept \( F \) is fixed by a sentence \( \Phi \) just in case \( \Phi^F \) and for every concept \( X \), if \( \Phi^X \) then \( F \preceq X \). And \( F \) is bounded if there is a sentence \( \Phi \) of pure second-order logic and a concept \( G \) such that \( F \preceq G \) and \( G \) is fixed by \( \Phi \).

Suppose that there were a formula \( \Psi(X) \) in the pure second-order language such that for each concept \( F \), \( \Psi(F) \) if and only if \( F \) is bounded. Then there would be a sentence that fixes the smallest unbounded concept, the Löwenheim number for the pure second-order language. But if there were a concept that large, then it, too, would be bounded, which is impossible.

This is only to be expected. We are working around the inchoate thought that the universe is ineffable, and we took that to be something like “not characterized by logical resources alone”. So we cannot expect to be able to describe the universe using those same logical resources. To be sure, the notion of “bounded” is definable in ordinary set theory. That is why we were able to characterize the models of the theory in the previous section, within ZFC. But, of course, we cannot just assume ZFC in trying to set up our set theory.

One option is to ascend to a third-order language. Let \( E \) be a variable ranging over concepts of concepts. There is a formula \( DEF(E) \) of “pure” third-order logic that “says” that there is a sentence \( \Phi \) in the pure second-order language such that for every \( X \),

\[ \mathcal{E}X \text{ if and only if } \Phi^X. \]

One can construct \( DEF \) by mimicking a Tarskian explicit definition of truth for the second-order fragment of the language. We have that a concept \( F \) is fixed if and only if:

\[ \exists \mathcal{E}(DEF(\mathcal{E}) \& \mathcal{E}F \& \forall Y (\mathcal{E}Y \rightarrow F \preceq Y)), \]

and \( F \) is bounded if and only if

\[ \exists X (F \preceq X \& \exists \mathcal{E}(DEF(\mathcal{E}) \& \mathcal{E}F \& \forall Y (\mathcal{E}Y \rightarrow F \preceq Y)). \]

Call this last formula \( BD(F) \). Now we do have the resources to say that a concept has an extension if and only if it is bounded:

\[ (RF+) \quad \forall X \exists x (\text{EXT}(x, X)) \equiv BD(X) \]
Let FZBB+ be the theory whose axioms are (RF+) and (RV).\footnote{Suppose we introduce a term \( \tilde{x}Fx \), which represents a function from concepts to objects. The idea is that if \( X \) is GOOD, then \( \tilde{x}Xx \) is its extension. Our two axioms (RF+) and (RV) can then be combined:

\[ \forall X \forall Y (BD(X) \& BD(Y) \rightarrow (\tilde{x}Xx = \tilde{y}Yy \equiv \forall z (Xz = Yz))) \]

Since this last has the form of a (restricted) abstraction principle, it is a candidate for Scottish neo-logicism.}

Since each instance of (RF) is a consequence of FZBB+, the results from previous sections carry over, and some can be sharpened. The theory FZBB+ entails the principles of null extensions, singletons, pairs, infinity, and replacement. As usual, the local axiom of choice follows from AC, and foundation holds on the hereditarily well-founded extensions. Since every GOOD concept is bounded, we also have an unrestricted power extension principle: if a concept has an extension, then so does its power-concept. So FZBB+ entails all of the axioms of ZFC, with the single exception of union.

The omission of union from the consequences of FZBB+ is glaring, and this defect cannot be easily remedied. For each standard model of FZBB+, foundation, and (AC), there is a strong inaccessible \( \kappa \) (or \( \Omega \)) such that a subset \( a \) of the domain has an extension if and only if \( |a| < l_\kappa \), where \( l_\kappa \) is the \( \kappa \)-limit (as defined in the previous section). Since the \( \kappa \)-limits all have cofinality \( \omega \), there is no standard model of FZBB+, foundation, and (AC) which satisfies the (countable) union principle. Too bad.

The real problem is that we have contravened the inchoate intuition that the universe of extensions is ineffable. Since the background language, earlier in the article, is second-order, we glossed the inchoate intuition as something like “not bounded by a sentence of the second-order language”. Here, we manage to say exactly which concepts are not bounded by a sentence of the pure second-order language, by moving to a third-order language. But the inchoate thought that the universe is ineffable should also entail that it cannot be bounded by a sentence of the pure third-order language either. We can approximate that with a schema in the third-order language. In fact, just about all of the entire development above, including the meta-theory, can be recapitulated by replacing phrases like pure second-order with pure third-order. Let (RF3) be the extension (RF) to include the sentences from the pure third-order language, and let FZBB3 be (RF3) plus (RV).

But now we are back to having only a sufficient condition for a concept to be GOOD. That is because the third-order language cannot characterize which concepts are bounded by sentences of that language. We could remedy that with an explicit definition in a fourth-order language. But once we do that, the inchoate thought suggests that the universe is not bounded by a sentence in the fourth-order language.

And on it goes—into the transfinite if the good reader so desires. Notice also that ZFC entails the consistency of each of the theories FZBB3, FZBB4, \ldots \), right up to the limits of what can be expressed in that language and theory.
We have here, it seems, an instance of the phenomenon that gets reflection going in the first place—and keeps it going. Recall how [35] 555, put it:

Any time we try to capture the universe from what we positively possess (or can express) we fail the task and the characterization is satisfied by certain (large) sets.

This applies especially here. Every time we add new expressive resources, we push the sky up further: we can establish the existence of extensions larger than any we could envision before. But we cannot think that we have it all. The very act of thinking about what we have—so far—gives us more than we think.\textsuperscript{18}

In an insightful article on Cantor and the historical and philosophical foundation of set theory, [31] §4, writes in a similar spirit:

Under what conditions should we admit the extension of a property of transfinite numbers to be a set—or equivalently, what transfinite numbers are there? No answer is final, in the sense that, given any criterion for what counts as a set of numbers, we can relativize the definition of $\Omega$ [i.e., transfinite number] to sets satisfying that criterion and obtain a class $\Omega'$ of numbers. But there would be no grounds for denying that $\Omega'$ is a set: the preceding argument that $\Omega$ is not a set merely transforms in the case of $\Omega'$ into a proof that $\Omega'$ does not satisfy the criterion in question. So . . . we can go on. In the foundations of set theory, Plato’s dialectician, searching for the first principles, will never go out of business.

\textbf{Acknowledgements.} We are grateful to audiences at the final Abstraction Workshop at Arché (Status Belli) and at the \textit{Mathematical Methods in Philosophy} in Banff, where earlier versions of this paper were presented. Thanks to John Burgess, Øystein Linnebo, Marcus Rossberg and Crispin Wright for helpful comments and discussion.

\textbf{REFERENCES}


\textsuperscript{18}This is the reason we find tension between Scottish neo-logicism and the inchoate thought that the universe of sets is ineffable. Suppose that there were an abstraction principle that give necessary and sufficient conditions for a a concept to be \textit{GOOD}, using only the resources of logic or previously defined logical abstractions. Then we could define a “bound” for the universe of extensions, using those resources. It would be the smallest concept that is not \textit{GOOD}.\textsuperscript{18}


THE OHIO STATE UNIVERSITY
COLUMBUS, OH 43210, USA
and
UNIVERSITY OF ST ANDREWS
ARCHÉ
E-mail: shapiro.4@osu.edu

UNIVERSITY OF OXFORD
PEMBROKE COLLEGE
OXFORD OX1 1DW, UK
E-mail: gabriel.uzquiano@philosophy.ox.ac.uk