Review of Leonard Bolc and Piotr Borowik: *Many-valued Logics: 1. Theoretical Foundations*, Berlin: Springer, 1991

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In recent years, interest in many-valued logics for computer science has increased considerably, and there is certainly a need for good and thorough textbooks on the subject geared towards that potential area of application, namely, computer science. The authors of the volume *Many-valued Logics*[1], Leonard Bolc and Piotr Borowik, have set out to do just that. The volume under review here is the first volume of a two-volume set. Volume 1 is devoted to the theoretical foundations of many-valued logics, the second volume to follow shall be devoted to applications of many-valued logics in computer science, in particular automated deduction in many-valued logics. A review of the first volume should of course take into account the second volume, so the present review can only be partial and tentative.

So what should be expected of a volume on the theoretical foundations of many-valued logic? Certainly it should deal with the basics of many-valued logics (matrices and axiomatizations), survey (at least the most important) many-valued systems considered in the literature, and present the most important mathematical results. The major textbook available on the subject so far, Rescher's *Many-valued Logic* [1969] lacks in the third aspect, and also does not place any emphasis on computer science related aspects. Another, more recent, book by Gottwald [1989] is an excellent work on the foundations of many-valued logics; it is unfortunately only available in German.

Chapter 1 ("Preliminaries") introduces and makes precise the formal notions used in the book. In the last two pages of this chapter the central concept of many-valued logic is hidden: the definition of a logical matrix. These two pages also contain the statement of Lindenbaum's theorem that every propositional logic closed under substitution can be characterized by an infinite matrix, and the product construction for matrices, attributed to Jaśkowski (both without proof).

Chapter 2 ("Many-valued Propositional Calculi") then takes up the topic of calculi for many-valued logic. Seemingly, the term "calculus" is used in a different sense as usual, namely, not only denoting an axiomatic system, but as a synonym for "logical system." The authors use what they call the algebraic approach to defining a propositional calculus: A propositional calculus is a quadruple $\mathcal{X} = (M, L, v, Pr)$, where M is a minimal finite-valued matrix, L

[1] Hereafter referred to as B.B.

is the language (given by a set of propositional variables V and the set of well-formed formulas S), $v: V \to \{1, \ldots, n\}$ is a valuation of the propositional variables, and *Pr* is a consequence relation. The consequence relation has to satisfy the usual (Tarskian) conditions. Furthermore, a calculus is said to be *axiomatically given* iff *Pr* is given by a set of axioms and rules. The axioms are required to be finite or recursively countable (which can mean recursive or recursively enumerable); the rules are $m+1$-ary relations over S (m premises, one conclusion) which are, curiously, not required to satisfy any conditions. *Pr* then is the usual proof relation based on this axiom system. The definition given here is different from the definition of a calculus given in chapter 1 (p. 20f)

This definition of a calculus evidently has slight problems. For one, the requirement that there is a finite adequate matrix precludes logics which are not finite-valued (e.g., intuitionistic logic) from having a calculus in the sense of the definition. It is also not entirely clear what the relation between the consequence relation *Pr*, and the matrix M actually is. For instance, there is no requirement that M and *Pr* are compatible. The authors also claim (p. 24) that this definition of calculus is more fundamental than the axiomatic approach, be it in the framework of Hilbert-style axiom systems, Gentzen's sequent calculus or natural deduction, or Beth's tableaux (finitely generated trees), and argue that these formalisms all require completeness proofs, while the algebraic definition of a calculus does not. Indeed, if a calculus is *defined* to be an (axiomatically given) consequence relation which is compatible (i.e., sound and complete) for a finite matrix[2], then the completeness proof is only changed from a result relating syntax and semantics to a proof that something actually *is* a calculus according to the definition. The authors would have been well advised to keep, as is usual, syntax (the axiomatic system) and semantics (the matrix, and maybe the consequence relation) apart. This would (maybe) have made algebraic results about the logic less elegant. This is, however, a book with the intended audience of computer scientists, and not algebraists. It would also have made clear the fundamental problem faced with in foundations of many-valued logic (as in modern symbolic logic in general) of *finding* a relation between syntax and semantics, and in particular the problem of axiomatization of many-valued logics. At this point it seems as if, from the point of view of the authors, the axiomatic system comes before the semantics. Historically, and in fact in practice, it is the other way around (with many-valued logics at least): first, we have the matrix, then we look for an axiom system for it.

In subsection 2.3, the finite-valued calculi for Łukasiewicz logic are introduced. The following subsection 2.4, although also entitled "Finitely Valued Calculi of Łukasiewicz," does not deal with them at all, but rehashes the precise definition of propositional languages and homomorphisms from a language (as an

[2]This is not required in the definition the authors give. Without a relation between the matrix M and the consequence relation *Pr*, however, the definition would not make much sense. Why the valuation function v—usually the notion of an interpretation, and kept apart from the definition of the logic per se—is part of the calculus, remains dark.

algebra) to other algebras. The following two subsections present the algebraic characterizations of finite-valued Łukasiewicz and Post logics by Łukasiewicz and Post algebras respectively, and follow closely the seminal work of Rasiowa [1974]. Why these are treated before the logics (in particular, Post logic) are presented in detail is not motivated. In subsection 2.6.6 the functional completeness of Post logic is elaborated. Functional completeness is a very important property in the theory of many-valued logic, and is also of key importance in some applications of many-valued logic (e.g., in many-valued circuit design). It would have deserved more extensive treatment in more generality.

Chapter 3 is a "Survey of Three-valued Propositional Calculi." For a number of three-valued logics considered in the literature, matrices and (if applicable) axiom systems are given (and not much more). For readers interested in applications it would probably have been interesting to know more about the motivations for these systems. In particular, Kleene's logic was originally developed as a logic of (terminating and non-terminating) computations. As such, it was a natural choice for three-valued semantics for logic programming (to provide semantics for negation as failure). In section 3.8 the B.B. expound the unorthodox view that Heyting's propositional calculus arose ("accidentally," p. 95) as the effort to axiomatize a certain three-valued logic. This logic of (purportedly—no reference is given) Heyting is in fact the three-valued Gödel logic (considered in 4.3). Heyting [1930] *does* use several many-valued matrices, although not the one presented in the book as Heyting's three-valued logic, to prove his axioms independent (a method introduced in [Bernays, 1926]).

Chapter 4 continues in a similar vein, surveying some n-valued systems of propositional logic. In view of the attention the intuitionistic logic is afforded elsewhere in the book (a separate chapter 5), Gödel's logics get little attention in comparison to their importance for the former. In particular, it would have deserved to be mentioned that using these logics Gödel proved that Heyting's intuitionistic calculus cannot be characterized by a finite matrix. Dummett's [1959] work is relevant here as well.

As mentioned, chapter 5 gives an introduction to intuitionistic logic. The authors present a Hilbert-style axiom system, a natural deduction system (in the sequent variation), and the characterizations by pseudo-Boolean algebras and Kripke trees. Completeness is stated without proof.

In chapter 6 the notions introduced and surveyed so far are generalized to (the) "First-Order Predicate Calculus for Many-Valued Logics." The fundamental syntactical and semantical notions of predicate logics are defined (first-order language, free and bound variables, interpretation and models). In the second half of the chapter, calculi for the n-valued first-order Post logics are given.

The topics so far are well known and are also contained in other textbooks (including proofs). From chapter 7 onwards B.B. consider newer material which is also of importance especially to computer science. Here, proofs are also provided for the main theorems.

Chapter 7 presents a method for axiomatizing finite-valued first-order logics which is based on the tableau method of Beth. This method has been generalized to the finite-valued (propositional) case by Surma (1977) and the first-order case by Carnielli (1987). In equivalent formulations as many-sided sequent calculi it has been considered even earlier: Schröter [1955] for the propositional case, Rousseau [1967] and Takahashi [1967] for the first-order case (Rousseau dealt with distribution quantifiers in full generality, Takahashi considered only ∀ and ∃). The particular approach of Carnielli has been extended, improved (e.g., to *sets* of truth values as signs), and *implemented* by Hähnle [1991; 1993]—hopefully this work makes its way into the second volume.

The presentation follows the paper of Carnielli closely, with minor notational differences. The idea of the method of finitely-generated trees is as follows: We introduce signs s_i for every truth value i. A signed formula is an expression of the form $s_i(\alpha)$, where α is a formula; intuitively, it stands for "α takes the value i." The tableaux rules give conditions for a signed (composite) formula to be true in terms of signed subformulas. Starting with a particular formula γ, a tree is constructed by applying these rules: every disjunct in the rule defines a successor branch of γ, and the conjuncts in every disjunct are the first nodes on the new branch. If a branch contains two signed formulas $s_i(\alpha)$ and $s_j(\alpha)$ with $i \neq j$, it is said to be *closed*. A tree with only closed branches is also called closed. If there is a closed tree with root $s_i(\gamma)$ the completeness theorem states that under no valuation does γ take the truth value i. Based on this definition it is possible to define a syntactic consequence relation which is complete for the usual semantic consequence in many-valued logics; the compactness theorem follows. The method is extended to the first-order case in sections 7.4 and 7.5. The brand of first-order logics considered here is more general than the one introduced in chapter 6: Here, *distribution quantifiers* are allowed, not only ∀ and ∃. For this class of first-order systems, tableau rules can also be given. Unfortunately, B.B. perpetuate an oversight of Carnielli's [Smith, 1988] which leaves the calculus incomplete. Carnielli gave a correction in [Carnielli, 1991].

Chapter 8 is called "Fuzzy Propositional Calculi" and the authors admit that their approach is "based entirely on Pavelka's papers, which seem to be of most interest." There is no doubt that Pavelka's papers [Pavelka, 1979] are of crucial importance for Fuzzy Logic; unfortunately they are rather unknown and their good presentation would be a great service. But the authors' presentation is far from being good for two main reasons: (i) Pavelka's work is presented in total isolation from related development in many-valued an fuzzy logic and (ii) Pavelka's papers are incorporated into the book in a strikingly mechanical way. We shall document this.

Whereas chapter 1 to 7 deal with finitely-valued logics here the authors abruptly turns to logics with values in a complete lattice, favorite examples being finite chains and the real interval [0,1]. Pavelka's completeness theorems (8.20 and 8.21 in B.B.) must be related with similar completeness theorems for finite and infinite Łukasiewicz's logics; only then one sees the place and strength of Pavelka's theorems in a proper light. But no mention is made in the book of infinitely-valued Łukasiewicz logic, and neither of the celebrated completeness theorem by [Rose and Rosser, 1958]. Pavelka seems to be have been ignorant of [Rose and Rosser, 1958] and cannot be blamed for it; but the authors could have put Pavelka's work into proper context and used the four axioms of Rose-Rosser to simplify drastically Pavelka's choice of axioms.

The second important omission is complete silence on the extensive literature on fuzzy logic; this is fooling the reader. The immense literature on fuzzy logic is admittedly of very varying quality; but at least a reference to the survey papers [Dubois and Prade, 1988] or [Dubois and Prade, 1991] is obligatory. Such a reference could have been critical (distinguishing the broad use of the term "fuzzy logic" from the narrow sense dealing with formal logical systems; but complete silence is hard to justify).

A comparison of chapter 8 with Pavelka's original papers shows that the basic part of the text of chapter 8 corresponds to Pavelka sentence by sentence, with the following changes made: (a) symbols are changed (x to α etc.), (b) words and word groups are replaced by synonymous ones (c) references to axioms and other formulas are replaced by the formulas themselves; and this is done without sufficient checking of the sense. Let us document this by two examples: First, in Pavelka's original there is an unpleasant misprint in axiom σ_{15} (p. 131). It reads $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow \chi)$ but *obviously* should be $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \ \& \ \psi) \Rightarrow \chi)$. B.B. copy the mistaken version of σ_{15} (B.B. p.167) and then multiplicate the mistake by replacing Pavelka's reference to σ_{15} by $((\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow \chi)$ on at least three places (pp. 178, 181, 186). For example [Pavelka, 1979], p. 456(e) reads: "Moreover, if $\varphi \leq \psi \Rightarrow \chi$ holds for some $\chi \in F$ then we may apply r_1 to $\chi \vdash_1 \varphi \Rightarrow (\psi \Rightarrow \chi)$ and $\chi \vdash_1 \sigma_{15}(\varphi, \psi, \chi)$ and obtain $\chi \vdash_1 (\varphi \,\&\, \psi) \Rightarrow \chi$." In the book this is transformed to (p. 181) "Further, assuming there exists $\gamma \in S$ with $\alpha \leq \beta \Rightarrow \gamma$, we can apply detachment to $\chi \vdash_1 \alpha \Rightarrow (\beta \Rightarrow \gamma)$ and $\chi \vdash_1 (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow \gamma)$ [sic] to obtain $\chi \vdash_1 (\alpha \,\&\, \beta) \Rightarrow \gamma$." Here the mistake is difficult to overlook.

Second example: On p. 463 Pavelka writes: "therefore $(jc' \rightarrow x)^n \rightarrow je' \in \mathcal{G}$, which clearly rules out $jc' \rightarrow x \in \mathcal{G}$." B.B. have on the corresponding place (p. 192) "Consequently $(j(y) \rightarrow \alpha)^n \rightarrow j(u) \in \mathcal{G}$ and thus [sic] $j(y) \rightarrow \alpha \in \mathcal{G}$". (They should have "and thus $\ldots \notin \mathcal{G}$" since "rules out $\ldots \in \mathcal{G}$" means "not $\ldots \in \mathcal{G}$".)

There are other misprints and errors, e.g., in the proof on p. 192 above besides the mistake just mentioned: $\frac{z-(u-v)}{n}$ should be $z - \frac{u-v}{n}$; $(\overline{z} \Rightarrow \beta) \Rightarrow \overline{v}$ should be $(\overline{z} - \beta)^n \rightarrow v$.

It is clear that the preparation of a volume like this requires a lot of time and effort. Still, a more thorough proof-reading would have greatly benefited the book. Main objections remain: the lack of references to important literature, the derivative style of at least chapters 7 and 8, and the repetition of known or evident mistakes from the sources. With more care, the book could have been a nice, although far from complete, collection of important aspects in the foundations of many-valued logic. As it is, it can serve as little more than the preliminaries to the second volume.

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