Investment under Uncertainty, Debt and Taxes

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Abstract

We present a capital budgeting valuation framework that takes into account both personal and corporate taxation. This has implications even for all-equity-financed projects. It is also important when the firm or project is partially financed by debt, of course. The setting is a Miller equilibrium economy with differential taxation of debt and equity income that is generalized to allow cross-sectional variation in corporate tax rates. We show broad circumstances under which taxes do not affect the martingale operator (the martingale operator is the same before and after personal taxes, which we call "valuation neutrality") and in which there are no tax-timing options.

One implication of this is that the appropriate discount rate for riskless equity-financed flows (martingale expectations or certainty-equivalents) is an equity rate that differs from the riskless debt rate by a tax wedge. This tax wedge factor is the after-tax retention rate for the corporate tax rate that corresponds to tax neutrality in the Miller equilibrium. We then extend this result to the valuation of the interest tax shield when the firm has an exogenous debt policy, where the debt may or may not have default risk. Interest tax shields accrue at a net rate corresponding to the difference between the corporate tax rate that will be faced by the project and the Miller equilibrium tax rate. Depending on the financing system, interest tax shields can be incorporated by using a tax-adjusted discount rate or by implementing an APV-like approach with additive interest tax shields.

We also analyze the effect of uncertainty and debt financing on the value of investment real options and on the exercise policy, including the effect of default risk. For low uncertainty, a rise in leverage reduces the time value of the real option and increases the probability of being exercised. This last effect on the exercise policy is completely offset when the firm is close to default (i.e, a high coupon). In this situation, more debt or more uncertainty reduces the probability of investing.

Keywords: Investment under uncertainty, real options, capital structure, risk-neutral valuation, corporate and personal taxation, default risk, interest tax shields, cost of capital, tax-adjusted discount rates.
JEL classifications: G31, G32, C61

1 Introduction

The most widely used models for valuing or optimizing capital investment decisions under uncertainty are based on cash flows discounted at a risk-adjusted rate. Nevertheless, the presence of leverage, discretion and asymmetry forces a valuation approach based on computation of certainty-equivalent, rather than expected, cash flows. This leads to the *real options approach* for capital budgeting valuation.

The market maker who sets the relative prices of financial derivatives and their underlying instruments is typically taxed at the same rate on all financial instruments. Thus, the common practice of discounting financial derivatives (puts and calls on stocks, for example) at the riskless debt rate is appropriate. However, we show that it is incorrect to carry this practice over to a real options setting because the marginal investor of a capital investment project is likely to face a differential taxation according to the type of instrument (i.e, equity or debt). In fact, the certainty equivalent rate of return for equity funds is typically lower than that of debt funds.[1]

In addition, the interaction of personal and corporate taxes on various financing instruments generates a net interest tax shield that may have positive or negative value. This interaction is so often mishandled that few people recognize that it could just as likely have a negative value as a positive value.

However, the following features commonly found in modern capital budgeting problems suggest that the best valuation approach is to use certainty equivalent operators, tax-adjusted discount rates for equity and a more sophisticated analysis of interest tax shields:

1. Projects are financed partially by equity and debt.

[1]The fact that a financial market maker does not use the same discount rate to value an investment as a long-term capital investor would use does seem to lead to some arbitrage opportunities that we do not explore. It may be difficult to take advantage of these arbitrage opportunities because a dynamic hedge based on the long term capital value may generate adverse tax consequences, or may be hard to form because of incomplete markets.

On the other hand, it might be possible to synthesize the cash flows of a commodity producer, such as an oil company with very predictable reserves and production by a strip of forward or futures contracts on oil and a strip of bonds. Both of these instruments are financial instruments and would be valued by discounting at the bond rate. They could be sold to an investor to replace an equity position in such a firm, and that investor may discount the flows that the riskless equity rate, which is less than the riskless debt rate. This could represent a simple tax arbitrage.

2. Personal and corporate taxes compound, and taxation of debt income differs from that of equity income.

3. Some cash flows from a project, such as the tax advantage to debtt are contingent on the cash flows of the project and are lost the event of default.

In this paper we first introduce a continuous-time valuation framework for cash flows emerging in capital budgeting that takes into account personal and corporate taxes. We show broad circumstances under which taxes neither affect the equivalent martingale measure (valuation neutrality) nor introduce tax timing options due to the taxation of capital gains (holding period neutrality).

Second, the paper shows how to adjust the value of discretionary and asymmetric cash flows, used later on in the real options approach, to compute the value of debt and taxes under two different types of settings: default-free and defaultable debt.

We can assume that debt is default-free if management constantly revises the debt level of the firm to maintain a constant debt ratio.[2] This assumption, first due to Miles and Ezzel (1980, 1985), is popular in capital budgeting, and is considered as the benchmark case here. More generally, we investigate other exogenous debt policies in which the debt level is adjusted to avoid default, as in Grinblatt and Liu (2002). We find that the value of the interest tax shield can be calculated either by an additive term that separates the value under different financing scenarios (an adjusted present value or APV treatment), or by adjusting the discount rate to reflect the tax wedge that separates the riskless market returns for instruments of different tax classes in what we will call a tax-adjusted discount rate (TADR) approach.

Later, we assume that debt is defaultable and incorporate the effect of the probability of default into the valuation. In this respect, our work extends previous research by Merton (1974), incorporating tax shields; by Brennan and Schwartz (1978), Leland (1994) and Leland and Toft (1996), considering personal and corporate taxation.

[2]Stewart Myers has pointed out in private communication that if debt is continuously rebalanced to keep a constant debt ratio, there can be no default on the debt. As the firm's cash flows fall, its value falls, so the firm must repurchase debt and replace it with equity to keep the debt ratio constant. By the time the firm approaches bankruptcy, there is no debt upon which to default.

In discrete time, there could be default even with a policy of rebalancing debt to keep a constant debt ratio at the rebalance dates.

The paper concludes by analyzing the influence of debt financing on the value of the project and on the decision to invest. We show that debt financing, in the default-free and defaultable cases, has a significant impact both on the value of the option to invest and on the probability of investing.

The are few other contributions in the real options literature that deal with the interaction between investment and financing decisions. Trigeorgis (1993) analyzed the option to default on debt, noting potential interactions with operating flexibility, but with no reference to tax benefits from debt financing. Mauer and Triantis (1994) presented a real options model of a flexible production plant with a capital structure changing over time as a consequence of an optimal dynamic financing policy, but with the important limitation that default is ruled out by the ability to reduce the debt. They do not find any influence of debt financing on the investment policy.[3]

Our model is more general than these contributions. We incorporate personal taxation. We do not assume that debt level is a predetermined function independent of the operating policy (as in Mello and Parsons (1992)) or of the strategic policy (as in Mauer and Ott (2000) and Childs et al. (2000)).

An outline of our work is as follows. In Section 2, we provide an equilibrium valuation approach in continuous-time for real and financial assets in an economy with personal and corporate taxes, where tax rates on bonds and stocks are different. This is the generalized Miller model. In Section 3 we present a continuous-time capital budgeting valuation approach for levered and unlevered real assets. We start with the case of no default risk, and then we consider defaultable debt. Debt policy is taken as exogenous in the sense that it is not subject to optimization. In Section 4 we introduce the basic real option to delay an investment under the assumption that the incremental debt to finance the real asset is issued conditional on the decision to invest. The debt policy is still exogenous. In Section 5 we analyze the effect of debt financing on the value of the real option and on the optimal investment policy. The results are sharply different in the default-free case and in the defaultable case. In the first, a higher leverage for a supra-marginal firm/project (i.e., a firm/project which has a tax advantage to debt) increases the intrinsic value of the option to delay investment and

[3]A third line of research was started by Mello and Parsons (1992), who studied the effect of agency problems of debt (underinvestment or overinvestment) on the optimal operating policy for the firm with different operating modes and provides a measure of the agency cost of debt. Mauer and Ott (2000) and Childs et al. (2000) extend these results to the case of expansion options. We do not analyze potential conflicts of interest between equity-holders and debt-holders. Thus, in this paper, investment is implemented under a first-best investment policy

increases the probability of investing, thus reducing the time-value. In the second case, the positive effect of debt for a supra-marginal firm/project, because of the tax benefit on interest payments, can be completely offset when uncertainty is high and the firm is close to default. Section 6 provides concluding remarks.

Proofs of the results are in the Appendix.

2 Asset valuation in a generalized Miller economy

2.1 Tax equilibrium

This section discusses the valuation framework for capital budgeting purposes in a continuous-time Miller (1977) economy that is generalized to allow for cross-sectional variation in corporate tax rates. In general, the personal tax rate for bond investment income is different from the personal tax rate for stock investment income. Miller assumes that there is cross-sectional variation in personal tax rates, but not corporate tax rates. Thus, in his tax equilibrium, all corporations are indifferent about capital structure, but investors have a tax-induced preference for debt or equity, leading to tax clienteles. By allowing cross-sectional variation in corporate tax rates, as in Sick (1990), only firms at the margin are indifferent (on a tax basis) between issuing debt and equity, and supra-marginal firms with a tax rate below the marginal rate prefer to issue equity and infra-marginal firms with a tax rate above the marginal rate prefer to issue equity.

A cash flow is valued according to an *equivalent martingale measure* (EMM) that, in principle, may be specific to the form in which the cash flow is conveyed to the investor. We establish circumstances where the tax system is neutral, in the sense that it does not affect the EMM (*valuation neutrality*). We also establish conditions where the holding-period timing options (as in Constantinides (1983)) have no value (*holding-period neutrality*). This is crucial when valuing corporate investments, because their cash flow streams must be valued from the point of view of bond-holders and equity-holders of the corporation. Although there are several neutral personal tax systems,[4] we will achieve a neutral tax system with a linear personal taxation scheme (symmetric in gains in losses, with a constant marginal tax rate) that also has a mark-to-market feature that taxes capital gains as accrued rather than when they are realized.[5] Of course, the relevant

[4]See Auerbach and Bradford (2001).

[5]In Jensen (2003), a proof of valuation neutrality and holding-period neutrality is also provided for other cash flow-based taxation schemes.

personal tax system for any capital budgeting valuation may not be exactly neutral, but it will often be close enough to neutral that the techniques here are the appropriate starting point for valuation and policy decisions.

We assume an economy with complete financial markets that has personal and corporate taxes. The time horizon is T, the underlying complete probability space is $(\Omega, \mathcal{F}, \mathbb{P})$, where the set of possible realizations of the economy is Ω, the σ-field of distinguishable events at T is $\mathcal{F}$, and the actual probability on $\mathcal{F}$ is $\mathbb{P}$. We denote by $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ the augmented filtration or information generated by the process of security prices, with $\mathcal{F}_T = \mathcal{F}$.

We denote by τ^c the marginal tax rate for a company; τ^b the personal tax rate for income from bonds, and τ^e the personal tax rate for income from equities. These are adapted to $\mathbb{F}$, and hence may be stochastic.[6] We assume that capital gains and coupons in bond markets are taxed at the same rate, and capital gains and dividends in equity markets are taxed at the same rate, for any given investor. The investor or personal tax operators are assumed to be linear at any date t, in the sense that income and losses from a particular investment are taxed, or generate tax relief, at the same rate.[7] On the other hand, the tax scheme for corporations need not be linear. We allow τ^c, τ^b and τ^e to be $\mathbb{F}$-adapted stochastic processes, i.e. they are determined as a function of the (stochastic) factors underlying the economy, but they must have continuous sample paths, almost surely. In general, τ^b and τ^e are different. Ross (1987) established the existence of equilibrium and of an EMM for an economy with personal taxes and a convex tax schedule. These assumptions are satisfied in our setting.

Consider a firm with tax rate τ^c that is deciding whether to issue debt or equity to an investor with tax rates τ^e and τ^b on equity and debt, respectively. If

$$(1 - \tau^b) < (1 - \tau^c)(1 - \tau^e), \tag{2.1}$$

then the firm and the investor has incentives to issue equity and the marginal investor has incentives to buy equity. To see this, suppose riskless debt has

[6]For the discussion below, we suppress the time subscript for brevity. The stochastic nature of tax rates is not essential to achieve the results of this paper and readers may find it more convenient to think of the tax rates as being deterministic or even constant.

[7]In general, invididual investors do have a progressive tax structure, with increasing rates at increasing levels of income. What we are assuming here is that a particular investment that the investor is pricing does not have income variations so large that it moves the investor to higher or lower tax brackets. The key assumption for this paper is that the investment gains and losses for a particular asset are taxed at the same rate and that there is no kink in the tax curve as an asset goes from a gain to a loss.

a yield of r^f. Then, if equation (2.1) holds, it is possible to choose a rate of return r^z for equity,[8] so that the after-corporate-tax cost of equity paid by the company is less than that of debt cost and, simultaneously, the after-all-tax rate of return for equity received by the investor is higher than that of debt. That is, we can choose r^z so that

$$\frac{1-\tau^b}{1-\tau^e} r^f < r^z < (1-\tau^c) r^f .$$

On the one hand, this implies that $(1-\tau^b)r^f < (1-\tau^e)r^z$, so that the investor achieves a higher after-all-tax return on equity than debt. On the other hand, it implies that $r^z < (1-\tau^c)r^f$, so that the cost of equity to the firm is lower than the after-tax cost of debt. Thus, if (2.1) holds for the marginal investor and the marginal firm, the economy can't be in equilibrium, since the firm and the marginal investor can both be better off by switching some debt to equity.

Similarly, the firm has incentives to issue debt rather than equity if

$$(1-\tau^b) > (1-\tau^c)(1-\tau^e)$$

In equilibrium, there will be no further incentives for the firm to issue debt to retire equity, or to issue equity to refund debt. Denoting the marginal firm's tax rate by τ^m, we have the Miller (1977) equilibrium relationship amongst the marginal tax rates of the marginal investor and marginal firm:

$$(1-\tau^b) = (1-\tau^m)(1-\tau^e)$$

We can define the marginal firm's tax rate τ^m by rearranging this as:

$$1-\tau^m = \frac{1-\tau^b}{1-\tau^e} \tag{2.2}$$

Using the equilibrium condition for the marginal investor, we can characterize (2.2) in terms of market rates of return on equity and debt:

$$r^z = (1-\tau^m) r^f \,. \tag{2.3}$$

This establishes a tax wedge $(1-\tau^m)$ between debt and equity returns. Thus, it is appropriate to refer to r^z as a *tax-adjusted discount rate*. Equation (2.3) provides a way to estimate the marginal tax rate τ^m, since r^z and r^f can either be observed or derived from security prices.

[8] We set aside the risk premium for a moment, so that r^z is a riskless yield. We will address this point later on.

The equilibrium rates of return on the money market and stock market are the same after all taxes for the marginal investor. Defining this common after-all-tax return to be $r^{z,\mathrm{at}} \equiv r^{f,\mathrm{at}}$, we have

$$r^{f,\mathrm{at}} = r_t^f(1-\tau_t^b) = r_t^z(1-\tau_t^e) = r^{z,\mathrm{at}}\,. \tag{2.4}$$

2.2 Tax clienteles

This model is really based on the marginal investor and her tax rates by the Miller equilibrium. The marginal firm's tax rate is derived from her rates, and the marginal firm is not really identified, so it could have some unusual properties. But, we can say that generally, the marginal firm's tax rate $\tau^m \geq 0$, because the marginal investor's debt tax rate usually exceeds her equity tax rate, $\tau^b \geq \tau^e$. In this situation, the riskless debt return exceeds the riskless equity return: $r^f \geq r^z$. If $\tau^b = \tau^e$, then $\tau_m = 0$ and thus $r^z = r^f$. The assumption that $\tau^b = \tau^e$ is common in the literature. It was used by Modigliani and Miller (1963) and underlies most of the standard exposition of the CAPM, such as Sharpe (1964) and APT as in Ross (1976). Others, such as Sick (1990) and Taggart (1991) have taken the return differentials implicit in Miller (1977) to get $r^f \geq r^z$.

We choose the notation r^z to bring the analogy with the zero-beta rate of return in the Black (1972) version of the CAPM. There may be no riskless equity security, but in many circumstances it is the intercept term in such a CAPM. It is the appropriate discount rate for certainty-equivalents that are all-equity financed. We shall use it as the discount rate in martingale and PDE valuation models for all-equity financed cash flows.

There may be clientele effects whereby the marginal investors for different types of securities have different tax classes. Thus, it may be appropriate to use different discount rates to value these securities. For example, in the financial derivatives literature and actual practice, there is rarely any consideration that a tax wedge should be applied to the riskless debt rate in valuation models. This is justified, since the marginal investor for a derivative is likely an incorporated market maker that trades frequently and thus takes capital gains and dividends as ordinary income. For such a marginal investor, $\tau^b = \tau^e$ and hence $r^z = r^f$ even if they finance their positions entirely with equity.

On the other hand, for real assets, the marginal investor is likely a taxable individual who holds shares in a corporation. If the real option is entirely financed with equity, then it is appropriate to use a riskless rate $r^z < r^f$.

In what follows, we only require that all tax rates be between 0 and 1: $0 \leq \tau^b, \tau^b \tau^c, \tau^m < 1$.

Tax arbitrage has a tendency to make all firms behave as if they have the same tax rate. At the corporate level, an arbitrage scheme could involve a highly taxed firm, with $\tau^c > \tau^m$ or $(1-\tau^b) > (1-\tau^c)(1-\tau^e)$, issuing debt to acquire its own equity, for example. Or, it could involve a low-tax firm, with $\tau^c < \tau^m$, issuing equity to buy back debt. We assume that there are tax laws and agency costs[9] that prevent a firm from undertaking such an arbitrage transaction. Thus, there will generally be supra- and infra-marginal *firms* in this generalized Miller tax equilibrium.

It is more difficult to generate tax arbitrage opportunities that would have all investors facing the same effective personal tax rate as that held by the marginal investor. This is because personal tax laws identify the individual and generally change when a financial entity (such as a corporation, mutual fund or trust) is inserted between the investor and the investment vehicle. There will generally be supra- and infra-marginal *investors* in this generalized Miller tax equilibrium. Indeed, Miller (1977) also assumed this. For example, suppose an investor pays little or no tax on any investment (e.g. a pension fund), but the marginal investor pays higher tax on debt than on equity. The untaxed infra-marginal investor would prefer the tax benefits of debt, but this will prevent the investor from earning risk premia paid on equity investments. We assume that any attempt to convert an equity investment with a risk premium to a debt instrument for tax purposes is prevented by tax law.

Taxes introduce a risk-sharing mechanism between government and taxed investors. This could cause a difference between the equilibrium EMM pricing measure with and without personal taxes. Or, the EMM could differ for different tax clienteles. We will establish a *valuation neutrality*, which says that the equilibrium pricing measure is unchanged by the presence of taxation.

For valuation purposes, this means that the expectation (under the EMM) of the pre-tax cash flow stream of a security, discounted at a pre-tax rate, is equal to the expectation (under the same EMM) of the after-tax cash flow stream, discounted at an after-tax rate.

A second important issue introduced by taxation of security returns is the presence of timing options related to taxation of capital gains, as pointed out by Constantinides (1983). Taxation of capital gains produces timing options due to a (rational) delay of liquidation of positions in financial securities, until a date of forced liquidation, if the accrued capital gain is positive and the anticipation of liquidation of the position if the capital gain is negative,

[9] See for instance, Myers and Majluf (1984).

to take advantage of the tax credit.

We assume a *holding-period neutral tax scheme*,[10] i.e. a tax scheme that does not introduce any timing options related to taxation of capital gains and so it does not change portfolio strategies of investors. Auerbach (1991) and Auerbach and Bradford (2001) describe a generalized tax scheme that prevent tax arbitrages and realizes holding-period neutrality.[11] Within this class of valuation and holding-period neutral tax schemes, we will consider the mark-to-market personal taxation. *Mark-to-market taxation* of a security consists of accrual of taxes on capital gains on a separate account *as if* the security were actually traded. The realization of the accrued tax on capital gain is deferred until the date the security is actually traded. There are several practical problems related to this scheme, especially when related to illiquid (or inefficiently traded) securities. As far as a no-arbitrage financial market is concerned, the main drawback of mark-to-market taxation is related to the liquidity constraints it imposes on the investors. Nevertheless, mark-to-market taxation is a reasonable benchmark and has been used for this in the public finance literature.[12]

[10]To the best of our knowledge, the definition of holding-period neutrality is due to Auerbach (1991, p. 169):

> "A realization-based tax system is *holding-period neutral* if it leads each investor in an asset to require a before-tax return having a certainty-equivalent value that is not a function of the length of holding period or the asset's past pattern of return."

[11]Jensen (2003) provides an exhaustive characterization of the class of taxation schemes that satisfy the requirements of valuation and holding-period neutrality.

[12]An extension of the results presented below to other neutral taxation scheme is beyond the scope of this paper. We will provide our results in a continuous-time Miller (1977) setting, where the tax rate for bonds and the tax rate for stocks can be different even if they are traded by the same agent. Our results significantly extend Jensen (2003) results to an economy where personal tax rates can be cross-sectionally different and bonds are taxed differently from bonds.

2.3 Security price dynamics

Suppose $\mathbb{Q}$ is an EMM on $\mathcal{F}$ that prices securities.[13,14] Denote the risk-neutral expectation operator with respect to $\mathbb{Q}$ by $\mathbb{E}^{\mathbb{Q}}[\cdot \mid \mathcal{F}_t] = \hat{\mathbb{E}}_t[\cdot]$.

The market price of a stock is an adapted Itô process $\{S_t\}$ with dynamics (under $\mathbb{Q}$):

$$dS_t = (r^z_t S_t - X_t)\, dt + \sigma_t dZ_t \qquad S(0) = S_0. \tag{2.5}$$

where r^z_t is the time-t instantaneous rate of return for riskless equity (after corporate taxes and before-personal taxes) and X_t is the flow rate of dividends. We assume that r^z, X_t and σ_t are adapted processes that satisfy the usual integrability conditions.

Note that r^z may be subject to the tax clientele effects discussed in Subsection 2.2. If the asset in question is a real asset (such as a real option) financed by equity, then this is likely to be less than the bond rate. But, if it is a financial asset, it is likely to equal the bond rate.

The riskless money market account[15] earns interest at r^f_t, and has value B with dynamics (before investor tax on the debt income):

$$dB_t = r^f_t B_t dt \qquad B_0 = 1\ . \tag{2.6}$$

where the riskless rate of return or riskless return on debt, r^f_t, is an adapted process (and satisfies the standard conditions). thus

$$B_t = \exp\left(\int_0^t r^f_u du\right)\ .$$

[13] We suppress the description of the other technical hypotheses needed for the existence of an EMM for valuing the securities also in an infinite horizon. Interested readers can refer to the standard references, such as Duffie (2001).

[14] Since in principle the EMM can be affected by personal taxation, our choice of $\mathbb{Q}$ is arbitrary for the time being. As we will prove later, under the mark-to-market linear tax scheme, the EMM is not affected by taxation. Hence, $\mathbb{Q}$ will be legitimately defined as *the* equilibrium EMM.

[15] We will deal with defaultable bonds in Section 3.3, where we introduce exogenous default and the valuation formula will be more easily introduced.

2.4 Valuation neutrality and holding-period neutrality

Using the martingale valuation results,[16] given a maturity date[17] $T' > t$ the market price of the stock at time t (i.e., before personal taxes) is

$$S_t = B_t^z \hat{\mathbb{E}}_t \left[\frac{S_{T'}}{B_{T'}^z} + \int_t^{T'} \frac{X_u}{B_u^z} du \right], \tag{2.7}$$

which is the value of the stock at the market level (i.e., before personal taxes), given by the CE price at T' discounted at the riskless return on equity. Let

$$B_t^z = \exp\left(\int_0^t r_u^z du \right)$$

be the time-t value of one dollar invested at time 0 and earning the riskless return on equity r_u^z. That is, the accumulated value of an investment in the riskless equity account. Let

$$B_t^{z,\mathrm{at}} \equiv \exp\left(\int_0^t r_u^z (1 - \tau_u^e) du \right)$$

be the time-t value of one dollar invested at time 0 and earning the riskless rate of return on equity r_u^z net of personal taxes τ_u^e for $0 \leq u \leq t$.

Define the *after-all-tax stock value* S_t^{at} by[18]

$$S_t^{\mathrm{at}} = B_t^{z,\mathrm{at}} \hat{\mathbb{E}}_t \left[\frac{S_{T'}}{B_{T'}^{z,\mathrm{at}}} + \int_t^{T'} \frac{(1 - \tau_u^e) X_u}{B_u^{z,\mathrm{at}}} du - \int_t^{T'} \frac{\tau_u^e}{B_u^{z,\mathrm{at}}} dS_u^{\mathrm{at}} \right]. \tag{2.8}$$

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[16] Ross (1987) shows that there is a martingale pricing operator in the presence of taxes. His results are general and, in our setting, establish a risk-neutral expectation operator (or, equivalently, a certainty-equivalent operator) after all taxes, as well as risk-neutral expectation operators for debt flows before personal tax and for equity flows before personal tax. There is no immediate guarantee that these pricing operators are related or equivalent. Sick (1990) raised this question in a discrete-time setting and showed that the certainty equivalent operators associated with these pricing operators are all identical to each other. That is, taxes and tax shields do not generate a risk premium. We establish an equivalent result in continuous time, but initially, we must be careful to distinguish the source of any pricing operator. For the time being, we take the martingale expectations operator $\mathbb{E}^{\mathbb{Q}}$ that prices equity cash flows after corporate tax and before personal tax.

[17] It may be that T' is stochastic, in which case we need to assume that T' is a stopping time with respect to $\mathbb{F}$.

[18] Equation (2.7) defines a market-level price that is based on a martingale prior to personal (investor) taxes. An after-all-tax value is a valuation that depends on the after-all-tax flows. Since this is the stream of flows that generate consumption for the investor, one may find it more convenient to start with this valuation operator, rather than the market-level operator. We proceed this way to make the mathematics simpler.

This is a recursive definition, initially defining $S_t^{\mathrm{at}} = S_{T'}$ for $t = T'$ and working backwards for earlier t. The first term in square brackets is the terminal value, the second term is the present value of the after-personal-tax dividend stream and the third term is the present value of the taxes on all capital gains on the *market-level price*. The definition uses the same risk-neutral expectations $\hat{\mathbb{E}}_t[\cdot]$ and probability measure $\mathbb{Q}$ as is used for the pretax value S_t. We will show that (2.8) is the same as the pre-personal-tax value of the stock, so that there is no cause for concern that we are mixing market-level values S_t with after-all-tax values S_t^{at}.

Equation (2.8) implements a *mark-to-market* taxation rule. That is, each period, capital gains (or losses, if negative) are taxed as accrued even though they are not realized. This tax scheme requires that, at the date of liquidation, the capital gains in the interval $[t, T']$ are considered at the date they occur and then interest accrues on the tax on capital gains at the after-personal tax rate of return until T'.

We denote the value of a unit-dollar amount on the money market account, accrued at the after-personal-tax rate $r_t^f(1-\tau_t^b)$ by

$$B_t^{f,\mathrm{at}} = \exp\left(\int_0^t r_u^f(1-\tau_u^b)du\right) .$$

Since the equilibrium tax relation in (2.4) holds at any t, we have $B^{f,\mathrm{at}} = B^{z,\mathrm{at}}$ for all t. Analogous to definition (2.8) for stocks, we define the *after-all-tax bond value* B_t^{at} by

$$B_t^{\mathrm{at}} = B_t^{f,\mathrm{at}}\hat{\mathbb{E}}_t\left[\frac{B_{T'}}{B_{T'}^{f,\mathrm{at}}} - \int_t^{T'} \frac{\tau_u^b}{B_u^{f,\mathrm{at}}}dB_u^{\mathrm{at}}\right] . \tag{2.9}$$

Proposition 1. *Under the linear mark-to-market taxation rule:*

1. *the value of the stock at the market level equals the value of the stock after personal taxes: $S_t = S_t^{\mathrm{at}}$, and*

2. *the value of the money market account before personal taxes is equal to the value of the money market account after personal taxes: $B_t = B_t^{\mathrm{at}}$.*

Proof. See the Appendix. □

Thus, we can say that the system of taxation that is *value neutral*. The next result is an extension of Sick (1990, Proposition 1) to a continuous-time setting.

Corollary 2. *The martingale expectation operator $\hat{\mathbb{E}}$, or equivalently, the risk-neutral measure $\mathbb{Q}$ correctly values* both *equity and money market payoffs that are after personal taxes as well as equity and money market payoffs that are before personal taxes. In other words, the martingale expectation operator (or certainty-equivalent operator) is the same for debt and equity markets and is the same before and after personal tax.*

Proof. See the Appendix. $\square$

We have implicitly assumed that the personal equity tax rate τ_t^e is the tax rate of the marginal investor, because we have been developing relationships for equilibrium prices. However, the same result holds true for any investor, whether or not he is marginal, provided that the tax schedule is *locally linear*. By local linearity, we mean that the marginal payoffs provided by the investment under consideration will cause variation along a portion of the tax schedule that has a constant marginal tax rate.

Corollary 3. *If an investor has a tax schedule that is* locally linear *in equity (bond) income at rate τ_t^e (τ_t^b), then the personal valuation of the risk-neutral expectation of a pretax stream of equity (money market account) flows discounted at the CE discount rate for equity r_t^z (at the bond rate r_t^f) has the same value as a stream of equity (money market account) net of mark-to-market capital gains tax, if discounted at the investor's personal after-all-tax equity (bond) rate $r_t^{z,\text{at}}$ ($r_t^{f,\text{at}}$).*

We have discussed the valuation neutrality property of a linear mark-to-market taxation scheme. We now show that such a scheme is *holding-period neutral*, in the sense that it does not change the set of opportunities available to the investor by inducing timing options.

Proposition 4. *Under the linear mark-to-market taxation rule and in equilibrium, for the marginal investor no further incentives for liquidating a position are introduced by taxation; i.e., tax options are worthless.*

Proof. See the Appendix $\square$

3 Capital budgeting with exogenous capital structure in a generalized Miller economy

In the framework described in Section 2, we introduce capital budgeting valuation, assuming that the corporation has an exogenous capital structure.

The capital structure can be the one proposed by Modigliani and Miller (1958), with constant level of debt, or the one proposed by Miles and Ezzel (1985), with constant debt proportion.[19] We will examine both the default-free case (constant debt proportion or Miles and Ezzel (1980) debt policy) and the case with defaultable debt (constant debt level or Modigliani and Miller (1958) debt policy), assuming that the default threshold is predetermined.

For capital budgeting valuation, since the interest payments on debt are tax deductible, it is important to evaluate the interest tax shield to determine the cost of capital. The tax shield is a function of the leverage of the firm/project and is contingent on the cash flow process, since it is lost in case of financial distress.[20] Hence, a proper valuation[21] of tax shield can only be done in a risk-neutral setting.[22] Moreover, the interest tax shield can be positive (for a supra-marginal firm, $\tau^c > \tau^m$) or negative (for an infra-marginal firm, $\tau^c < \tau^m$) as a consequence of the level of the equilibrium marginal tax rate, τ^m.[23] This makes the framework described in Section 2 appropriate for capital budgeting valuation.

Hence, the goal of this section is to deeply analyze the value creation process for debt under different financing policies. In addition, we will compute the company and debt values that will be inputs for Section 4.

3.1 Valuation of tax shields

Let there be given a firm/project,[24] with marginal corporate tax rate τ^c. We assume that τ^c is independent of earnings.

The firm/project has duration T^p (possibly, infinite) and the EBIT

[19]We will not consider model with endogenous capital structure, as Fisher et al. (1989), Mauer and Triantis (1994), Goldstein et al. (2001), Dangl and Zechner (2003), Christensen et al. (2002). Capital budgeting valuation with endogenous financial decisions will be the subject for future research.

[20]This is witnessed also by Leland (1994, p. 1220): under the U.S. tax code, the tax benefit on coupon payment is allowed only if EBIT is grater than coupon.

[21]We do not want to mean it is impossible from a conceptual standpoint, rather what we refer to is the difficulty in finding the correct tax as well as risk-adjusted discount rate.

[22]Of course, there can be other contingent claims in capital budgeting, like real options. The extension of the results of this section to real options are in Section 4.

[23]Although we will focus on the supra-marginal case, our results holds true also for the infra-marginal case.

[24]Note that, for the time being, and in the sake of simplicity, our setting is different from the case of an ongoing company with its own capital structure and growth options. We will show that the cost of capital for the option to delay investment in the project is independent of the current capital structure of the company, so our choice is legitimate.

(Earning Before Interest and Taxes, i.e. a before-corporate taxes free cash flow) rate, X_t. The EBIT rate follows the adapted process, under the EMM

$$dX_s = g(X_s, s)ds + \sigma(X_s, s)dZ_s, \qquad X_t = x, \tag{3.1}$$

where g is the instantaneous CE growth rate and σ is the diffusion.

For convenience, we derive first the value of the firm/project assuming it is unlevered. The after-personal tax total cash flow from the firm/project is $X_t(1-\tau_t^c)(1-\tau_t^e)$. Since in Propositions 1 and 4 we established the equivalence of a before- and after- personal taxes valuation for equity income, assuming that the after corporate tax earning are immediately paid to equity-holders, the value of the unlevered firm/project, denoted U, by straightforward applications of the pricing relations is Section 2, is

$$U(t,x) = B_t^z \hat{\mathbb{E}}_t \left[\int_t^{T^p} \frac{X_s(1-\tau_s^c)}{B_s^z} ds \right] \tag{3.2}$$

for $0 \leq t \leq T^p$.

In what follows we determine the Adjusted Present Value (APV), i.e. the value of the firm/project including the value of the tax advantage to debt, in case operations are financed also with debt: we will analyze first the default-free case and than we will consider the case with defaultable debt.

3.2 Default-free debt

In this section we will assume also that there is no default, because of the ability of the management to constantly adapt, at zero adjustment costs, the debt level in order to avoid default. This implies that the coupon rate for corporate bonds is the risk free rate.

Since the firm/project is financed also by issuing coupon bonds, assuming interest on debt is paid out immediately, given the total coupon rate paid to bond-holders R_t, then the (instantaneous) cash flow to equity-holders is $(X_t - R_t)(1-\tau_t^c)(1-\tau_t^e)$ and to debt-holders is $R_t(1-\tau_t^b)$, so that the total cash flow from the project is, after-personal taxes,

$$\begin{aligned}(X_t - R_t)(1-\tau_t^c)(1-\tau_t^e) + R_t(1-\tau_t^b)& \\ = X_t(1-\tau_t^c)(1-\tau_t^e) + \tau_t^* R_t(1-\tau_t^e),&\end{aligned} \tag{3.3}$$

for $0 \leq t \leq T^d$, where $\tau^* = \tau^c - \tau^m$ and τ^m is the marginal tax rate defined in (2.2). We assume that the principal is paid back at $T^d \leq T^p$. Alternatively, if we are valuing an infinite-horizon project we will assume that it is financed by issuing consol bonds ($T^d = \infty$). Moreover, for simplicity we assume that debt is always valued at par.[25] Since debt is default-free, $R_t = r_t^f D(t, x)$, where $D(t, x)$ is the market value of debt. Hence, R_t is an adapted process.

In equation (3.3), while $X_t(1 - \tau_t^c)(1 - \tau_t^e)$ coincides with the after personal taxes flow for an all-equity financed project, the (net) tax shield, $\tau_t^* R_t$, is taxed at the equity rate because it accrues to equity-holders.[26] Hence, the right-hand-side of (3.3) must be discounted using the after-personal taxes discount factor for stocks, $B^{z,\text{at}}$. From Propositions 1 and 4, we know that this is equivalent to a before-personal taxes valuation using the discount factor B^z. This intuition is made precise in the subsequent proposition. We denote V the APV of the levered firm/project. Hence, the levered project (and in particular, the tax shield) is a contingent claim of the free cash flow.

Proposition 5. *The APV, incorporating the value of the tax shield, satisfies equation*

$$\frac{1}{2}\sigma^2(X, t)\frac{\partial^2 V}{\partial X^2} + g(X, t)\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + X_t(1 - \tau_t^c) + \tau_t^* R_t = r_t^z V. \tag{3.4}$$

with boundary conditions $V(T^d, X_{T^d}) = U(T^d, X_{T^d})$ *and* $U(T^p, X_{T^p}) = 0$, *and is*

$$\begin{aligned} V(t, x) =& B_t^z \hat{\mathbb{E}}_t \left[\int_t^{T^d} \frac{X_s(1 - \tau_s^c) + \tau_s^* R_s}{B_s^z} ds + \frac{U(T^d, X_{T^d})}{B_{T^d}^z} \right] \\ &= U(t, x) + B_t^z \hat{\mathbb{E}}_t \left[\int_t^{T^d} \frac{\tau_s^* R_s}{B_s^z} ds \right], \end{aligned} \tag{3.5}$$

for $0 \leq t \leq T^d$, *where the second term in the right-hand-side is the tax shield.*

[25] Obviously, this equality derives from the fact that debt is default-free and its coupon has been set at the level of the risk free rate.

[26] Sick (1990) was the first to point out this fact. A possible interpretation of (3.3) is the following: a bond can be interpreted as a swap between equity-holders and the marginal investor, who is indifferent between receiving cash flow from equities and cash flows from bonds. At $t = 0$, equity-holders swap B dollars of equity for an equivalent amount (at the market value) of debt, so, the initial net position for equity-holders is equal to 0. At every coupon date, they save to pay a required rate of return $r^z B$ and pay $r^f(1 - \tau^c)B$ to debt-holders, so the net position for equity-holders is $r^z B - r^f(1 - \tau^c)B$. Since the generalized Miller equilibrium relation (2.3) holds for the marginal investor, the net position for equity-holders is $r^f B(\tau^c - \tau^m)$, where $r^f B$ is the CE coupon (which coincides with the actual coupon at the current date), derived from the valuation equation for debt.

Proof. See the Appendix. □

Equation (3.5) implies that when computing the APV from the total cash flows generated by a company, the appropriate tax-adjusted riskless equity rate of return is r_z no matter what its capital structure is. Then, in our setting, the problem of finding the correct risk-neutral discount rate that takes into consideration the different stakeholders taxation rates has been solved.[27] In addition, for supramarginal companies, since $\tau^* = \tau^c - \tau^m > 0$, debt always creates value.

The APV in (3.5) can be computed in the general case, for general processes for X_t and R_t, for the risk-free rate and for the tax rates, using some numerical methods.[28] Nevertheless, to obtain a default-free debt, we have to properly define the debt policy: in particular, we assume that $D(t, x)$ is a *linear function of* APV: $D = LV$, where L is the constant debt ratio, $0 \leq L < 1$. This is the Miles and Ezzel (1980, 1985) (M-E) debt policy.[29]

By replacing this condition in equation (3.4) we obtain

$$\frac{1}{2}\sigma^2(X,t)\frac{\partial^2 V}{\partial X^2} + g(X,t)\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + X_t(1-\tau_t^c) = \rho_t V \tag{3.6}$$

where

$$\rho_t = r_t^z - \tau^* r_t^f L = (1-L)r_t^z + L(1-\tau_t^c)r_t^f \tag{3.7}$$

is the *tax-adjusted certainty equivalent cost of capital* (TADR), i.e., the cost of capital under the EMM that incorporates also the effect of tax shield. In contrast to what happened for equation (3.5), if free cash flows are used to compute the value of a levered company with a (M-E) debt policy, the tax-adjusted CE discount rate, ρ_t, is not independent of the level of debt. In fact, from the right-hand-side of equation (3.7) we notice that ρ_t is also a weighted average cost of capital (wacc) under the EMM. This is an extension of the tax-adjusted rate of return in Sick (1990) to a continuous-time setting. Our result is in accordance to the general practice of computing the NPV by discounting the expected free cash flows under the actual probability at the average cost of capital.

[27] Among the studies that, under the same assumptions on personal and corporate taxation, use r^f as the tax-adjusted discount rate (TADR), we can cite Mello and Parsons (1992), Mauer and Triantis (1994), Goldstein et al. (2001), and Titman and Tsyplakov (2005).

[28] Monte Carlo simulation is particularly suited for solving this types of problems.

[29] We will prove below that M-E debt policy is actually implies the assumption of absence of default.

Applying conditions $V(T^d, X_{T^d}) = U(T^d, X_{T^d})$ and $U(T^p, X_{T^p}) = 0$, and considering that, for $t > T^d$, $D \equiv 0$ (since $R \equiv 0$),[30] we have that the APV of the firm/project is[31]

$$V(t,x) = B_t^w \hat{\mathbb{E}}_t \left[\int_t^{T^d} \frac{X_s(1-\tau_s^c)}{B_s^w} ds \right] + B_t^z \hat{\mathbb{E}}_t \left[\frac{U(T^d, X_{T^d})}{B_{T^d}^z} \right], \qquad (3.8)$$

for $0 \leq t \leq T^d$, where $B_t^w = \exp \int_0^t \rho_u du$ is the time-t value of one dollar accrued at the tax-adjusted CE discount rate. Equation (3.8) when compared to (3.2) clarifies the role of the tax-adjusted CE cost of capital, ρ_t, as the stochastic instantaneous discount rate that generates the levered asset value when applied to the unlevered cash flow process.[32] In particular, the randomness of ρ_t derives from the randomness of r_t^z and r_t^f, but not from the leverage ratio. Hence, equation (3.8) provides a time-consistent pricing operator for levered cash flows.

Equation (3.8) can be used, by employing some numerical methods, to compute the APV of the firm/project using a stochastic wacc under general hypotheses on the relevant stochastic processes.

3.3 Defaultable debt

If debt is default free, the effect of debt for a supra-marginal firm is always positive, and only the equilibrium of the financial market (and potential agency costs and tax laws) prevents the realization of a tax arbitrage, as in Miller (1977). On the other hand, if default can actually happen, then the cost of debt is influenced by the credit risk and this tends to reduce the tax advantage to debt. We will show that this affects also the investment policy.

We will limit the analysis to the case of exogenous default; i.e., we assume that there is a given barrier (coupon coverage), denoted x_D, such that, when the EBIT process X_t breaches the barrier, the firm is in default.[33] In this case, the bond-holders file for bankruptcy and receive the value of the unlevered asset net of bankruptcy costs. Bankruptcy costs are assumed to be proportional to the unlevered asset value, with known proportion α.

As long as the EBIT process X_t is above the default threshold x_D, the total cash flow generated by the firm is, after personal taxes, as in equation

[30]The argument is the same used in the proof of Proposition 5.

[31]If $L = 0$, then $\rho_t = r_t^z$ and 3.8 and 3.2 coincide.

[32]Grinblatt and Liu (2002) have an analogue definition of wacc, although they limit the analysis to a less general setting with constant rate of returns and with $\tau^m \equiv 0$.

[33]See Leland (1994) and more recently Titman and Tsyplakov (2005) for a discussion on what a plausible level for x_D should be.

(3.3). On the other hand, if at date t, $X_t = x_D$, the firm defaults on the coupon payment and so the tax shield is lost. In this case, the value of the levered asset is used to pay bond-holders, who receive $(1-\alpha)U(t, x_D)$. When bond-holders file for bankruptcy, they become owners of the firm. Hence they receive the present value of EBIT after corporate taxes and after personal taxes for equity flows. From Proposition 1 and 4, the valuation on an after personal tax basis is equivalent to a before-personal taxes valuation using the discount factor B^z. This is stated in the following proposition.

Proposition 6. *The APV, incorporating the value of tax shield, satisfies equation* (3.4) *with boundary conditions* $V(T^d, x) = U(T^d, x)$ *and* $U(T^p, x) = 0$ *and* $V(s, x_D) = (1-\alpha)U(s, x_D)$ *for all* $t \leq s \leq T^d \leq T^p$ *and is, assuming* $x > x_D$,

$$V(t,x) = U(t,x) + B_t^z \hat{\mathbb{E}}_t \left[\int_t^{T_D \wedge T^d} \frac{\tau_s^* R_s}{B_s^z} ds \right] - \alpha B_t^z \hat{\mathbb{E}}_t \left[\chi_{\left\{\exists s \in [t, T^d], X_s = x_D\right\}} \frac{U(T_D, x_D)}{B_{T_D}^z} \right] \quad (3.9)$$

where $T_D = \inf\left\{s \in [t, T^d], X_s = x_D\right\}$, *is the first time* $X_t = x_D$, *and* χ_A *is the indicator function for event* A.[34]

Proof. See the Appendix. □

In the right-hand-side of equation (3.9), the second term is the tax shield taking into account the risk it is lost in case of default, i.e. the first time the process X_t hits x_D from above; the third term is the value of the bankruptcy costs incurred at the date of default, which are proportional to the current value of the unlevered asset. Equation (3.9) can be easily implemented using some numerical methods.

In subsequent sections we will need also D, the market value of corporate bonds, in the generalized Miller equilibrium economy. As above, we denote R the coupon payment and P the principal payment (face value) paid back at maturity T^d. Under the assumption of exogenous default at the threshold x_D, debt is a contingent claim on X, the EBIT of the firm. The next proposition determines, under the hypothesis of a generalized Miller equilibrium, the value of defaultable debt.

[34] $\chi_A(\omega) = 1$ if $\omega \in A$, $\chi_A(\omega) = 0$ otherwise. We denote $x \wedge y = \inf\{x, y\}$.

Proposition 7. *The market value of debt satisfies*

$$\frac{1}{2}\sigma^2(X,t)\frac{\partial^2 D}{\partial X^2} + g(X,t)\frac{\partial D}{\partial X} + \frac{\partial D}{\partial t} + R_t = r^f D \tag{3.10}$$

with boundary conditions $D(T^d, X_{T^d}) = P$, $D(s, x_D) = (1-\alpha)U(s, x_D)$ *for all* $t \le s \le T^d \le T^p$ *and, assuming* $x > x_D$, *is*

$$D(t,x) = B_t^f \hat{\mathbb{E}}_t \left[\int_t^{T_D \wedge T^d} \frac{R_s}{B_s^f} ds\right] + B_t^f \hat{\mathbb{E}}_t \left[\chi_{\left\{\forall s \in [t,T^d], X_s > x_D\right\}} \frac{P}{B_{T^d}^f}\right] + (1-\alpha) B_t^f \hat{\mathbb{E}}_t \left[\chi_{\left\{\exists s \in [t,T^d], X_s = x_D\right\}} \frac{U(T_D, x_D)}{B_{T_D}^f}\right] \tag{3.11}$$

where $T_D \le T^d$ *is the first time* $X_t = x_D$.

Proof. See the Appendix. □

4 Valuation of real options when the debt policy is exogenous

This section addresses real options valuation under the general framework introduced in Section 2, assuming that the tax shield may be valued according to the analysis of Section 3. In particular, we will differentiate our analysis according to the cases of default-free debt and defaultable debt.

We have the opportunity to delay investment in a project,[35] whose incremental EBIT follows the stochastic process in (3.1) under the EMM. The project has duration T^p (possibly, $T^p = \infty$), so that the project starts from the date the option is exercised, T_I, and ends at $T_I + T^p$. The cost to implement the project is I (an adapted process) and we have the opportunity to delay the investment until date T^o (possibly, $T^o = \infty$).[36]

[35]For definiteness, we will discuss the prototypical case of the option to delay an investment decision, although our approach can be applied to a broader class of real options. See Dixit and Pindyck (1994) and Trigeorgis (1996) for a general classification of real options.

[36]To simplify our analysis, we assume perfect information of shareholders and equity-holders and the absence of agency costs between shareholders and equity-holders and equity-holders and management. Hence, investment is implemented under a *first-best investment policy* (i.e., a policy aiming at maximizing the total project/firm value as opposed to a policy in the sole interest of shareholders) by the managers. The role of agency costs of debts on investment decisions have been analyzed among others by Mello and Parsons (1992), Mauer and Ott (2000) and Childs et al. (2000).

We assume that the capital expenditure to implement the project is also partially financed with incremental debt, and the incremental debt is issued if and when the option to invest is exercised.[37] This assumption is realistic since there would be no reason to raise capital before investment, so incurring in a (useless) opportunity cost of capital. The optimal exercise policy depends on X_t, the EBIT process, and consequently the date we will issue debt is a stopping time. Issuance of incremental debt is contingent on the decision to invest, so yielding that the financing decision is influenced by the investment decision. Conversely, the investment decision is influenced by the financing decision, since the former is made if the return from the project compensates its financing cost. The debt policy is alternatively the one in Section 3.2 (default-free) or the one in Section 3.3 (defaultable). The debt has duration T^d, so that it is issued at T_I and is paid-back at $T_I + T^d$.

The value of the levered project, at the date it is implemented, is $V(t, X_t)$ from equation (3.5) in the case with default-free debt or from equation (3.9) with defaultable debt. In case default is possible, given the above assumption that debt is issued conditional on the investment decision, we assume that default can happen only after the investment date. Let Π denote the payoff of the option at the exercise date,

$$\Pi(t, X_t) = \max\{V(t, X_t) - I, 0\}, \tag{4.1}$$

and let $F(t, X_t)$ denote the value of the investment project including the time-value of the option to postpone the decision.

Proposition 8. *The value of the option to delay investment satisfies equation*

$$\frac{1}{2}\sigma^2(X_t, t)\frac{\partial^2 F}{\partial X^2} + g(X_t, t)\frac{\partial F}{\partial X} + \frac{\partial F}{\partial t} = r^z F. \tag{4.2}$$

with boundary condition $F(T_I, X_{T_I}) = \Pi(T_I, X_{T_I})$, and is

$$F(t, x) = B_t^z \sup_{T_I} \hat{\mathbb{E}}_t \left[\frac{\Pi(T_I, X_{T_I})}{B_{T_I}^z} \right], \tag{4.3}$$

where $T_I \leq T^o$ is the investment (stopping) time, i.e., the first time $F(t, X_t) = \Pi(t, X_t)$.

Proof. See the Appendix. □

[37] Mauer and Ott (2000) and Childs et al. (2000) assume that the incremental investment is financed only with equity. Their assumption is included in our framework by posing that no incremental debt is issued at the time the investment decision is made.

From equation (4.3) it is worth stressing that, while the option to delay investment is still alive, the appropriate CE discount factor to be applied to its expected payoff is the one for equity flows, B^z. Obviously, this CE discount factor is independent of both the *current* capital structure, in case the investment option is owned by an ongoing firm, and the capital structure *after* the project is implemented.

In addition, equation (4.3) suggests that the option to invest in a marginal project (i.e., a project with corporate tax rate, τ^c, equal to the marginal tax rate, τ_m, and no tax shield $\tau^* = 0$) is evaluated according to Black, Scholes, and Merton's formula, but using r_z instead of r_f. Note that in our setting, since a project cannot be all-debt financed, r_f is *never used* but when $\tau^e = \tau^b$ (which implies $\tau_m = 0$).

5 Debt and investment policy

In this section we will analyze the impact of the exogenous debt policy on the value of the investment option, and hence on the investment policy. In addition, we will also analyze the impact of debt on the time-value of the option as well as on the probability of investing, and therefore, on the exercise policy.

Differently from the previous sections, the analysis in the current section will rely as much as possible on closed-form formulas for the value of the tax shields and for the value of the options. For this reason we will confine ourselves to the case with τ^c, τ^e, τ^b, r^f, r^z constant, and in (3.1), $g(X,t) = gX$, $\sigma(X,t) = \sigma X$, with constant g and σ. This is done in the interest of simplicity, since otherwise we would have to present numerical valuations with no substantial additional economic insight. Moreover, since the tax shield is valuable when $\tau^c > \tau_m$, we will spell out the results for a supra-marginal firm/project, noticing that the result are of the opposite sign for an infra-marginal firm/project ($\tau^c < \tau_m$).

In relation to our first goal, to analyze the impact of an exogenous debt policy on the investment policy, starting from Proposition 8, which states the valuation principle for the investment option assuming that the project is partially debt financed, we provide an approximate solution for equation (4.3) applying an analytic approximation akin the one in MacMillan (1986) and Barone-Adesi and Whaley (1987) proposed for American options.

Assuming that the maturity of debt is equal to the maturity of the project, $T^d = T^p$, given $V(t,x)$, the value of the levered project from equations (A.18) and (A.19) (or, if the M-E debt policy is assumed, $D = LV$, from (A.20)) in the default-free case or from equation (A.22) in the defaultable case, we can approximate the value of the option to invest, $F(t,x)$ in (4.3), with payoff $\Pi(t,X_t)$ from (4.1) with constant I and maturity T^o, using an analytic approximation. In the default-free case, we assume also that $k(t)$ in (A.19) is strictly positive, so that $V(t,x)>0$.[38] In defaultable case, we assume that $x>x_D$, so that $V(t,x)>0$.

Proposition 9. *The value of the option to invest is approximated by*

$$\widetilde{F}(t,x)=\begin{cases} f(t,x)+\varphi x^{\eta}\left(1-e^{-r^z(T^o-t)}\right) & \text{if } x<x_t^* \\ V(t,x)-I & \text{if } x\geq x_t^* \end{cases} \tag{5.1}$$

for given constants $\eta>1$, $\varphi>0$ and x_t^*0, where

$$f(t,x)=e^{-(r_z-g)(T^o-t)}\mathcal{N}(m_1)V(t,x)-e^{r_z(T^o-t)}\mathcal{N}(m_2)I$$

is the value of the (related) European *option with maturity T^o and payoff* $\Pi(t,X_t)$ *from* (4.1), *with*

$$m_1=\frac{\log\frac{V(t,x)}{I}+\left(g+\frac{\sigma^2}{2}\right)(T^o-t)}{\sigma\sqrt{T^o-t}}, \qquad m_2=m_1-\sigma\sqrt{T^o-t}.$$

Proof. See the Appendix. □

Note that x_t^* in Proposition 9 depends on t. This means that the above approximation must be done at any time $0\leq t\leq T^o$ to properly define a time-dependent investment policy, $\{x_t^*\}$.

In addition, with respect to the second goal, to analyze the effect of the debt policy on the exercise policy, we compute the time-value of the investment option,[39] as well as the probability of investing (assuming that currently the opportunity is still available) within the time horizon T^o. Since the probability we are interested in is under the actual measure, and not under the EMM, we have to compute the risk premium, denoted Φ, for the stochastic process X_t.[40] We will denote $\hat{g}=g+\Phi$ the drift for process X_t under the actual probability measure.

At t, assuming that the option to defer has not been exercised yet, this is equivalent to compute the probability that X_t touches (from below) the

[38] Sufficient conditions for $k(t)>0$ for a supra-marginal firm/project are $r^z-r^f\tau^*L>g$ and $\tau^*>0$. See also footnote 55.

[39] It is computed in the regular way: option value minus intrinsic value.

[40] In Appendix A, we present a way to determine the risk premium for the EBIT process, X_t.

investment threshold $\{x_s^* \mid t < s \leq T\}$, as computed using the approximation introduced in Proposition 9. According to Harrison (1985, pp. 11–14) this probability is[41]

$$H(T^o, x_t^*, x, t) = \mathcal{N}(p_1) + \mathcal{N}(p_2)\left(\frac{x_t^*}{x}\right)^{2\hat{g}/\sigma^2 - 1} \tag{5.2}$$

where

$$p_1 = \frac{\log\frac{x}{x_t^*} + \left(\hat{g} - \frac{\sigma^2}{2}\right)(T^o - t)}{\sigma\sqrt{T^o - t}}, \qquad p_2 = p_1 - \left(\frac{2\hat{g}}{\sigma^2} - 1\right)\sigma\sqrt{T^o - t}.$$

Equations (5.1) and (5.2), the value of the option and the probability of investing respectively, simplify if we assume that the time-horizon for the project, the debt, and the investment option is infinite: $T^p = T^d = T^o = \infty$.

Hence, in the default-free case, (5.1) becomes[42]

$$F(x) = \begin{cases} \frac{Kx^*}{\eta}\left(\frac{x}{x^*}\right)^\eta & x < x^* \\ Kx - I & x \geq x^* \end{cases}, \quad \text{where} \quad x^* = \frac{\eta}{\eta - 1}\frac{I}{K} \quad \text{is no longer} \tag{5.3}$$

time dependent and $K = (1 - \tau^c)/(\rho - g)$ from equation (A.20).

In case debt is defaultable, (5.1) becomes

$$F(x) = \begin{cases} \frac{1}{\beta_1}\left(\frac{x(1-\tau^c)}{r^z - g} - \beta_2\left(\frac{x}{x_D}\right)^{\beta_2}\left(\frac{\tau^* R}{r^z} + \frac{\alpha x_D(1-\tau^c)}{r^z - g}\right)\right)\left(\frac{x}{x^*}\right)^{\beta_1} & \text{if } x_D \leq x < x^* \\ V(t,x) - I & \text{if } x > x^*, \end{cases} \tag{5.4}$$

where $V(t,x)$ is defined in equation (A.28) and x^* is the root of equation[43]

$$\frac{1}{\beta_1}\left(\frac{x(1-\tau^c)}{r^z - g} - \beta_2\left(\frac{x}{x_D}\right)^{\beta_2}\left(\frac{\tau^* R}{r^z} + \frac{\alpha x_D(1-\tau^c)}{r^z - g}\right)\right) = V(t,x) - I,$$

and is independent of t.

[41] Actually, H is the probability of the first time $X_t = x_t^*$, with initial condition $x_t < x_t^*$. Nevertheless, x_t^* is the current investment threshold. Since x_s^* is decreasing over time, H in equation (5.2) is actually a lower bound for the probability of investing.

[42] Note that in this case the solution is exact and not approximated: $F^\infty = F$.

[43] This equation must be solved numerically.

Similarly, equation (5.2) considerably simplifies with the assumption of $T^p = T^d = T^o = \infty$. By straightforward algebra, the actual probability of investing simplifies to[44]

$$H(x^*, x) = \begin{cases} 1 & \text{if } \hat{g} - \sigma^2/2 \geq 0 \\ \left(\frac{x^*}{x}\right)^{\frac{2\hat{g}}{\sigma^2}-1} & \text{if } \hat{g} - \sigma^2/2 < 0, \end{cases} \tag{5.5}$$

when $x < x^*$, with $\hat{g} = g + \Phi$, where x^* can be either the one in (5.3) in the default-free case or the one in (5.4) in the defaultable case.

In the infinite horizon case ($T^p = T^d = T^o = \infty$), it can be checked that, for a supra-marginal project/firm (τ^*0), assuming that debt is default-free, the value of the option to invest and the probability of investing are increasing function of leverage, L. In fact, F and x^* in equation (5.3) are respectively increasing and decreasing with respect to K, and K is an increasing function of L. Moreover, H from equation (5.5) is a decreasing function of x^*. The opposite is true for an infra-marginal project. As we can see from subsequent numerical analysis, this is confirmed also in the finite horizon case. Things are completely different in the case with defaultable debt. To show this we will resort to numerical analysis.

In the finite horizon setting, and for the the supra-marginal case $\tau^* > 0$,[45] we will analyze the effect of debt and uncertainty on the APV, on the value of the option to defer, on its time-value and on the exercise policy. We will do so by discussing a numerical example and running a sensitivity analysis on the afore enumerated key parameters: the uncertainty of EBIT, σ and the level of debt (i.e., the total coupon payment, R, in the defaultable case and leverage, L, in the default-free case). Although we chose a specific set of parameters (see base case parameters in Table 1) for presentation purposes, the results we show are general.

[Table 1 about here]

For these parameters, $\tau^* = \tau^c - \tau^m > 0$. Moreover, the APV of the project in the default-free case is, from equation (A.20), $V = 5.014$ and the market value of debt is $D = L \cdot V \approx 2.858$. On the other hand, in the defaultable case, the APV of the project, from equation (A.22), is $V = 5.197$

[44]Since the investment threshold, x^*, is independent of t, the valuation formula for probability is exact.

[45]It is easy to check that $\widetilde{F}$ and $\widetilde{P}$ are not affected by the debt policy when $\tau^* = 0$. This is a motivation for having introduced the general Miller equilibrium framework where tax shields may be valuable.

and the market value of debt is, from equation (A.27), $D = 2.960$. We set the face value of debt $P \approx R/r^f$. To make the values in the defaultable case comparable to the values in the default-free case, we set the same initial debt proportion (at market values), $L = D/V = 0.569 \approx 0.57$, for both cases.

In the subsequent analyses we will make debt (i.e., total coupon payment, R in the defaultable case and leverage, L, in the default-free case) vary in a given range: accordingly, in the defaultable case, we will assume that the default threshold x_D is always equal to R. Hence, for the given EBIT rate, $x = 1$, the higher R the closer to default is the firm/project.

The first analysis concerns the effect of uncertainty and debt on the APV and on the value of the tax shield. From Figure 1 (above) we can see that in the default-free (M-E) case, leverage has always a positive effect and uncertainty has no role in valuation.[46] The outcome is completely different in the defaultable case, Figure 1 (below). Here both uncertainty and leverage have a role: for any level of uncertainty, there is a debt level (represented by total coupon payment, R) that maximizes the APV. Moreover, the lower the uncertainty the higher the optimal level of debt. For high uncertainty and debt, the APV becomes negligible. For further clarification, in Figure 2 (above) we plot the tax shield from equation (A.26) for the same range of values for σ and R. This shows that the humped-shape of the APV is due to the tax shield. Actually, the base value (Figure 2 (below)), defined as the APV less the tax shield, is almost constant when R is low and default is unlikely and become negligible when R is large and default is almost certain. The above represents a generalization of the results provided by Brennan and Schwartz (1978).

[Figure 1 about here]

[Figure 2 about here]

We next explore the effect of uncertainty and debt on the value of the investment option along with on its time-value. Starting with the default free case, from Figure 3 (above), it can be seen that volatility has a positive effect on the value of the investment option, whereas leverage has no influence at all. The positive effect of volatility derives from its positive effect on the time-value of the option (see Figure 4). Likewise, leverage has a negative

[46] In our computations we are assuming that the additional uncertainty does not affect the risk premium, and hence does not change the value of the project. On the other hand, if there was an increment in systematic risk, the APV would be reduced. We are not interested here to capture this effect in our analysis and concentrate on the effect of uncertainty on the tax shield.

impact on the time-value. This can be explained by observing that while leverage has no effect on the value of the option, it does have a positive effect on APV, and then, on the intrinsic value of the option, i.e. $APV - I$.

In case of defaultable debt (see Figure 3 (below)), the effect of debt and uncertainty differs from that of the default-free case. Still, for every level of uncertainty, there is an optimal debt level which maximizes the value of the option to invest. This is mostly due to the effect of the tax shield on the APV of the project, which is the underlying asset of the option. Yet, for low leverage, the value of the option to invest is an increasing function of volatility, since the increase in the time-value (see Figure 4 (below)) overweighs the drop in the value of tax shield, and hence on the APV, due to the higher probability of default. However, for high leverage, the value of the option to delay investment is not a monotonic function of volatility of EBIT. When volatility is low, an increase on this parameter causes a drop in the option value, whereas the opposite positive effect holds for the case of high volatility. The former positive behavior can be explained by observing that, this time, the negative effect of volatility on APV is greater than the positive effect on the time-value of the option. The latter positive effect can be explained in the same manner, noticing that now the domination is reversed. Finally, it is worth noticing that any level of σ a rise in time-value is a convex function of leverage. This behavior is exactly matched by a corresponding behavior (with the opposite sign) of the probability of investing with respect to leverage that will be described below.

[Figure 3 about here]

[Figure 4 about here]

As noted above, both in the default-free case and in the defaultable case, uncertainty increases the time-value of the option, and debt reduces the time-value in the default-free case, whereas it has a positive effect in the defaultable case. So it is natural to analyze the effect of uncertainty and leverage on the probability of investing. In Figure 5 we plot the probability of investing before maturity in the default-free case (above) and defaultable case (below). In the first case, leverage increases the probability of investing for any level of uncertainty, because the higher the leverage, the higher the value of the underlying asset of the option; on the other hand, for a given leverage, uncertainty first has a positive and then a negative effect on the probability of investing.[47] In the defaultable case (Figure 5 (below)), the

[47]This behavior of the probability of investing with respect to volatility of the value-

effect of debt is positive only when σ is low. As mentioned before, this effect corresponds to the negative effect on the time-value. Otherwise we have that, when the probability of default is large (high coupon), more debt reduces the probability of investing (increases the time-value) and uncertainty has only a negative effect (a positive effect on the time-value).

[Figure 5 about here]

6 Concluding Remarks

We have shown how to implement the value of interest tax shields in a real options setting and investigated the implications of these on real options development policy.

First, we had to study interest tax shields more rigourously than is common in the literature because the differential taxation of debt and equity income at the personal level has the effect of making interest tax shields less valuable than is commonly believed. Indeed, for firms with low tax rates, interest tax shields can have negative value. In a generalized Miller tax equilibrium, there will be marginal firms and investors who are indifferent between the tax implications of debt and equity. Their tax rates can be used to determine the differential rate of return on riskless debt and riskless equity that is needed to sustain such a tax equilbrium.

This allowed us to characterize interest tax shields in terms of the difference between a firm's tax rate and the marginal firm's tax rate.

Having cash flows at a pre-corporate tax level, after-corporate tax level and an after-all-tax level leads to natural questions of the level at which cash flows should be valued. The most appropriate valuation occurs at the after-all-tax level for the marginal investor. Unfortunately, such cash flows are not readily observable. Fortunately, we have been able to show that if there is linear taxation at the personal level, a linear valuation operator after all taxes leads to a linear valuation operator after corporate tax but prior to personal tax. Moreover, the certainty-equivalent operators or risk-neutral expectation operators of these pretax and after-tax cash flows are the same. The only difference in valuation is the choice of discount rates: a riskless equity rate is correctly used to discount risk-neutral expected equity flows, a riskless debt rate is correctly used to discount risk-neutral expected debt flows and an after-all-tax rate is used to discount risk-neutral expected flows after all tax.

driver is well known. See Sarkar (2000) and Cappuccio and Moretto (2001) for more details.

In terms of the fundamental pde for valuing risky payoffs or interest tax shields, this means that the only change needed to reflect that the flows go to equity or debt investors is to change the riskless discount rate in the risk-neutral required return. The coefficients corresponding to risk adjustments are the same for the equity and debt pde, despite the differential taxation.

This allows us to value interest tax shields with and without default risk on the debt.

It also allows us to study the impact of interest tax shields on the decision of when to develop a real option. Using an extension of the analytic approximation to the value of an American option by MacMillan and Barone-Adesi and Whaley, we were able to calculate the value of a real option and optimal real option strategy in the presence of differential taxes on debt and equity and interest tax shields. Taking the special case of a corporation with a higher tax rate than the marginal firm, we find that interest tax shields are always valuable. When there is no offseting default risk, the firm has incentives to become fully levered. However, when there is a probability of default, we find that there is an optimal debt ratio.

These results extend to the real option to invest. Without default risk, this real option increases in value as leverage increases, but in the presence of default risk, there is an optimal choice of debt policy that maximizes real option value.

Option value also increases with volatility and time to maturity of the option in the presence of these taxes.

However, the probabilty of investing by developing the real option has some non-monotonic features. In the default-free case, the probability of exercise initially increases with volatility and eventually declines with volatility. In the defaultable debt case, there is a coupon rate that maximizes the probability of investment.

Thus, interest tax shields and debt do have an effect on real asset valuation and real option development strategy.

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A Proofs of propositions

Proof of Proposition 1

We first prove the Proposition for equity assuming that no dividend is paid. Then we extend the result to the case that dividend are actually paid. Finally, we extend the result to the money market account.

Setting $t = T'$ in Equation (2.8) establishes that $S_{T'} = S_{T'}^{\text{at}}$. If there is no dividend ($X = 0$), we will show that the price in (2.7) and the price in (2.8) obey the same dynamic equation (under $\mathbb{Q}$) with the riskless discount rate for equity r^z:

$$\hat{\mathbb{E}}_t \left[dS_t \right] = r_t^z S_t dt \tag{A.1}$$

and

$$\hat{\mathbb{E}}_t \left[dS_t^{\text{at}} \right] = r_t^z S_t^{\text{at}} dt \ . \tag{A.2}$$

Then, by the Feynman-Kac solution to the valuation equation (see, e.g. Duffie (2001, Ch. 5)), the security prices at a date $t \leq T'$ are the risk-neutral expectations of the common terminal value $S_{T'}^{\text{at}} = S_{T'}$

$$\begin{aligned} S_t &= B_t^z \hat{\mathbb{E}}_t \left[\frac{S_{T'}}{B_{T'}^z} \right] \\ &= S_t^{\text{at}} \ . \end{aligned}$$

The proof that S_t in equation (2.7) satisfies condition (A.1) is the standard martingale result. Thus, it remains to prove that S_t^{at} defined in equation (2.8) satisfies the equivalent relation (A.2). From equation (2.8) we have, for small Δt,

$$\begin{aligned} \hat{\mathbb{E}}_t \left[S_{t+\Delta t}^{\text{at}} \right] &= \hat{\mathbb{E}}_t \left[B_{t+\Delta t}^{z,\text{at}} \hat{\mathbb{E}}_{t+\Delta t} \left[\frac{S_{T'}}{B_{T'}^{z,\text{at}}} - \int_{t+\Delta t}^{T'} \frac{\tau_u^e}{B_u^{z,\text{at}}} dS_u^{\text{at}} \right] \right] \\ &= \hat{\mathbb{E}}_t \left[B_{t+\Delta t}^{z,\text{at}} \left(\frac{S_{T'}}{B_{T'}^{z,\text{at}}} - \int_{t}^{T'} \frac{\tau_u^e}{B_u^{z,\text{at}}} dS_u^{\text{at}} + \int_{t}^{t+\Delta t} \frac{\tau_u^e}{B_u^{z,\text{at}}} dS_u^{\text{at}} \right) \right] \\ &= \left(1 + r_t^{z,\text{at}} \Delta t \right) \left(S_t^{\text{at}} + B_t^{z,\text{at}} \hat{\mathbb{E}}_t \left[\int_{t}^{t+\Delta t} \frac{\tau_u^e}{B_u^{z,\text{at}}} dS_u^{\text{at}} \right] \right) + o(\Delta t) \ . \end{aligned} \tag{A.3}$$

Here $o(\Delta t)$ denotes terms that converge to zero as $\Delta t \to 0$. The second equality comes from breaking out the tax term for the interval $[t, t+\Delta t]$ and applying $\hat{\mathbb{E}}_t\left[\hat{\mathbb{E}}_{t+\Delta t}[\cdot]\right] = \hat{\mathbb{E}}_t[\cdot]$. The third equality comes from substituting for S_t^{at} from (2.8) after using the following Taylor approximation for the riskless equity account value:[48]

$$\begin{aligned} B_{t+\Delta t}^{z,\text{at}} &= \exp(r_t^{z,\text{at}}\Delta t) B_t^{z,\text{at}} + o(\Delta t) \\ &= B_t^{z,\text{at}}\left(1 + r_t^{z,\text{at}}\Delta t\right) + o(\Delta t)\ . \end{aligned}$$

Since the tax rate τ_u^e varies continuously over time, we have that

$$\int_t^{t+\Delta t} \frac{\tau_u^e}{B_u^{z,\text{at}}} dS_u^{\text{at}} = \frac{\tau_t^e\left(S_{t+\Delta t}^{\text{at}} - S_t^{\text{at}}\right)}{B_t^{z,\text{at}}} + o(\Delta t)\ ,$$

so we can rewrite (A.3) as:

$$\hat{\mathbb{E}}_t\left[S_{t+\Delta t}^{\text{at}}\right] = \left(1 + r_t^{z,\text{at}}\Delta t\right)\left(S_t^{\text{at}} + \tau_t^e\,\hat{\mathbb{E}}_t\left[S_{t+\Delta t}^{\text{at}} - S_t^{\text{at}}\right]\right) + o(\Delta t)\ .$$

Rearranging, we get

$$\begin{aligned} (1-\tau_t^e)\,\hat{\mathbb{E}}_t\left[S_{t+\Delta t}^{\text{at}} - S_t^{\text{at}}\right] &= r_t^{z,\text{at}} S_t^{\text{at}}\Delta t + o(\Delta t) \\ &= r_t^z(1-\tau_t^e) S_t^{\text{at}}\Delta t + o(\Delta t)\ . \end{aligned}$$

Since $\tau_t^e < 1$, dividing by $(1-\tau_t^e)$ gives

$$\hat{\mathbb{E}}_t\left[S_{t+\Delta t}^{\text{at}} - S_t^{\text{at}}\right] = r_t^z \Delta t S_t^{\text{at}} + o(\Delta t)\ .$$

By taking $\Delta t \to 0$ we get condition (A.2).

If dividends are paid, $(X \neq 0)$, the only thing that changes in the proof is that instead of (A.1), the dynamic condition is

$$\hat{\mathbb{E}}_t\left[dS_t\right] + X_t dt = r_t^z S_t.$$

We must establish the analogue of (A.2), which is

$$\hat{\mathbb{E}}_t\left[dS_t^{\text{at}}\right] + X_t dt = r_t^z S_t^{\text{at}}\ .$$

[48]We are also using the fact that $r_t^{z,\text{at}}$ and τ_t^e are adapted.

Inserting the after-tax dividend term in the analogue of (A.3) yields

$$
\begin{aligned}
\hat{\mathbb{E}}_t\left[S^{\mathrm{at}}_{t+\Delta t}\right] = &\left(1+r^{z,\mathrm{at}}_t\Delta t\right)\Bigg(S^{\mathrm{at}}_t \\
&+B^{z,\mathrm{at}}_t\hat{\mathbb{E}}_t\left[-\int_t^{t+\Delta t}\frac{(1-\tau^e_u)X_u}{B^{z,\mathrm{at}}_u}du+\int_t^{t+\Delta t}\frac{\tau^e_u}{B^{z,\mathrm{at}}_u}dS^{\mathrm{at}}_u\right]\Bigg) \\
&+o(\Delta t) \\
= &\left(1+r^{z,\mathrm{at}}_t\Delta t\right)\Bigg(S^{\mathrm{at}}_t \\
&+B^{z,\mathrm{at}}_t\hat{\mathbb{E}}_t\left[-(1-\tau^e_t)\int_t^{t+\Delta t}\frac{X_u}{B^{z,\mathrm{at}}_u}du+\int_t^{t+\Delta t}\frac{\tau^e_u}{B^{z,\mathrm{at}}_u}dS^{\mathrm{at}}_u\right]\Bigg) \\
&+o(\Delta t)\ .
\end{aligned}
$$

The first equality follows from the development of (A.3) and the second comes from the continuity of the tax rate τ^e_u.

Rearranging as before, we have

$$
(1-\tau^e_t)\,\hat{\mathbb{E}}_t\left[S^{\mathrm{at}}_{t+\Delta t}-S^{\mathrm{at}}_t+\int_t^{t+\Delta t}\frac{X_u}{B^{z,\mathrm{at}}_u}du\right]=r^z_t(1-\tau^e_t)S^{\mathrm{at}}_t\Delta t+o(\Delta t)\ .
$$

Dividing by $(1-\tau^e_t)$ and letting $\Delta t\to 0$ gives the desired result.

For the money market account, it is now straightforward to follow the same steps as above and show that

$$
\hat{\mathbb{E}}_t\left[dB_t\right]=dB_t=r_fB_tdt \tag{A.4}
$$

and

$$
\hat{\mathbb{E}}_t\left[dB^{\mathrm{at}}_t\right]=r_fB^{\mathrm{at}}_tdt \tag{A.5}
$$

The first is simply our definition of bond price dynamics. The second proof comes from appropriate modifications of the analogous result we just established for stocks. Since we defined the after-all-tax bond value so that it equals the bond value at T' (i.e. $B_{T'}=B^{\mathrm{at}}_{T'}$, we can take the risk-neutral expectations to get the desired result:

$$
B_t=B_t\hat{\mathbb{E}}_t\left[\frac{B_{T'}}{B_{T'}}\right]=B^{\mathrm{at}}_t
$$

□

Proof of Corollary 2

Note that the proof of Proposition 1 started with a risk-neutral measure $\mathbb{Q}$ for stock S_t before personal taxes and evaluated payoffs after personal taxes to derive a process of after-personal-tax stock prices S_t^{at} using the same risk-neutral measure. This means that the CE operator is the same before and after tax. $\square$

Proof of Proposition 4

We will phrase the proof for a stock, but the argument is the same also for a money market account.

Since the argument for proving holding-period neutrality is not affected by the presence of dividends, because they are taxed at the date they are paid, we will assume for simplicity $X = 0$ in equation (2.8). We define the *tax account* at t as the value of the taxes on capital gains accrued until t at the after-tax rate of return:

$$A_t = B_t^{f,\text{at}} \int_0^t \frac{\tau_u^e}{B_u^{f,\text{at}}} dS_u.$$

By definition, equation (2.8) can be written as

$$\frac{S_t}{B_t^{f,\text{at}}} - \frac{A_t}{B_t^{f,\text{at}}} = \hat{\mathbb{E}}_t \left[\frac{S_T}{B_T^{f,\text{at}}} - \frac{A_T}{B_T^{f,\text{at}}} \right]. \tag{A.6}$$

Assume that at the (arbitrarily chosen) stopping time t, $0 \leq t < T$ the investor decides to liquidate the position. The net proceeds are the price of the stock net of the taxes on capital gains accrued from $t = 0$ until that date: $S_t - A_t$.

If $A_t > 0$, then he can borrow an amount A_t at an after-personal tax cost[49] $\{r_u^f(1 - \tau_u^b)\}_{u=t}^T$ and with the proceeds he can buy the same stock at S_t. At T, by liquidating the position in the stock, he will receive $S_T - B_T^{f,\text{at}} \int_t^T \tau_u^e / B_u^{f,\text{at}} dS_u$ (i.e., the stock price less the tax on capital from t to T), and paying back the loan, $-A_t B_T^{f,\text{at}} / B_t^{f,\text{at}}$, so that the net payoff is

$$\begin{aligned} S_T - B_T^{f,\text{at}} \int_t^T \frac{\tau_u^e}{B_u^{f,\text{at}}} dS_u - A_t \frac{B_T^{f,\text{at}}}{B_t^{f,\text{at}}} \\ = S_T - B_T^{f,\text{at}} \int_t^T \frac{\tau_u^e}{B_u^{f,\text{at}}} dS_u - B_T^{f,\text{at}} \int_0^t \frac{\tau_u^e}{B_u^{f,\text{at}}} dS_u = S_T - A_T \end{aligned}$$

[49] This means that borrowing generates a tax saving proportional to τ^b. Note that, from equilibrium, the after tax cost of borrowing is equal to $r^z(1 - \tau^e)$.

which is equal to the payoff of the buy-and-hold strategy over the interval $[0, T]$.

On the other hand, if $A_t < 0$, then by liquidating the position at t, the amount $-A_t$ can be invested at the (after-personal tax) rate $\{r_u^f(1-\tau_u^b)\}_{u=t}^T$ over the interval $[t, T]$ and S_t can be used to buy the same stock. By applying the same argument, at T the net payoff is again $S_T - A_T$, which is the same of the buy-and-hold strategy. $\square$

Proof of Proposition 5

Consider the process for EBIT, under the actual probability measure,

$$dX_t = \hat{g}(X_t, t)dt + \sigma(X_t, t)dZ_t.$$

Since the EBIT process is not traded, for valuation purposes we assume that there is a spanning (twin) security/portfolio, whose price is denoted S, with process, under the actual probability measure,

$$dS_t = \alpha(X_t, t)S_t dt + \beta(X_t, t)S_t dZ_t \tag{A.7}$$

and dividend rate $\delta(X_t, t)$. Moreover, a money market account is available, with value B^f, following the process $dB^f = r_t^f B_t^f dt$.

To derive the valuation equation

$$\frac{1}{2}\sigma^2(X,t)\frac{\partial^2 V}{\partial X^2} + g(X,t)\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + X_t(1-\tau_t^c) + \tau^* R_t = r_t^z V. \tag{3.4}$$

we employ the standard replicating argument introduced by Modigliani and Miller (1958) and extended to a continuous-time framework by Merton (1973), but on an after personal tax basis.

The rate of return of V,[50] using Itô's Lemma, is (for notational simplicity, we drop functional dependence when it is clear)

$$\begin{aligned}
\frac{dV + X(1-\tau^c) + \tau^* R}{V}(1-\tau^e) = \\
\frac{1}{V}\left(X(1-\tau^c) + \tau^* R + \frac{\partial V}{\partial t} + \hat{g}\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial X^2}\right)(1-\tau^e)dt \\
+ \frac{1}{V}\frac{\partial V}{\partial X}\sigma(1-\tau^e)dZ_t.
\end{aligned} \tag{A.8}$$

[50]Note that, from (3.3), the value of the levered asset accrues to equity-holders, and hence is taxed at the tax rate for equity flows, τ^e.

Taking a long position in a portfolio with price $W = n_1 B^f + n_2 S$ we obtain a rate of return

$$\left(1 - \frac{n_2 S}{P}\right) r^f (1-\tau^b) dt + \frac{n_2 S}{W}\left(\alpha + \frac{\delta}{S}\right)(1-\tau^e) dt + \frac{n_2 S}{W}\beta(1-\tau^e) dZ_t \quad \text{(A.9)}$$

Putting

$$\frac{n_2 S}{W} = \frac{\sigma}{\beta}\frac{\partial V/\partial X}{V}$$

in (A.8), given the generalized Miller equilibrium relation (2.4) and after some manipulations, we have

$$\frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X^2} + \left(\hat{g} - \sigma \frac{\alpha + \delta/S - r^z}{\beta}\right)\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + X(1-\tau^c) + \tau^* R = r^z V.$$

Considering the unit risk premium computed from the twin security/portfolio

$$\lambda = \frac{\alpha + \delta/S - r^z}{\beta} \quad \text{(A.10)}$$

we obtain equation (3.4) under the EMM, with $g = \hat{g} - \lambda\sigma$.

With condition $V(T^d, X_{T^d}) = U(T^d, X_{T^d})$ and using the Feynman-Kac solution of (3.4), see Duffie (2001, pp. 340–346), we have (3.5). □

Proof of Proposition 6

The proof that the APV satisfies equation (3.4) while $X_t > x_D$ is the same as in Proposition 5. Considering the boundary conditions $V(T^d, x) = U(T^d, x)$ and $U(T^p, x) = 0$ and $V(s, x_D) = (1-\alpha)U(s, X)$ for all $t \le s \le T^d$, the existence of the solution of this problem follows from the application of a version of Feynman-Kac result for partial differential equations on open bounded sets, see Lamberton and Lapeyre (1996, Th. 5.1.9).

For verification of (3.9), we define

$$Y_s = \frac{B_t^z}{B_s^z} V(s, X_s).$$

By applying Itô's Lemma, we have

$$Y_{T_D \wedge T^d} = V(t,x) + \int_t^{T_D \wedge T^d} \frac{B_t^z}{B_s^z}\left(\frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X^2} + g\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} - r^z V\right) ds + \int_t^{T_D \wedge T^d} \frac{B_t^z}{B_s^z}\frac{\partial V}{\partial X}\sigma dZ_s. \quad \text{(A.11)}$$

We take expectations of both sides of (A.11) and note that:

$$\mathbb{E}_t \left[\int_t^{T_D \wedge T^d} \frac{B_t^z}{B_s^z} \frac{\partial V}{\partial X} \sigma dZ_s \right] = 0;$$

$$\mathbb{E}_t \left[\int_t^{T_D \wedge T^d} \frac{B_t^z}{B_s^z} \left(\frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial X^2} + g \frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} - r^z V \right) ds \right]$$
$$= -\mathbb{E}_t \left[\int_t^{T_D \wedge T^d} \frac{B_t^z}{B_s^z} \left(X_t (1 - \tau_t^c) + \tau_t^* R_t \right) ds \right],$$

from equation (3.4), and

$$\mathbb{E}_t \left[Y_{T_D \wedge T^d} \right] = \mathbb{E}_t \left[\chi_{\left\{ \forall s \in [t, T^d], X_s > x_D \right\}} \frac{B_t^z}{B_{T^d}^z} V(T^d, X_{T^d}) \right]$$
$$+ \mathbb{E}_t \left[\chi_{\left\{ \exists s \in [t, T^d], X_s = x_D \right\}} \frac{B_t^z}{B_{T_D}^z} V(T_D, X_{T_D}) \right].$$

Hence, from equation (A.11) and the boundary conditions of the problem, we have

$$V(t,x) = \mathbb{E}_t \left[\chi_{\left\{ \forall s \in [t, T^d], X_s > x_D \right\}} \frac{B_t^z}{B_{T^d}^z} U(T^d, X_{T^d}) \right]$$
$$+ (1 - \alpha) \mathbb{E}_t \left[\chi_{\left\{ \exists s \in [t, T^d], X_s = x_D \right\}} \frac{B_t^z}{B_{T_D}^z} U(T_D, X_{T_D}) \right]$$
$$+ \mathbb{E}_t \left[\int_t^{T_D \wedge T^d} \frac{B_t^z}{B_s^z} X_t (1 - \tau_t^c) ds \right] + \mathbb{E}_t \left[\int_t^{T_D \wedge T^d} \frac{B_t^z}{B_s^z} \tau_t^* R_t ds \right]$$

that is, after few manipulations, equation (3.9). □

Proof of Proposition 7

To derive the valuation pde (3.10) for D, we follow the same argument we used in the proof of Proposition 5 and create a portfolio whose return replicates the return of the bond.

The return of the bond is, by Itô's Lemma

$$\frac{dD + Rdt}{D}(1-\tau^b) = \frac{1}{D}\left(\frac{1}{2}\sigma^2\frac{\partial^2 D}{\partial X^2} + \hat{g}\frac{\partial D}{\partial X} + \frac{\partial D}{\partial t} + R\right)(1-\tau^b)dt + \frac{1}{D}\frac{\partial D}{\partial X}\sigma(1-\tau^b)dZ_t$$

A long position in a portfolio with price $W = n_1 B^f + n_2 S$, where B^f is the value of the money market account and S, as of (A.7), is the price of a (twin) security/portfolio for X have a rate of return as in equation (A.9). Putting

$$\frac{n_2 S}{W} = \frac{\partial D/\partial X}{D}\frac{\sigma}{\beta}\frac{(1-\tau^b)}{(1-\tau^e)}$$

we obtain, after a few manipulations and using the generalized Miller equilibrium relation (2.4),

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial X^2} + \left(\hat{g} - \sigma\frac{\alpha + \delta/S - r^z}{\beta}\right)\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + R = r^f V$$

that is (3.10) when $g = \hat{g} - \lambda\sigma$ in (A.10).

The existence of the solution for this problem can be derived from Lamberton and Lapeyre (1996, Th. 5.1.9), using the same argument we used before. We skip it in the interest of brevity. □

Proof of Proposition 8

The proof that F satisfies equation (4.2) is the same as for the proof of Proposition 5. The solution in (4.3) is derived using standard results (see Duffie (2001, pp. 182–186)). □

Proof of Proposition 9

The value of the *European* option with maturity T^o and payoff $\Pi(t, X_t)$ from (4.1) is

$$f(t, X) = e^{-(r_z - g)(T^o - t)}\mathcal{N}(m_1)V(t, x) - e^{r_z(T^o - t)}\mathcal{N}(m_2)I$$

with

$$m_1(x) = \frac{\log\frac{V(t,x)}{I} + \left(g + \frac{\sigma^2}{2}\right)(T^o - t)}{\sigma\sqrt{T^o - t}}, \qquad m_2 = m_1 - \sigma\sqrt{T^o - t}.$$

where $V(t,x) = xK(t)$ from equations (A.17), (A.18) and (A.19) in the default-free case, with

$$\begin{aligned} K(t) = (1-\tau^c)&\frac{1-e^{-(r^z-g)(T^p-t)}}{r^z-g} \\ &+ (1-\tau^c) r^f \tau^* L \frac{\left(1-e^{-(r^z-g)(T^d-t)}\right)}{(r^z-g)(r^z-r^f\tau^*L-g)} \\ &\qquad + (1-\tau^c) e^{-(r^z-g)(T^d-t)} \frac{\left(1-e^{(r^f\tau^*L)(T^d-t)}\right)}{(r^z-r^f\tau^*L-g)}, \end{aligned}$$

and $V(t,x)$ as defined in equation (A.22) in the defaultable case.

The early-exercise premium of the real option to delay investment, denoted $E(t,x) = F(t,x) - f(t,x)$, satisfies equation

$$\frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 E}{\partial X^2} + g X_t \frac{\partial E}{\partial X} + \frac{\partial E}{\partial t} = r^z E. \tag{A.12}$$

E is approximated by

$$\widetilde{E}(t,x) = \varphi x^\eta h(T^o - t),$$

where $h(s) = 1 - e^{-r_z s}$ and φ and η are parameters to be determined. By replacing $\widetilde{E}$ in equation (A.12), we get

$$\eta(\eta-1) + 2\frac{g}{\sigma}\eta - 2\frac{r^z}{\sigma^2 h(T^o-t)} = 0.$$

η is known and is the positive root of the above equation:

$$\eta(r^z) = \eta = \frac{1}{2} - \frac{g}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{g}{\sigma^2}\right)^2 + 2\frac{r_z}{\sigma^2 h(T^o-t)}} > 1. \tag{A.13}$$

At the critical level for the cash flow rate, denoted x_t^*, the value of the real options, F, must satisfies the value-matching and smooth-pasting conditions. Hence, for the approximation $\widetilde{F}(t,x) = f(t,x) + \widetilde{E}(t,x)$ the following conditions hold:

$$f(t,x_t^*) + \varphi\,(x_t^*)^\eta\, h(T^o-t) = V(t,x_t^*) - I \tag{A.14}$$

$$\left.\frac{\partial f(t,x)}{\partial X}\right|_{x=x_t^*} + \varphi\eta\,(x_t^*)^{\eta-1}\, h(T^o-t) = \left.\frac{\partial V(t,x)}{\partial X}\right|_{x=x_t^*}. \tag{A.15}$$

We solve the above system for φ and x_t^*. In the defaultable case, we have to rely on a numerical solution for both φ and x_t^*. Existence and uniqueness of solution is given by the strict monotonicity of $V(t,x)$ with respect to x, for $x > x_D$.

On the other hand, in the default-free case, equation (A.15) can be simplified as follows

$$e^{-(r_z-g)(T^o-t)}\mathcal{N}(m_1(x_t^*))K(t) + \varphi\eta\,(x_t^*)^{\eta-1}\,h(T^o-t) = K(t). \tag{A.16}$$

From (A.16) we solve for φ:

$$\varphi = \frac{K(t)\left(1 - e^{-(r^z-g)(T^o-t)}\mathcal{N}(m_1(x_t^*))\right)}{(x^*)^{\eta-1}\eta h(T^o-t)}.$$

By replacing this in (A.14) we solve equation

$$f(t,x_t^*) + \left(1 - e^{-(r^z-g)(T^o-t)}\mathcal{N}(m_1(x_t^*))\right)\frac{K(t)x_t^*}{\eta} = V(t,x_t^*) - I$$

for x_t^* (using a numerical method).

This completely determines the approximation $\widetilde{F}$ as in equation (5.1).

□

Closed form solutions for the APV, the debt and the tax shield

For convenience, in this Appendix we derive the closed-form solution for the APV, the tax shield and the debt under more restrictive assumptions on the stochastic process for X and the model parameters. These closed form solutions are used to carry on the analysis in Section 5. In details we assume here that: the corporate (τ^c) and the personal (τ^e and τ^b) tax rates are constant; the CE rates for bonds (r^f) and equities (r^z) are constant;[51] for the EBIT process in (3.1), the growth rate, $g(X,t) = gX$, and the volatility, $\sigma(X,t) = \sigma X$, where g and σ are given constant.

Default-free debt

Under these hypotheses, the value of the unlevered firm/project in (3.2) becomes

$$U(t,x) = x(1-\tau^c)\frac{1 - e^{-(r^z-g)(T^p-t)}}{r^z-g}. \tag{A.17}$$

[51] This hypothesis implies in particular that the yield curve for bonds is flat and hence maturity for corporate bonds is irrelevant.

In order to derive the APV of the firm/project, we observe that the solution is of the type

$$V(t,x) = U(t,x) + k(t)x \tag{A.18}$$

(where k is a function of time to be determined) and hence debt tax shield at t is related to the current value of X_t. By replacing (A.18) in (3.4) (and considering that $D = LV$) we have[52]

$$k(t) = (1-\tau^c)r^f\tau^*L\frac{\left(1-e^{-(r^z-g)(T^d-t)}\right)}{(r^z-g)(r^z-r^f\tau^*L-g)}+ \\ + (1-\tau^c)e^{-(r^z-g)(T^d-t)}\frac{\left(1-e^{(r^f\tau^*L)(T^d-t)}\right)}{(r^z-r^f\tau^*L-g)} \tag{A.19}$$

for $0 \leq t \leq T^d$. We remark that for a marginal firm/project $\tau^c = \tau^m$ (or $\tau^* = 0$), and hence $k \equiv 0$. In addition, as it is expected, $k \equiv 0$ if the company has no debt. Replacing $k(t)$ from (A.19) in (A.18) and after few manipulations we obtain

$$V(t,x) = x\left((1-\tau^c)\frac{1-e^{(\rho-g)(T^d-t)}}{\rho-g} \right. \\ \left. + e^{-(r^z-g)(T^d-t)}(1-\tau^c)\frac{1-e^{-(r^z-g)(T^p-T^d)}}{r^z-g}\right) = \gamma(t)x, \tag{A.20}$$

with $\rho = \rho_t$ for all t from equation (3.7), which is the analogue of equation (3.8) under the current more restrictive assumptions.[53]

In capital budgeting it is customary to use the (current) wacc to discount expected free cash flows in order to obtain the levered asset value. Under the current more restrictive hypotheses, wacc from equation (3.7) is non-stochastic and hence it can be used for valuation purposes.[54]

---

[52]To obtain equation (A.19) we note that U from equation (3.2) satisfies the pde

$$\frac{1}{2}\sigma^2(X,t)\frac{\partial^2 U}{\partial X^2} + g(X,t)\frac{\partial U}{\partial X} + \frac{\partial U}{\partial t} + X_t(1-\tau^c) = r_t^z U.$$

Moreover, we consider the terminal condition $k(T^d) = 0$.

[53]Equation (A.20) and condition $D = LV$ permit us to prove that the M-E debt policy is consistent with the assumption of absence of default, since R_t is a linear function of X_t: $R_t = \gamma(t)X_t$, where $\gamma(t)$ is determined by the results stated above. Hence, in case $X \to 0$, also $R \to 0$, and the debt is default-free.

[54]This is in contrast with the conclusions of Grinblatt and Liu (2002), who show that

If the firm/project (and debt) is infinite-lived, $T^d = T^p = \infty$, assuming that the conditions for convergence, $r^z - r^f \tau^* L > g$ and $r^z > g$,[55] hold true, then from equation (A.20) (note that $D = LV$ and $R_t = r^f D(x)$)

$$V(x) = U(x) + \frac{\tau^* R_t}{r^z - g}, \tag{A.21}$$

with $U(x) = x(1-\tau^c)/(r^z - g)$. Equation (A.21) is the analogue of equation (3.5) when debt is default-free, using the M-E constant leverage ratio and the EBIT process is a geometric Brownian motion with constant parameters.

Defaultable debt

In this case, besides the assumptions introduced above, we assume that the operations are financed with a coupon bond with constant coupon payment, R, and principal, P, payable at maturity $T^d \leq T^p$. This is the Modigliani and Miller (1958) constant debt extended to the case of defaultable debt.

Under these assumptions, from (3.9) and considering the analytic formula for the density of the hitting time T_D, the value of the firm is, for $x > x_D$,

$$\begin{aligned} V(t,x) = U(t,x) + \frac{\tau^* R}{r^z} - \left(\frac{\tau^* R}{r^z} + \frac{\alpha x_D (1-\tau^c)}{r^z - g} \right) G(T^d, x_D, x, t, r^z) \\ + \frac{\alpha x_D (1-\tau^c)}{r^z - g} e^{-(r^z - g)(T^p - t)} G(T^d, x_D, x, t, g) \\ - \frac{\tau^* R}{r^z} e^{-r^z (T^d - t)} \left(1 - H(T^d, x_D, x, t) \right) \end{aligned} \tag{A.22}$$

where $U(t,x)$ is defined in equation (A.17),

$$H(T^d, x_D, x, t) = \mathcal{N}(-p_1)) + \mathcal{N}(-p_2) \left(\frac{x}{x_D} \right)^{2g/\sigma^2 - 1} \tag{A.23}$$

$$p_1 = \frac{\log \frac{x}{x_D} + \left(g - \frac{\sigma^2}{2} \right)(T^p - t)}{\sigma \sqrt{T^d - t}},$$

the circumstances for a non-stochastic wacc are, besides the fact that debt is default-free, X following a geometric Brownian motion with constant parameters and constant risk-free rate, that the firm/project is perpetual. Here we have just proved that the constant (initial) wacc can be used, under the M-E scheme, also if debt has finite maturity.

[55] Note that, for a supra-marginal firm, $r^z - r^f \tau^* L > g$ implies $r^z > g$.

$$p_2 = \frac{\log \frac{x}{x_D} - \left(g - \frac{\sigma^2}{2}\right)(T^p - t)}{\sigma\sqrt{T^d - t}},$$

with $\mathcal{N}(\cdot)$ denoting the cumulative Normal distribution, and where

$$G(T^d, x_D, x, t, r) = \left(\frac{x}{x_D}\right)^{\beta_1(r)} \mathcal{N}\left(-q_1(r)\right) + \left(\frac{x}{x_D}\right)^{\beta_2(r)} \mathcal{N}\left(-q_2(r)\right), \tag{A.24}$$

with

$$\beta_1(r) = -\left(\frac{g}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{g}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}} > 1,$$

$$\beta_2(r) = -\left(\frac{g}{\sigma^2} - \frac{1}{2}\right) - \sqrt{\left(\frac{g}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}} < 0,$$

$$q_1(r) = \frac{\log \frac{x}{x_D} + \sqrt{2r\sigma^2 + \left(g - \frac{\sigma^2}{2}\right)^2}(T^d - t)}{\sigma\sqrt{T^d - t}},$$

$$q_2(r) = \frac{\log \frac{x}{x_D} - \sqrt{2r\sigma^2 + \left(g - \frac{\sigma^2}{2}\right)^2}(T^d - t)}{\sigma\sqrt{(T^d - t)}}.$$

Note that in equation (A.22), for $x > x_D$, $V(t, x) > 0$.

Equation (A.22) can be derived as follows. From (3.9), we have

$$\begin{aligned} V(t, x) = x(1 - \tau^c)&\frac{1 - e^{-(r^z - g)(T^p - t)}}{r^{z-g}} \\ &+ \tau^* R \int_t^{T^d} e^{-r^z(s-t)} \left(1 - H(s, x_D, x, t)\right) ds \\ &- \int_t^{T^d} e^{-r^z(s-t)} \alpha x_D (1 - \tau^c) \frac{1 - e^{-(r^z - g)(T^p - s)}}{r^z - g} h(s, x_D, x, t) ds \end{aligned}$$

where $h(s, x_D, x, t)$ is the density and $H(s, x_D, x, t)$ is the cumulative distribution function of the first hitting time s to x_D for a geometric Brownian motion, under the EMM, with starting point $X_t = x$. Integrating by parts the second term in the right-hand-side of the above equation and after few manipulations we obtain

$$
\begin{aligned}
V(t,x) = U(t,x) &+ \frac{\tau^* R}{r^z} \\
&- \left(\frac{\tau^* R}{r^z} + \frac{\alpha x_D(1-\tau^c)}{r^z - g}\right) \int_t^{T^d} e^{-r^z(s-t)} h(s, x_D, x, t) ds \\
&+ \frac{\alpha x_D(1-\tau^c)}{r^z - g} e^{(r^z-g)(T^p-t)} \int_t^{T^d} e^{-g(s-t)} h(s, x_D, x, t) ds \\
&\qquad - \frac{\tau^* R}{r^z} e^{-r^z(T^d-t)} \left(1 - H(s, x_D, x, t)\right)
\end{aligned}
$$

where U is defined in (A.17). We denote

$$
G(T^d, x_D, x, t, r) = \int_t^{T^d} e^{-r(s-t)} h(s, x_D, x, t) ds. \tag{A.25}
$$

G is explicitly computed in equation (A.24) (details in Reiner and Rubinstein (1991)) and H from equation (A.23) can be found in Harrison (1985, pp. 11–14).

In Section 5 we will need also the value of the tax shield and debt in the defaultable case under the more restrictive assumptions introduced above. As for the tax shield, from equation (A.22) we have

$$
\begin{aligned}
TS(t,x) = \frac{\tau^* R}{r^z} &- \frac{\tau^* R}{r^z} G(T^d, x_D, x, t, r^z) \\
&- \frac{\tau^* R}{r^z} e^{-r^z(T^d-t)} \left(1 - H(T^d, x_D, x, t)\right)
\end{aligned} \tag{A.26}
$$

where H and G are defined respectively in (A.23) and (A.24).

As for debt, from Proposition 7:

$$
\begin{aligned}
D(t,x) = \frac{R}{r^f} &- \frac{R}{r^f} G(T^d, x_D, x, t, r^f) \\
&+ e^{-r^f(T^d-t)} \left(P - \frac{R}{r^f}\right) \left(1 - H(T^d, x_D, x, t)\right) \\
&+ \frac{(1-\alpha) x_D(1-\tau^c)}{r^z - g} \Big(G(T^d, x_D, x, t, r^f) \\
&\qquad + e^{-(r^z-g)(T^p-t)} G(T^d, x_D, x, t, g + r^f \tau^m)\Big)
\end{aligned} \tag{A.27}
$$

where H and G are defined respectively in (A.23) and (A.24).

Equation (A.27) is derived as follows. From (3.11) we have

$$D(t,x) = R\int_t^{T^d} e^{-r^f(s-t)}\left(1 - H(s, x_D, x, t)\right) ds \\ + e^{-r^f(T^d-t)}P(1 - H(T^d, x_D, x, t)) \\ + \frac{(1-\alpha)x_D(1-\tau^c)}{r^z - g}\int_t^{T^d} e^{-r^f(s-t)}\left(1 - e^{-(r^z-g)(T^p-s)}\right) h(s, x_D, x, t)ds.$$

Integrating by parts the first term in the right-hand-side of the equation above and after few manipulations, we obtain

$$D(t,x) = \frac{R}{r^f} - \frac{R}{r^f}\int_t^{T^d} e^{-r^f(s-t)} h(s, x_D, x, t)ds \\ + e^{-r^f(T^d-t)}\left(P - \frac{R}{r^f}\right)\left(1 - H(T^d, x_D, x, t)\right) \\ + \frac{(1-\alpha)x_D(1-\tau^c)}{r^z - g}\left(\int_t^{T^d} e^{-r^f(s-t)} h(s, x_D, x, t)ds \right. \\ \left. + e^{-(r^z-g)(T^p-t)}\int_t^{T^d} e^{-(r^f-r^z+g)(s-t)} h(s, x_D, x, t)ds\right).$$

Using definition (A.25) and from the generalized Miller equilibrium relation in (2.3), we have (A.27).

The valuation formula for bond in (A.27) should be compared to the ones in Leland (1994, Eq. (7)) and Leland and Toft (1996, Eq. (3)). The first difference with respect to those formulas are that here the value driver is the EBIT process, whereas there the driver is the value of the asset. More important is the difference induced by the differential taxation of equity flows and bond flows.[56]

If the firm/project and the debt are perpetual, $T^d = T^p = \infty$, then (A.22) simplifies to[57]

$$V(x) = U(x) + \frac{\tau^* R}{r^z}\left(1 - \left(\frac{x}{x_D}\right)^{\beta_2(r^z)}\right) - \frac{\alpha x_D(1-\tau^c)}{r^z - g}\left(\frac{x}{x_D}\right)^{\beta_2(r^z)} \tag{A.28}$$

[56]In Leland (1994, Footnote 27), personal taxation is introduced into the model, and the effective tax advantage to debt, τ^*, is used in place of the corporate tax rate, τ^c. Nevertheless, the same risk-free rate is used for valuing equity flows, like the tax shield, and bond flows, like the coupon payment.

[57] Alternatively, equation (A.28) can be derived also considering that, under the simplifying assumptions introduced above and since V is independent of time because the

for $x > x_D$ and where $U(x) = x(1-\tau^c)/(r^z - g)$, and (A.27) is reduced to[58]

$$D(x) = \frac{R}{r^f} + \left(\frac{(1-\alpha)x_D(1-\tau^c)}{r^z - g} - \frac{R}{r^f}\right)\left(\frac{x}{x_D}\right)^{\beta_2(r^f)} \tag{A.29}$$

for $x > x_D$.[59,60]

horizon is infinite, then the valuation pde in (3.4) reduces to

$$\frac{1}{2}\sigma^2 X_t^2 V_{xx} + gX_t V_x + X_t(1-\tau^c) + \tau^* R = r^z V.$$

where subscripts of V denote partial derivatives. The general solution for this pde is $V(x) = Ax^{\beta_2(r^z)} + (1-\tau^c)x/(r^z - g) + \tau^* R/r^z$. By imposing the value-matching condition, for $X_t = x_D$, $V(x_D) = (1-\alpha)(1-\tau^c)x_D/(r^z - g)$, we solve for A and, by replacing A into the expression of the general solution we obtain (A.28).

[58]The derivation of equation (A.29) can be done following the same argument used in Footnote 57.

[59]It is possible to extend the discussion of the infinite horizon case to incorporate an endogenous default, by selecting a threshold x_D such that the value of equity is maximized. This entails first the valuation of debt and next the valuation of equity. Then by imposing the smooth-pasting condition on equity, the optimal value of x_D would be determined. For brevity we skip this part. The related equations are available from the Authors on request.

[60]In accordance with Brennan and Schwartz (1978), if $\alpha = 0$ in equation A.29, then, debt always adds value for supramarginal companies.

Table 1: Option to delay: base case parameters

r_z	CE return for stocks	0.07	
τ^c	corporate tax rate	0.4	
τ^e	tax rate for equity flows	0.1	
τ^b	tax rate for bond flows	0.2	
x	current EBIT rate	1	
g	risk-neutral growth rate of X_t	0.02	
σ	volatility of X_t	0.35	
Φ	risk-premium rate for X_t	0.05	
T^p	duration of the project	10	(years)
T^d	duration of debt	10	(years)
T^o	expiry of the option to delay	1	(years)
α	bankruptcy (proportional) costs	0.25	
L	leverage (in M-E)	0.57	
R	total coupon payment (in M-M)	0.3	($)
P	face value of bond (in M-M)	3.8	($)
x_D	exogenous default threshold (in M-M)	0.3	
I	capital expenditure	5	($)

For these parameters, from (2.2), $\tau^m = 0.111$, $\tau^* = \tau^c - \tau^m = 0.289$, from (2.3), $r^f = 0.079$, and, from equation (3.7), $\rho = 0.057$.

The APV of the project in the default-free case is, from equation (A.20), $V = 5.014$ and the market value of debt is $D = L \cdot V \approx 2.858$.

The APV of the project in the defaultable case is, from equation (A.22), $V = 5.197$ and the market value of debt is, from equation (A.27), $D = 2.960$. Note that the face value is set so that $P \approx R/r^f$. The initial leverage in the defaultable case is $L = D/V = 0.569 \approx 0.57$, that is the same for the the default-free case.

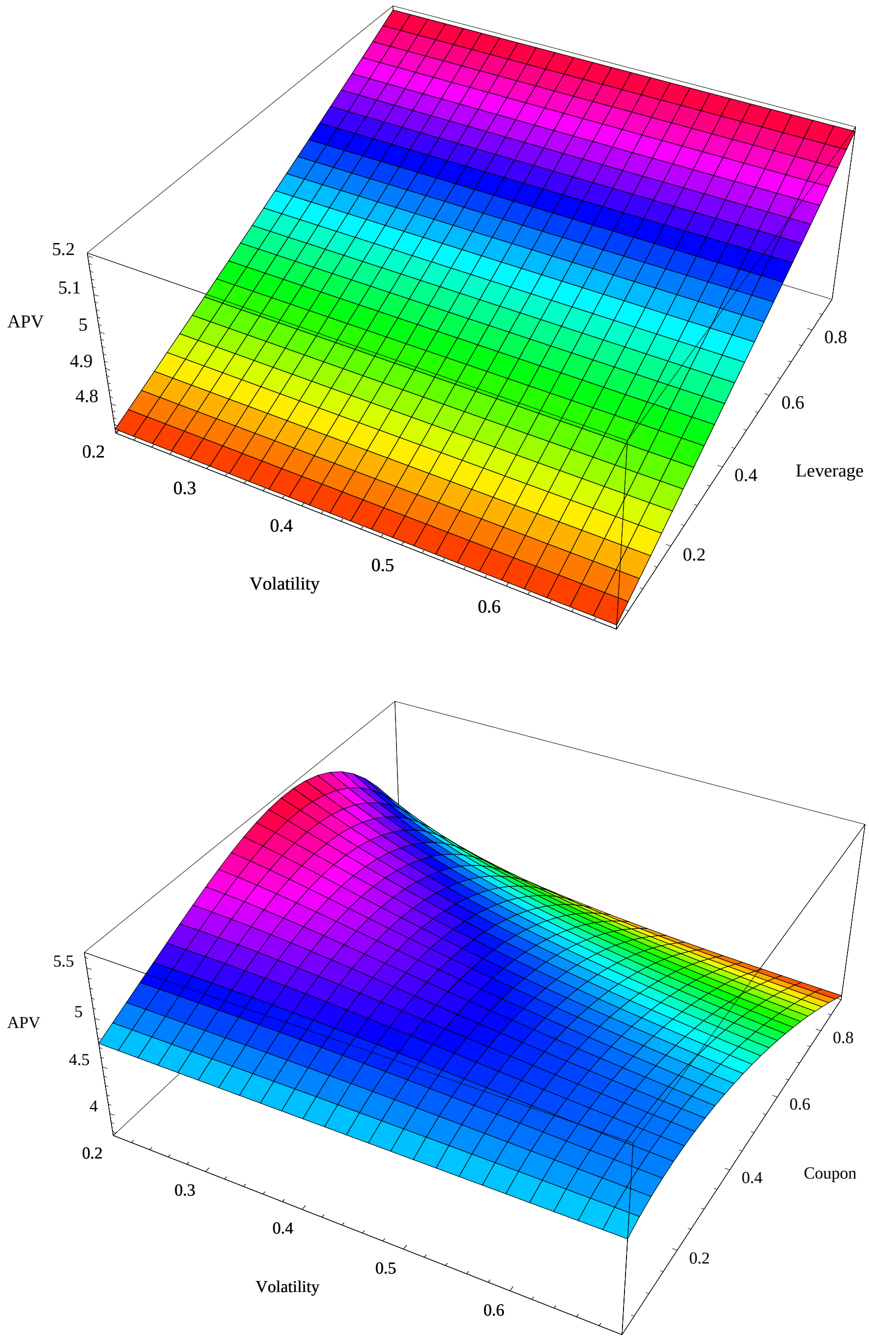


Figure 1: **APV of the project**. Value of the project, V, including the debt tax shield, vs volatility of EBIT process, σ, and debt level (represented by the coupon payment, R, in the defaultable case and by leverage, L, in the default-free case). Default-free case is above and defaultable case is below. The other parameters are from Table 1.

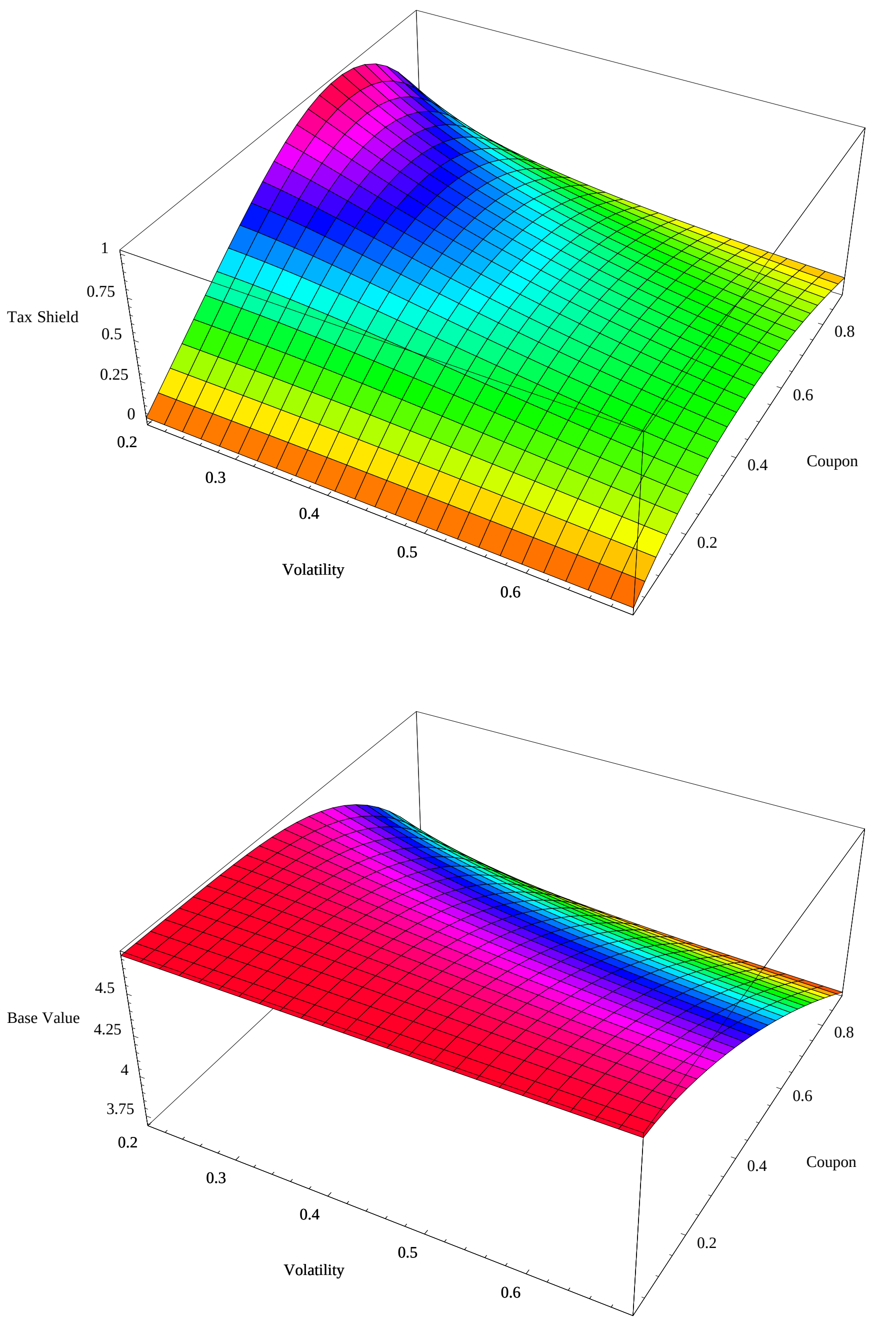


Figure 2: **Tax Shield and Base Value - defaultable case**. Value of the tax shield, TS, and base value (defined as the APV net of the tax shield), vs volatility of EBIT process, σ, and debt level (represented by the coupon payment, R). The other parameters are from Table 1.

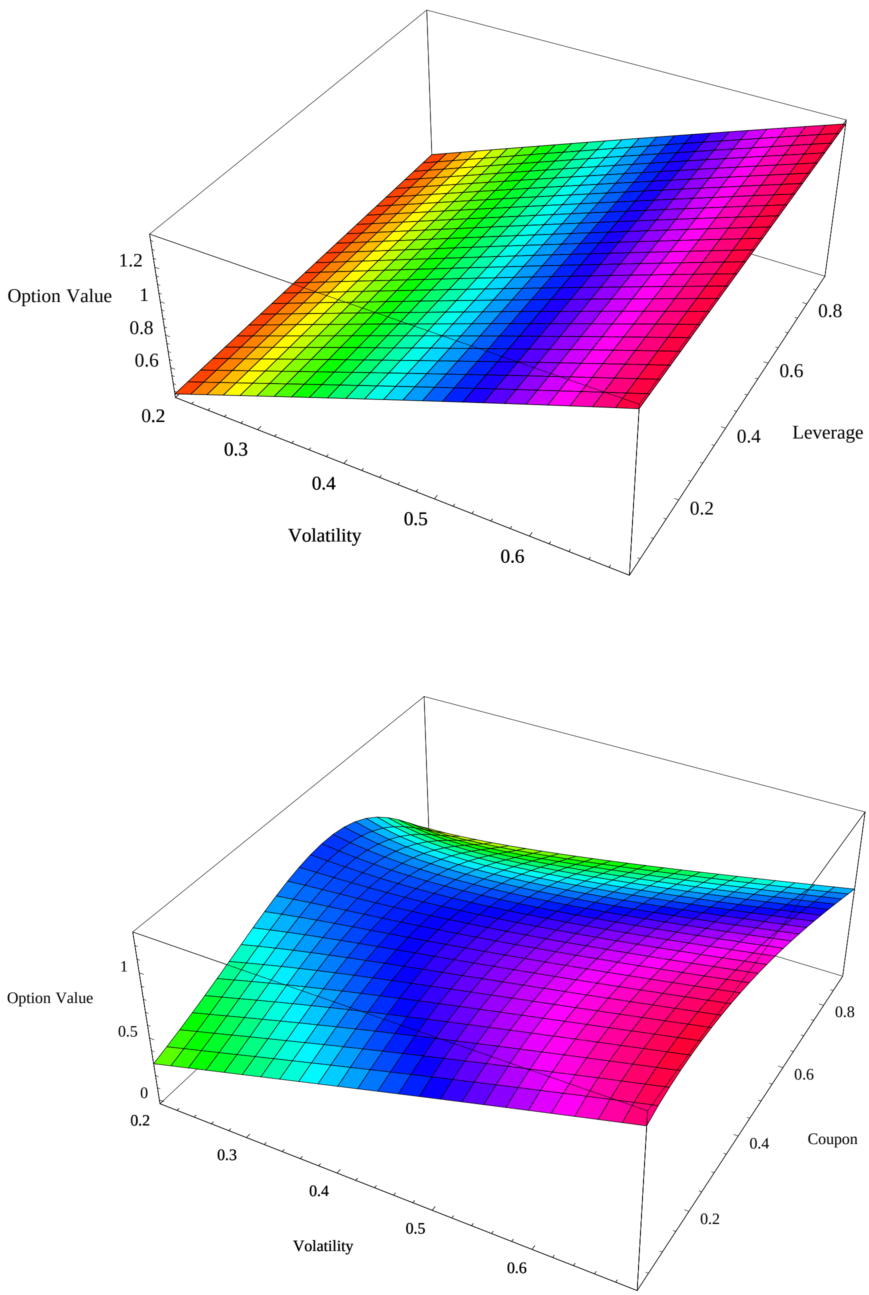


Figure 3: **Option to invest**. Value of the option to invest, $\widetilde{F}$, vs volatility of EBIT process, σ, and debt level (represented by the coupon payment, R, in the defaultable case and by leverage, L, in the default-free case). Default-free case is above and defaultable case is below. The other parameters are from Table 1.

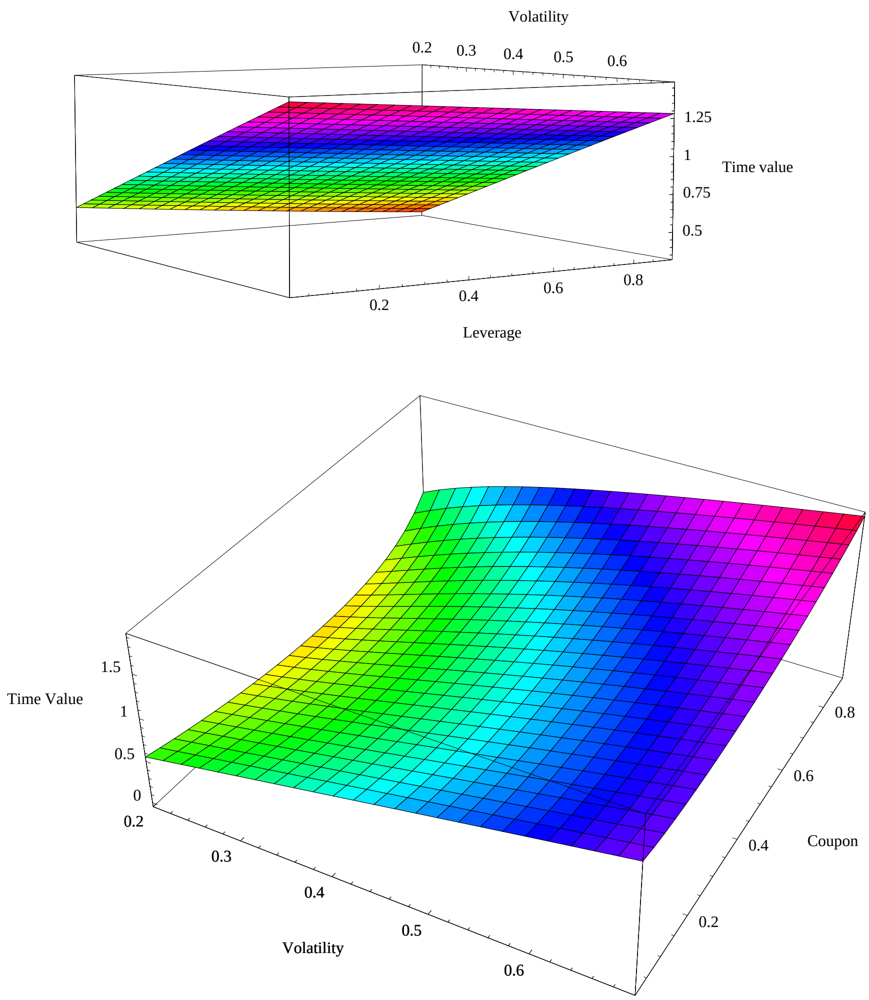
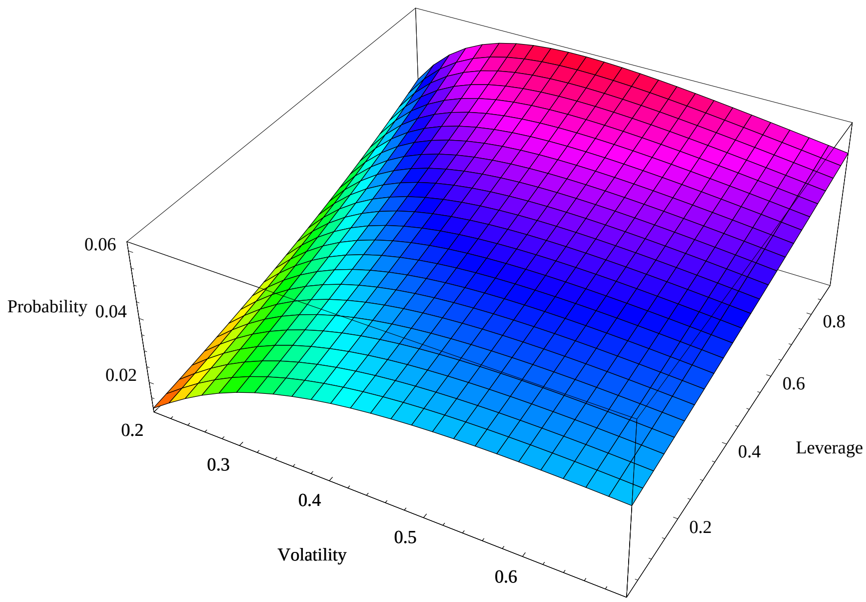


Figure 4: **Time value**. Time value of the option to invest, $\widetilde{F}-(APV-I)$, vs volatility of EBIT process, σ, and debt level (represented by the coupon payment, R, in the defaultable case and by leverage, L, in the default-free case). Default-free case is above and defaultable case is below. The other parameters are from Table 1.



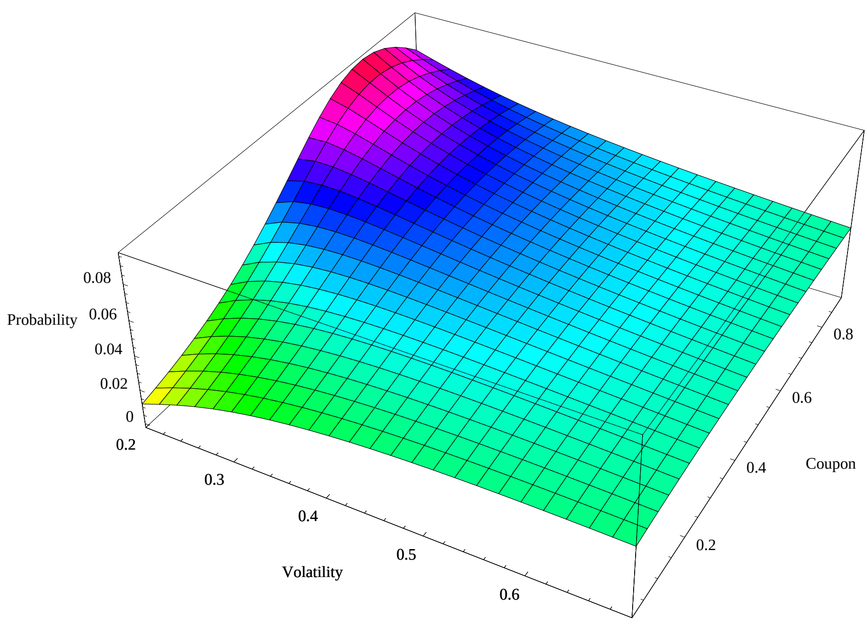


Figure 5: **Probability of investing**. Probability of exercising the option to invest, H, vs volatility of EBIT process, σ, and debt level (represented by the coupon payment, R, in the defaultable case and by leverage, L, in the default-free case). Default-free case is above and defaultable case is below. The other parameters are from Table 1.

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