

# Investment under Uncertainty, Debt and Taxes

current version, October 16, 2006

Andrea Gamba  
Department of Economics, University of Verona  
Verona, Italy  
email: [andrea.gamba@univr.it](mailto:andrea.gamba@univr.it)

Gordon A. Sick  
Haskayne School of Business, University of Calgary  
Calgary, Canada  
email: [sick@ucalgary.ca](mailto:sick@ucalgary.ca)

Carmen Aranda León  
University of Navarra  
Pamplona, Spain  
email: [sick@ucalgary.ca](mailto:sick@ucalgary.ca)

Corresponding author:

Gordon A. Sick  
Haskayne School of Business  
University of Calgary  
Calgary, Alberta  
Canada T2N 1N4  
email: [maranda@unav.es](mailto:maranda@unav.es)

# Investment under Uncertainty, Debt and Taxes

## Abstract

We present a capital budgeting valuation framework that takes into account both personal and corporate taxation. We show broad circumstances under which taxes do not affect the martingale expectations operator. That is, the martingale operator is the same before and after personal taxes, which we call “valuation neutrality”.

The appropriate discount rate for riskless equity-financed flows (martingale expectations or certainty-equivalents) is an equity rate that differs from the riskless debt rate by a tax wedge. This tax wedge factor is the after-tax retention rate for the corporate tax rate that corresponds to tax neutrality in the Miller equilibrium. We extend this result to the valuation of the interest tax shield for exogenous debt policy with default risk. Interest tax shields accrue at a net rate corresponding to the difference between the corporate tax rate that will be faced by the project and the Miller equilibrium tax rate. Depending on the financing system, interest tax shields can be incorporated by using a tax-adjusted discount rate or by implementing an APV-like approach with additive interest tax shields.

We also provide an illustrative real options application of our valuation approach to the case of an option to delay investment in a project, showing that the application of Black and Scholes formula may be incorrect in presence of personal and corporate taxes.

**Keywords:** Investment under uncertainty, real options, capital structure, risk-neutral valuation, corporate and personal taxation, default risk, interest tax shields, cost of capital, tax-adjusted discount rates.

**JEL classifications:** G31, G32, C61

# 1 Introduction

Many models for valuing or optimizing capital investment decisions under uncertainty are based on cash flows discounted at a risk-adjusted rate. However, the presence of leverage, discretion and asymmetry forces a valuation approach based on computation of certainty-equivalent, rather than expected, cash flows. This includes the *real options approach*.

The market maker who sets the relative prices of financial derivatives and their underlying instruments is typically taxed at the same rate on all financial instruments. Thus, the common practice of discounting the certainty equivalent payoff of financial derivatives (puts and calls on stocks, for example) at the riskless debt rate is appropriate. However, we show that it is incorrect to carry this practice over to a real options setting because the marginal investor of a capital investment project is likely to face a differential taxation according to the type of instrument (i.e, equity or debt). In fact, the certainty equivalent rate of return for equity funds is typically lower than that of debt funds.<sup>1</sup>

In addition, the interaction of personal and corporate taxes on various financing instruments generates a net interest tax shield that may have positive or negative value. This interaction is so often mishandled that few people recognize that it could just as likely have a negative value as a positive value.

However, the following features commonly found in capital budgeting problems suggest that the best valuation approach is to use certainty equivalent operators, tax-adjusted

---

<sup>1</sup>The fact that a financial market maker does not use the same discount rate to value an investment as a long-term capital investor would use does seem to lead to some arbitrage opportunities that we do not explore. It may be difficult to take advantage of these arbitrage opportunities because a dynamic hedge based on the long term capital value may generate adverse tax consequences, or may be hard to form because of incomplete markets.

On the other hand, it might be possible to synthesize the cash flows of a commodity producer, such as an oil company with very predictable reserves and production by a strip of forward or futures contracts on oil and a strip of bonds. Both of these instruments are financial instruments and would be valued by discounting at the bond rate. They could be sold to an investor to replace an equity position in such a firm, and that investor may discount the flows at the riskless equity rate, which is less than the riskless debt rate. This could represent a simple tax arbitrage.

discount rates for equity and a more sophisticated analysis of interest tax shields:

1. Projects are financed partially by equity and debt.
2. Personal and corporate taxes compound, and taxation of debt income differs from that of equity income.
3. Some cash flows from a project, such as the tax advantage to debt are contingent on the cash flows of the project and are lost in the event of default.

In this paper we first introduce a continuous-time valuation framework for cash flows emerging in capital budgeting that takes into account personal and corporate taxes. We show broad valuation neutrality circumstances under which taxes do not affect the equivalent martingale measure.

Second, the paper shows how to adjust the value of asymmetric cash flows to compute the value of debt and taxes under two different types of settings: default-free and defaultable debt.

We can assume that debt is default-free if management constantly revises the debt level of the firm to maintain a constant debt ratio.<sup>2</sup> This assumption, first due to Miles and Ezzel (1980, 1985), is popular in capital budgeting (e.g., Grinblatt and Liu (2002)), and is considered as the benchmark case here. We find that the value of the interest tax shield can be calculated either by an additive term that separates the value under different financing scenarios (an adjusted present value or APV treatment), or by adjusting the discount rate to reflect the tax wedge that separates the riskless market returns for instruments of

---

<sup>2</sup>Stewart Myers has pointed out in private communication that if debt is continuously rebalanced to keep a constant debt ratio, there can be no default on the debt. As the firm's cash flows fall, its value falls, so the firm must repurchase debt and replace it with equity to keep the debt ratio constant. By the time the firm approaches bankruptcy, there is no debt upon which to default.

In discrete time, there could be default even with a policy of rebalancing debt to keep a constant debt ratio at the rebalance dates.

different tax classes in what we will call a tax-adjusted discount rate (TADR) approach. Later, we assume that debt is defaultable and incorporate the effect of the probability of default into the valuation.

The paper concludes by discussing an application of our results to a real options setting. For clarity sake, we have selected the basic case of an option to delay an investment decision. We demonstrate that the the project can be evaluated according to the Black, Scholes, and Merton formula. However, using the riskless debt rate as the discount factor would be a mistake, because the personal tax rate for bond investment income and that for stock investment income are usually different.

The valuation of asymmetric cash flow is ubiquitous in corporate finance. To price corporate securities, Merton (1974) sought to develop a theory for pricing bonds along Black-Scholes lines with a significant probability of default. However, he did not include taxes of any kind in his analysis, so the Black-Scholes setting was appropriate. Later on, Brennan and Schwartz (1978), Leland (1994) and Leland and Toft (1996) became aware of the importance of corporate income taxes and, accordingly, included corporate taxation in their debt pricing and optimal capital structure models. Our work extends this previous research by considering personal as well as corporate taxation. Kane et al. (1984) did analyze the value of interest tax shields with personal taxes for risky debt where the unlevered firm value could jump to zero according to a Poisson process. They did not incorporate flexibility, so their valuation is a european option model of default. They have a riskless rate for equity-financed flows that differs from the debt rate, but they find that interest tax shields have a positive value. Our paper shows that interest tax shields can have a negative or a positive value. More recently, Cooper and Nyborg (2004) model tax- and risk-adjusted discount rates when the company follows the Miles-Ezzell leverage policy but debt is risky, and both corporate and personal taxes are considered. Likewise,

Fernandez (2004, 2005), Fieten et al. (2005) and Cooper and Nyborg (2006) debate the issue of whether or not the value of the debt tax saving is equal to the present value of tax shields. They use a risk-adjusted discount rate settings, and a Miles-Ezzell leverage policy while only considering corporate taxes. In contrast, we value general cash flows emerging in capital budgeting using a certainty equivalent approach.

Our analysis goes beyond these papers in the following ways. It incorporates personal and corporate taxes, a variety of debt policies and uses a certainty-equivalent approach to risk. It discusses the appropriateness of various tax models. It also considers non-linear cash flows that are not accurately valued with risk-adjusted discount rates, such as cash flows with optionality, like real options, or asymmetric cash flows, like tax shields.

An outline of our work is as follows. In Sections 2 and Section 3, we provide an equilibrium valuation approach in continuous-time for real and financial assets in an economy with personal and corporate taxes, where tax rates on bonds and stocks are different. This is the generalized Miller model. In Section 4 we present a continuous-time capital budgeting valuation approach for levered and unlevered real assets. Using continuous time allows us to apply the results directly to such valuation problems (including real options), but the results have natural interpretations in discrete time, as well. We start with the case of no default risk, and then we consider defaultable debt with exogenous default. Debt policy is taken as exogenous in the sense that it is not subject to optimization.<sup>3</sup> In Section 5 we introduce the basic real option to delay an investment under the assumption that the incremental debt to finance the real asset is issued conditional on the decision to invest. The debt policy is still exogenous. Section 6 provides concluding remarks.

---

<sup>3</sup>Footnotes 14 and 17 discuss how our approach can be extended to endogenous optimization of capital structure and default decisions.

## 2 Asset valuation in a generalized Miller economy

### 2.1 Tax equilibrium

We assume a continuous-time Miller (1977) economy that is generalized to allow for cross-sectional variation in corporate tax rates. In general, the personal tax rate for bond investment income is different from the personal tax rate for stock investment income. Miller assumes that there is cross-sectional variation in personal tax rates, but not corporate tax rates. Thus, in his tax equilibrium, all corporations are indifferent about capital structure, but investors have a tax-induced preference for debt or equity, leading to tax clienteles. By allowing cross-sectional variation in corporate tax rates, as in Sick (1990), only firms at the margin are indifferent (on a tax basis) between issuing debt and equity, and firms with a marginal tax rate  $\tau^c$  below the marginal firm's rate rate ( $\tau^c - \tau^m = \tau^* < 0$ ) prefer to issue equity and firms with a tax rate above the marginal firm's rate of the marginal firm ( $\tau^* > 0$ ) prefer to issue debt.

We assume that capital gains and coupons in bond markets are taxed at the same rate  $\tau^b$ , and capital gains and dividends in equity markets are taxed at the same rate  $\tau^e$ , for the marginal investor. The operators for investor or personal tax are assumed to be linear at any date  $t$ , in the sense that income and losses from a particular investment are taxed, or generate tax relief, at the same rate.<sup>4</sup> On the other hand, the tax scheme for corporations need not be linear. We allow  $\tau^c$ ,  $\tau^b$  and  $\tau^e$  to be  $\mathbb{F}$ -adapted stochastic processes, i.e. they are determined as a function of the (stochastic) factors underlying the economy, but they

---

<sup>4</sup>In general, individual investors do have a progressive tax structure, with increasing rates at increasing levels of income. What we are assuming here is that a particular investment that the investor is pricing does not have income variations so large that it moves the investor to higher or lower tax brackets. The key assumption for this paper is that the marginal investment gains and losses (at the investor level) for a particular asset are taxed at the same rate and that there is no kink in the tax curve as an asset goes from a gain to a loss.

Note, also, that we are speaking of the marginal tax rates of the marginal investor or firm, and assume the reader can distinguish which "marginal" to use in any particular context.

must have continuous sample paths, almost surely. In general,  $\tau^b$  and  $\tau^e$  are different. Ross (1987) established the existence of equilibrium and of an EMM for an economy with personal taxes and a convex tax schedule. These assumptions are satisfied in our setting.

Consider a firm with tax rate  $\tau^c$  that is deciding whether to issue debt or equity to an investor with tax rates  $\tau^e$  and  $\tau^b$  on equity and debt, respectively. If

$$(1 - \tau^b) < (1 - \tau^c)(1 - \tau^e), \quad (2.1)$$

then the firm has the ability and incentives to issue equity on terms that would be more favourable to the marginal investor than debt a issue. To see this, suppose riskless debt has a yield of  $r^f$ . Then, if equation (2.1) holds, it is possible to choose a rate of return  $r^z$  for equity,<sup>5</sup> so that the after-corporate-tax cost of equity paid by the company is less than that of debt cost and, simultaneously, the after-all-tax rate of return for equity received by the investor is higher than that of debt. That is, we can choose any value of  $r^z$  such that

$$\frac{1 - \tau^b}{1 - \tau^e} r^f < r^z < (1 - \tau^c) r^f.$$

On the one hand, this implies that  $(1 - \tau^b)r^f < (1 - \tau^e)r^z$ , so that the investor achieves a higher after-all-tax return on equity than debt. On the other hand, it implies that  $r^z < (1 - \tau^c)r^f$ , so that the cost of equity to the firm is lower than the after-tax cost of debt. Thus, if (2.1) cannot hold for the marginal investor and the marginal firm with an equilibrium economy, since the firm and the investor can both be better off by switching some debt to equity.

Similarly, the marginal firm has incentives to issue (and the marginal investor has an

---

<sup>5</sup>We set aside the risk premium for a moment, so that  $r^z$  is a riskless yield. We will address this point later on.

incentive to buy) debt rather than equity if

$$(1 - \tau^b) > (1 - \tau^c)(1 - \tau^e).$$

In equilibrium, there will be no further incentives for the firm to issue debt to retire equity, or to issue equity to refund debt. Denoting the marginal firm's tax rate by  $\tau^m$ , we have the Miller (1977) equilibrium relationship amongst the marginal tax rates of the marginal investor and marginal firm:

$$(1 - \tau^b) = (1 - \tau^m)(1 - \tau^e).$$

Thus, we have established the following proposition:

**Proposition 1.** *If the marginal firm has tax rate  $\tau^m$ , and the marginal investor has marginal tax rates  $\tau^e$  in equity and  $\tau^b$  on debt, then:*

$$1 - \tau^m = \frac{1 - \tau^b}{1 - \tau^e}. \quad (2.2)$$

Moreover, the market yields on riskless debt and equity are related by the following tax wedge:

$$r^z = (1 - \tau^m)r^f. \quad (2.3)$$

Thus, it is appropriate to refer to  $r^z$  as a *tax-adjusted discount rate*. Equation (2.3) provides a way to estimate the marginal tax rate  $\tau^m$ , since  $r^z$  and  $r^f$  can either be observed or derived from security prices.

The equilibrium rates of return on the money market and stock market are the same after all taxes for the marginal investor. Defining this common after-all-tax return to be

$r^{z,\text{at}} \equiv r^{f,\text{at}}$ , we have

$$r^{f,\text{at}} = r_t^f(1 - \tau_t^b) = r_t^z(1 - \tau_t^e) = r^{z,\text{at}}. \quad (2.4)$$

## 2.2 Tax clienteles

This model uses the marginal investor and her tax rates to derive the marginal firm's tax rate, which is not otherwise identified. We can say that the marginal firm's tax rate  $\tau^m \geq 0$ , because the marginal investor's debt tax rate usually exceeds her equity tax rate,  $\tau^b \geq \tau^e$ . In this situation, the riskless debt return exceeds the riskless equity return:  $r^f \geq r^z$ . If  $\tau^b = \tau^e$ , then  $\tau_m = 0$  and thus  $r^z = r^f$ . The assumption that  $\tau^b = \tau^e$  is common in the literature. It was used by Modigliani and Miller (1963) and underlies most of the standard exposition of the CAPM, such as Sharpe (1964) and APT as in Ross (1976). Others, such as Sick (1990) and Taggart (1991) have taken the return differentials implicit in Miller (1977) to get  $r^f \geq r^z$ .

We choose the notation  $r^z$  to bring the analogy with the zero-beta rate of return in the Black (1972) version of the CAPM. There may be no riskless equity security, but in many circumstances it is the intercept term in such a CAPM. It is the appropriate discount rate for certainty-equivalents that are all-equity financed. We shall use it as the discount rate in martingale and PDE valuation models for all-equity financed cash flows.

There may be clientele effects whereby the marginal investors for different types of securities have different tax classes. Thus, it may be appropriate to use different discount rates to value these securities. For example, in the financial derivatives literature and actual practice, there is rarely any consideration that a tax wedge should be applied to the riskless debt rate in valuation models. This is justified, since the marginal investor for a derivative is likely an incorporated market maker that trades frequently and thus takes capital gains and dividends as ordinary income. For such a marginal investor,  $\tau^b = \tau^e$  and

hence  $r^z = r^f$  even if they finance their positions entirely with equity.

On the other hand, for real assets, the marginal investor is likely a taxable individual who holds shares in a corporation. If the real option is entirely financed with equity, then it is appropriate to use a riskless rate  $r^z < r^f$ .

In what follows, we only require that all tax rates be between 0 and 1:  $0 \leq \tau^b, \tau^e, \tau^c, \tau^m < 1$ .

Tax arbitrage has a tendency to make all firms behave as if they have the same tax rate. At the corporate level, an arbitrage scheme could involve a highly taxed firm, with  $\tau^c > \tau^m$  or  $(1 - \tau^b) > (1 - \tau^c)(1 - \tau^e)$ , issuing debt to acquire its own equity, for example. Or, it could involve a low-tax firm, with  $\tau^c < \tau^m$ , issuing equity to buy back debt. We assume that there are tax laws and agency costs<sup>6</sup> that prevent a firm from undertaking such an arbitrage transaction. Thus, there will generally be firms with  $\tau^* > 0$  and firms with  $\tau^* < 0$  in this generalized Miller tax equilibrium.

It is more difficult to generate tax arbitrage opportunities that would have all investors facing the same effective personal tax rate as that held by the marginal investor. This is because personal tax laws identify the individual and generally change when a financial entity (such as a corporation, mutual fund or trust) is inserted between the investor and the investment vehicle. There will generally be investors who prefer debt to equity and investors who prefer equity to debt in this generalized Miller tax equilibrium. Indeed, Miller (1977) also assumed this. For example, suppose an investor pays little or no tax on any investment (e.g. a pension fund), but the marginal investor pays higher tax on debt than on equity. The untaxed investor would prefer the tax benefits of debt, but this will prevent the investor from earning risk premia paid on equity investments. We assume that any attempt to convert an equity investment with a risk premium to a debt instrument for

---

<sup>6</sup>See for instance, Myers and Majluf (1984).

tax purposes is prevented by tax law.

### 3 Stochastic model

We assume an economy with complete financial markets that has personal and corporate taxes. The time horizon is  $T$ , the underlying complete probability space is  $(\Omega, \mathcal{F}, \mathbb{P})$ , where the set of possible realizations of the economy is  $\Omega$ , the  $\sigma$ -field of distinguishable events at  $T$  is  $\mathcal{F}$ , and the actual probability on  $\mathcal{F}$  is  $\mathbb{P}$ . We denote by  $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$  the augmented filtration or information generated by the process of security prices, with  $\mathcal{F}_T = \mathcal{F}$ .

#### 3.1 Stochastic tax rates

Under many tax systems, such as the US tax code, the tax benefit of a bond coupon payment is fully allowed only if the firm's EBIT is greater than the coupon. If it is less, a tax loss occurs that can be carried forward or backwards against positive taxable income. If it is carried forward, the effective marginal tax rate is reduced by a loss of time value of money, as observed by Leland (1994, p. 1220). We assume that the uncertainty in the effective tax rate is caused by other cash flows of the firm that are independent of the project's tax rate. This allows us to model the carryforwards with a stochastic tax rate  $\tau^c$ . Because the tax rate is independent of the project under consideration, all of our proofs can be done conditional on the value of  $\tau^c$ , and then an expectation take over this risk. It is quite likely that many of the valuation results can be extended to the situation where tax rates are not independent of project cash flows.

We consider a firm that may have many projects that are independently financed with non-recourse or project financing. Thus, default and bankruptcy events are confined to each project and don't affect the rest of the firm. The shocks to earnings of projects can affect the tax rate faced by other projects. We assume that these shocks to the tax rate

$\tau^c$  are independent of the shocks to the cash flows to the project we analyze. This would happen for an arbitrarily small project or for one where the corporate tax rate is constant. Similarly, we can have the marginal firm's tax rate  $\tau^m$  to be stochastic, but we assume it to be independent of the project cash flows. The stochastic nature of tax rates is not essential to achieve the results of this paper and readers may find it more convenient to think of the tax rates as being deterministic or even constant.

Taxes introduce a risk-sharing mechanism between government and taxed investors. This could cause a difference between the equilibrium EMM pricing measure with and without personal taxes. Or, the EMM could differ for different tax clienteles. We will establish a *valuation neutrality* principle, which says that the equilibrium pricing measure is unchanged by the presence of taxation. For valuation purposes, this means that the expectation (under the EMM) of the pre-tax cash flow stream of a security, discounted at a pre-tax rate, is equal to the expectation (under the same EMM) of the after-tax cash flow stream, discounted at an after-tax rate.

A second important issue introduced by taxation of security returns is the presence of timing options related to taxation of capital gains, as pointed out by Constantinides (1983). Taxation of capital gains produces timing options due to a (rational) delay of liquidation of positions in financial securities, until a date of forced liquidation, if the accrued capital gain is positive and the anticipation of liquidation of the position if the capital gain is negative, to take advantage of the tax credit.

We avoid these timing options by assuming that capital gains are taxed as accrued, as if the investor were to have a mark-to-market cash flow that induces the capital gain. Taxation of accrued capital gains is one example of a *holding-period neutral tax scheme* in which the tax scheme does not introduce any timing options related to taxation of capital gains. Auerbach (1991), Auerbach and Bradford (2001) and Jensen (2003) describe taxa-

tion schemes that provide holding-period neutrality. It may be that our results generalize to these other tax schemes as well, but that is a topic for future research.

### 3.2 EBIT and asset-price dynamics

Suppose the process for EBIT, under the actual probability measure is

$$dX_t = g(X_t, t)dt + \sigma(X_t, t)dZ_t. \quad (3.1)$$

Consider a project of value  $V = V(X_t, t)$  that pays equity holders the after-corporate-tax cash flow at the rate

$$Y(X_t, \tau_t^c, t), \quad 0 \leq t \leq \theta^p, \quad (3.2)$$

where the project life is  $\theta^p \leq \infty$ . For an unlevered firm,  $Y(X_t, \tau_t^c, t) = X_t(1 - \tau_t^c)$ . For a levered firm, we will break out interest payments and interest tax shields by having a more general form for  $Y$  in equation (4.2).

Since the EBIT process is not traded, we will value the claim to the equity-financed stream  $Y$  by using an after-all-tax version of the consumption CAPM, as developed in Constantinides (1983, Theorem 6) and used by others, such as Kane et al. (1984). Ross (2005) also uses a general equilibrium after-all-tax model to get the market price of risk.<sup>7</sup> Then we will calculate a sum of values of individual cash flows, as in Constantinides (1978). To keep the notation convenient, suppose the market risk premium for the diffusion  $dZ$  is  $\lambda$  in an after-tax consumption CAPM.<sup>8</sup>

---

<sup>7</sup>We would get the same results with replication arguments using a traded security. Interestingly, the hedge ratios used for the replication do not need a tax adjustment.

<sup>8</sup>That is, suppose a security has price  $P$  following

$$dP = \mu(P)dt + \sigma(P)dZ$$

and pays a dividend at the rate  $P\delta(P)$ . Then, by the after-tax consumption CAPM, we have the instant-

By Itô's lemma, the after-all-tax wealth gain to the equity investor follows the process

$$\begin{aligned} & \left( \frac{\sigma^2(X, t)}{2} \frac{\partial^2 V}{\partial X^2} + g(X, t) \frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + Y(X, \tau_t^c, t) \right) (1 - \tau^e) dt \\ & + \sigma(X, t) \frac{\partial V}{\partial X} (1 - \tau^e) dZ_t. \end{aligned}$$

Here we take advantage of the fact that the capital gains are taxed as accrued to the investor and at the same rate as the dividend distribution  $Y$ .

Taking expectations under the actual probability measure  $\mathbb{P}$ , the expected after-all-tax growth rate in wealth from the investment is

$$\left( \frac{\sigma^2(X, t)}{2} \frac{\partial^2 V}{\partial X^2} + g(X, t) \frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + Y(X, \tau_t^c, t) \right) (1 - \tau^e). \quad (3.3)$$

In the after-tax consumption CAPM equilibrium, this must equal the riskless rate after tax plus the risk premium

$$(1 - \tau^e) \left( r^z + \lambda \frac{\partial V}{\partial X} \frac{\sigma(X, t)}{V} \right) V. \quad (3.4)$$

Equating (3.3) and (3.4) and simplifying yields the following proposition:

**Proposition 2.** *Under the assumptions of this section, including personal taxation, the neous required rate of return:*

$$\left( \frac{\mu(P)}{P} + \delta(P) \right) (1 - \tau^e) = r^z (1 - \tau^e) + \lambda \frac{\sigma(P)}{P} (1 - \tau^e).$$

Since dividends and capital gains are taxed at the same rate in our model, the factor  $(1 - \tau^e)$  factors out and this becomes the same as a pretax consumption CAPM:

$$\frac{\mu(P)}{P} + \delta(P) = r^z + \lambda \frac{\sigma(P)}{P}.$$

value of an equity-financed real asset satisfies the fundamental after-all-tax PDE:

$$\begin{aligned} & \left( \frac{1}{2} \sigma^2(X, t) \frac{\partial^2 V}{\partial X^2} + \hat{g}(X, t) \frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} \right) (1 - \tau^e) \\ & + Y(X_t, \tau_t^c, t) (1 - \tau^e) = r^{z, \text{at}} V \end{aligned} \quad (3.5)$$

where the risk-neutral growth rate is

$$\hat{g}(X, t) \equiv g(X, t) - \lambda \sigma(X, t), \quad (3.6)$$

whenever  $0 \leq t \leq \theta^p$ .

We also have the equivalent pretax PDE:

$$\frac{1}{2} \sigma^2(X, t) \frac{\partial^2 V}{\partial X^2} + \hat{g}(X, t) \frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + Y(X_t, \tau_t^c, t) = r^z V, \quad (3.7)$$

where the risk-neutral growth rate  $\hat{g}$  is the same as defined in (3.6).

**Proposition 3.** *If the assumptions of Proposition 2 hold and the project has residual terminal value at time  $\theta^p$  of*

$$V_{\theta^p} = V(X_{\theta^p}, \theta^p) = \text{TV}(X_{\theta^p}, \theta^p), \quad (3.8)$$

then

$$V(X_t, t) = B_t^z \hat{\mathbb{E}}_t \left[ \int_t^{\theta^p} \frac{Y(X_s, s)}{B_s^z} ds + \frac{\text{TV}(X_{\theta^p}, \theta^p)}{B_{\theta^p}^z} \right] \quad (3.9)$$

where the money market account for riskless equity is

$$B_u^z = \exp\left(\int_0^u r_s^z ds\right). \quad (3.10)$$

*Proof.* Since the capital gains on changes in  $V$  are taxed as accrued, there is no personal tax on the final distribution of  $\text{TV}(X_{\theta^p}, \theta^p)$  to the investor. We can use the Feynman-Kac solution<sup>9</sup> to the PDE (3.7) to obtain the result.  $\square$

Characterizing the value as the expectation of a discounted stream of payoffs is very useful, because it allows us to value projects by Monte Carlo simulation, even if they are american real options where the manager can adjust strategies in response to the arrival of information.

### 3.3 Valuation neutrality

Note that the risk-neutral growth rate  $\hat{g}$  and hence the risk-neutral expectation operator  $\hat{\mathbb{E}}$  is the same for the pre-personal-tax and after-all-tax valuation models. This means that taxation affects the valuation only through the cashflow process and the discount rate, but it does not affect the risk premium or certainty-equivalent calculation.<sup>10</sup>

We refer to this principle as *valuation neutrality*.

---

<sup>9</sup>See, for example, Duffie (2001, pp. 340–346).

<sup>10</sup>Ross (1987) shows that there is a martingale pricing operator in the presence of taxes. His results are general and, in our setting, establish a risk-neutral expectation operator (or, equivalently, a certainty-equivalent operator) after all taxes, as well as risk-neutral expectation operators for debt flows before personal tax and for equity flows before personal tax. There is no immediate guarantee that these pricing operators are related or equivalent. Sick (1990) raised this question in a discrete-time setting and showed that the certainty equivalent operators associated with these pricing operators are all identical to each other. That is, taxes and tax shields do not generate a risk premium. Proposition 2 extends this result to continuous time.

### 3.4 After-all-tax valuation

The mark-to-market taxation of capital gains results in a cash-flow stream that is similar to a dividend stream, and this has to be considered if we want to compute values from after-all-tax cash flows.

**Proposition 4.** *Under the assumptions of Propositions 2 and 3, the value of the real asset satisfies the following expectation:*

$$V_t = B_t^{z,\text{at}} \hat{\mathbb{E}}_t \left[ \int_t^{\theta^p} \frac{(1 - \tau_u^e) Y(X_u, u)}{B_u^{z,\text{at}}} du - \int_t^{\theta^p} \frac{\tau_u^e}{B_u^{z,\text{at}}} dV_u + \frac{V_{\theta^p}}{B_{\theta^p}^{z,\text{at}}} \right], \quad (3.11)$$

where the after-tax money market account for riskless equity is

$$B_u^{z,\text{at}} = \exp \left( \int_0^u r_s^{z,\text{at}} ds \right). \quad (3.12)$$

*Proof.* The proof is an extension of the proof of the Feynman-Kac result, as discussed, for example in Duffie (1988, Section 21). We will show that the solution to the martingale valuation (3.11), will also satisfy the fundamental valuation PDE (3.7). Then we appeal to the uniqueness of the solution of the PDE subject to the terminal condition (3.8).

Define the differential operator  $\mathcal{D}$  such that if a real-valued function  $f \in \mathcal{C}^{2,1}$ , then

$$\mathcal{D}f(x, t) = \frac{1}{2} \sigma^2(x, t) \frac{\partial^2 f(x, t)}{\partial x^2} + \hat{g}(x, t) \frac{\partial f(x, t)}{\partial x} + \frac{\partial f(x, t)}{\partial t}. \quad (3.13)$$

where  $0 \leq t \leq \theta^p$ . Define the after-all-tax discounted value of  $V$  as

$$W(x, \theta) \equiv \frac{V(x, \theta)}{B_\theta^{z,\text{at}}}, \quad (3.14)$$

for  $t \leq \theta \leq \theta^p$ . By equation (3.11),

$$\begin{aligned} W(X_t, t) &= \hat{\mathbb{E}}_t \left[ \int_t^\theta \frac{(1 - \tau_u^e)Y(X_u, u)}{B_u^{z,at}} du - \int_t^\theta \frac{\tau_u^e}{B_u^{z,at}} dV_u + \frac{V_\theta}{B_\theta^{z,at}} \right] \\ &= \hat{\mathbb{E}}_t \left[ \int_t^\theta \frac{(1 - \tau_u^e)Y(X_u, u)}{B_u^{z,at}} du - \int_t^\theta \frac{\tau_u^e}{B_u^{z,at}} dV_u \right] + \hat{\mathbb{E}}_t [W(X_\theta, \theta)]. \end{aligned} \quad (3.15)$$

Thus

$$\begin{aligned} \hat{\mathbb{E}}_t [W(X_\theta, \theta)] - W(X_t, t) &= -\hat{\mathbb{E}}_t \left[ \int_t^\theta \frac{(1 - \tau_u^e)Y(X_u, u)}{B_u^{z,at}} du - \int_t^\theta \frac{\tau_u^e}{B_u^{z,at}} dV_u \right] \\ &= -\hat{\mathbb{E}}_t \left[ \int_t^\theta \frac{(1 - \tau_u^e)Y(X_u, u)}{B_u^{z,at}} du - \int_t^\theta \frac{\tau_u^e}{B_u^{z,at}} \mathcal{D}V du \right] \end{aligned} \quad (3.16)$$

The second equality comes from evaluating  $dV_u$  by Itô's lemma and noting that the diffusion term has an expectation of zero.

Applying Itô's formula to  $W$  gives

$$W(X_\theta, \theta) - W(X_t, t) = \int_t^\theta \mathcal{D}W du + \int_t^\theta \frac{\partial W}{\partial x} \sigma(x_u) dZ_u. \quad (3.17)$$

Taking risk-neutral expectations, the second term on the right also becomes zero. The left side is the same as the left side of equation (3.16), so we can equate the right sides, getting:

$$\hat{\mathbb{E}}_t \left[ \int_t^\theta \left( \mathcal{D}W + \frac{(1 - \tau_u^e)Y(X_u, u)}{B_u^{z,at}} - \frac{\tau_u^e}{B_u^{z,at}} \mathcal{D}V \right) du \right] = 0.$$

This holds for all  $\theta \in [t, \theta^p]$ , including values arbitrarily close to  $t$ , and this can only happen if the integrand is identically zero. Thus

$$\mathcal{D}W + \frac{(1 - \tau_t^e)Y(X_t, t)}{B_t^{z,at}} - \frac{\tau_t^e}{B_t^{z,at}} \mathcal{D}V = 0. \quad (3.18)$$

Now, we can evaluate  $\mathcal{D}W$  by substituting the definition (3.14) for  $W$ , expanding the differential operator and applying the product rule and chain rule:

$$\begin{aligned}\mathcal{D}W &= \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 W}{\partial x^2} + \hat{g}(x,t)\frac{\partial W}{\partial x} + \frac{\partial W}{\partial t} \\ &= \frac{\sigma^2(x,t)}{2B_t^{z,\text{at}}}\frac{\partial^2 V}{\partial x^2} + \frac{\hat{g}(x,t)}{B_t^{z,\text{at}}}\frac{\partial V}{\partial x} + \frac{1}{B_t^{z,\text{at}}}\frac{\partial V}{\partial t} - \frac{r_t^z(1-\tau^e)B_t^{z,\text{at}}}{\left(B_t^{z,\text{at}}\right)^2}V \\ &= \frac{1}{B_t^{z,\text{at}}}\mathcal{D}V - \frac{r_t^z(1-\tau^e)}{B_t^{z,\text{at}}}V.\end{aligned}$$

Substituting for  $\mathcal{D}W$  in equation (3.18) and simplifying gives equation (3.7), which is the desired result.  $\square$

The second term in the expectation of the martingale valuation (3.11) is the present value of the mark-to-market capital gains taxation where the initial capital gains basis is  $V_t$ .<sup>11</sup> If we take the same initial basis, but assume that capital gains are only taxed when realized, then we have the following variant of the valuation (3.11), which is constructive:

$$V_t = B_t^{z,\text{at}}\hat{\mathbb{E}}_t \left[ \int_t^{\theta^p} \frac{(1-\tau_u^e)Y(X_u, u)}{B_u^{z,\text{at}}} du + \frac{(1-\tau_T^e)V_{\theta^p} + \tau_T^e V_t}{B_{\theta^p}^{z,\text{at}}} \right].$$

In this case, there is no simple pre-personal-tax equivalent valuation comparable to equation (3.9).

## 4 Valuation of levered and unlevered projects

We consider a project with life  $\theta^p \leq \infty$  and EBIT following (3.1). We assume there is no residual value at the end of the project, so  $\text{TV}(X_{\theta^p}, \theta^p) = 0$ .

---

<sup>11</sup>It may be more natural to take the basis to be  $V_0$ , since we allow  $t$  to vary. This just adds another term to (3.11) for the valuation of the capital gains from 0 to  $t$ , and we omit it for brevity.

## 4.1 Valuation of an unlevered project

For this section, suppose  $Y(X_t, \tau_t^c, t) = X_t(1 - \tau_t^c)$  is the unlevered earnings before interest but after corporate tax. The risk-neutral drift  $\hat{g}(X, t)$  is given in (3.6). The value of the unlevered project  $U(x, t)$  is given by Proposition 3:

$$U(x, t) = B_t^z \hat{\mathbb{E}}_t \left[ \int_t^{\theta^p} \frac{X_s(1 - \tau_s^c)}{B_s^z} ds \right] \quad (4.1)$$

for  $0 \leq t \leq \theta^p$ .

## 4.2 Valuation of a levered project

The sum of the values of the debt and equity claims against a project is called its Adjusted Present Value or APV. Suppose the firm is financed with debt paying a coupon at the rate  $R_t = R(X_t, t)$ . This can be a fixed coupon, or a growing or amortizing coupon. It can also be a risky coupon, where payments are a function of EBIT,  $X$ . We could have the coupon depend on another random variable, but for simplicity, we keep the dimensionality of the risk to one. The debt matures at time  $\theta^d \leq \theta^p$  with a promised principal repayment of  $P$  to the bondholders.

For  $t \leq \theta^d$ , the cash flow to equity-holders is at the rate  $(X_t - R_t)(1 - \tau_t^c)(1 - \tau_t^e)$  and that to debt-holders is at the rate  $R_t(1 - \tau_t^b)$ . After the debt is retired, the project is financed completely with equity and we can keep the same representation of cash flow to equity holders by setting  $R_t = 0$  for  $\theta^d < t \leq \theta^p$ . Thus, the total after-all-tax cash flow from the project to the equity and debt investors is (dropping the common subscript  $t$ ):

$$\begin{aligned} X(1 - \tau^c)(1 - \tau^e) + R(1 - \tau^b) &= (X(1 - \tau^c) + \tau^* R)(1 - \tau^e) \\ &\equiv Y(X, \tau^c)(1 - \tau^e), \end{aligned} \quad (4.2)$$

for  $0 \leq t \leq \theta^d$ , where the net rate of interest tax shield benefits is  $\tau^* = \tau^c - \tau^m$  and  $\tau^m$  is the marginal tax rate defined in (2.2). This formula has the following important features:

1. The term  $\tau^*R$  is the interest tax shield.
2. The factor  $(1 - \tau^e)$  emerges on the right side of the first equality. In effect, the tax shield accrues to equity investors and is taxed at their marginal rate.<sup>12</sup>
3. We can define the “after-corporate-tax levered cash flow”  $Y = X(1 - \tau^c) + \tau^*R$  by the last equality, since the factor  $(1 - \tau^e)$  is common in the second equality.

Note that  $\tau^* = \tau^c - \tau^m$  is the rate at which net tax shields accrue to investors. The traditional literature is based on interest tax shields having positive value, which requires  $\tau^* > 0$ . However, since  $\tau^*$  is based on the difference between the firm’s tax rate  $\tau^c$  and the marginal firm’s tax rate  $\tau^m$ , interest tax shields could just as easily have negative value as they have positive value. For high-tax firms, they have positive value, and for low-tax firms, they have negative value.

Since we have an after-all-tax cash flow that is taxed at the personal equity rate, we can apply Proposition 3 to obtain the following valuation result:

**Proposition 5.** *The APV, incorporating the value of the tax shield, satisfies the fundamental PDE:*

$$\frac{1}{2}\sigma^2(X, t)\frac{\partial^2 V}{\partial X^2} + \hat{g}(X, t)\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + X_t(1 - \tau_t^c) + \tau_t^*R_t = r_t^z V. \quad (4.3)$$

---

<sup>12</sup>Sick (1990) was the first to point out this fact, but it has been observed by others, such as Ross (2005). One interpretation of (4.2) is the following: the interest tax shield can be interpreted as a swap of debt for equity. The swap is valued by the marginal investor, who is indifferent between receiving cash flow from equities and cash flows from bonds. At  $t = 0$ , the firm swaps  $B$  dollars of riskless or zero-beta equity claims for an equivalent amount (at the market value) of debt, so, the initial net change in investment for investors is equal to 0. At every coupon date, the firm avoids having to remit the required rate of return  $r^z B$  to equity holders, but must pay  $r^f(1 - \tau^c)B$  to debt-holders. The net gain for equity-holders is  $r^z B - r^f(1 - \tau^c)B$ , which can be rewritten using the generalized Miller equilibrium (2.3) as  $r^f B(\tau^c - \tau^m)$ .

with boundary conditions  $V(\theta^d, X_{\theta^d}) = U(\theta^d, X_{\theta^d})$  and  $U(\theta^p, X_{\theta^p}) = 0$ , and is

$$\begin{aligned} V(x, t) &= B_t^z \hat{\mathbb{E}}_t \left[ \int_t^{\theta^d} \frac{X_s(1 - \tau_s^c) + \tau_s^* R_s}{B_s^z} ds + \frac{U(\theta^d, X_{\theta^d})}{B_{\theta^d}^z} \right] \\ &= U(x, t) + B_t^z \hat{\mathbb{E}}_t \left[ \int_t^{\theta^d} \frac{\tau_s^* R_s}{B_s^z} ds \right], \end{aligned} \quad (4.4)$$

for  $0 \leq t \leq \theta^d$ . The second term in the right side is the value of the stream of interest tax shields.

In line with Cooper and Nyborg (2006) and in contrast to Fernandez (2004), equation (4.4) implies that the value of interest tax shield can be calculated by an additive term in an adjusted present value or APV treatment. Equation (4.4) also implies that when computing the APV from the total cash flows generated by a company, the appropriate tax-adjusted discount rate (TADR) is the riskless equity rate of return,  $r_z$ , no matter what its capital structure is. In particular, it is inappropriate to discount the interest tax shield at a debt rate, which is an error commonly seen in the literature.<sup>13</sup>

### 4.3 APV with specific debt policies

Now we assume the corporation has an exogenous capital structure policy. Models with endogenous capital structure include Fisher et al. (1989), Mauer and Triantis (1994), Goldstein et al. (2001), Christensen et al. (2002), Dangl and Zechner (2003), Titman and Tsyplakov (2006), Strebulaev (2006), Hennessy and Whited (2005, 2006) but we will not consider these here.<sup>14</sup> They do not establish any relationship between pre-tax and post-tax

---

<sup>13</sup>Even under the assumption on personal and corporate taxation Mello and Parsons (1992), Mauer and Triantis (1994), Goldstein et al. (2001), and Titman and Tsyplakov (2006) use  $r^f$  as the tax-adjusted discount rate (TADR).

<sup>14</sup>Capital budgeting valuation with endogenous financial decisions entails the solution of a dynamic programming problem, where debt is a control variable. In this case, our approach could be extended to the use of a Hamilton-Jacobi-Bellman equation for optimization (as opposed to the partial differential valuation equation) of the problem, without modifying our model of personal and corporate taxation. A

valuation operators as we do here.

One exogenous policy is the constant debt policy of Modigliani and Miller (1958), which can generate bankruptcy. We will take the bankruptcy threshold to be exogenous. Another is the constant proportional debt policy of Miles and Ezzel (1985), which does not have default risk if debt is revised continuously and the unlevered cash flow is non-negative, as in a lognormal diffusion. Only in the case of a proportional debt policy, can we find a tax-adjusted discount rate  $\rho$  that can be applied to unlevered flows to correctly compute the value of levered flows.

#### 4.4 APV with a riskless debt policy

In this section we assume that management maintains a riskless debt policy by continuously buying and selling bonds in the open market as the firm value rises and falls. If the firm value falls, the bondholders can tender their bonds to management at a market price that rationally anticipates no default. Thus, they do not suffer a loss and the coupon rate for corporate bonds would be the risk free rate. One policy that achieves this is the policy of Miles and Ezzel (1980, 1985), which is that the firm maintains a constant debt ratio  $L$ .

We assume the debt consists of floating rate coupon bonds for a term  $\theta^d \leq \theta^p$ , at the end of which the debt is retired at par. If the project and debt are perpetual  $\theta^d = \theta^p = \infty$ , the firm issues consol bonds, which are never retired. The coupon is set at the instantaneous riskless rate, so the bonds are always priced at par. That is, the coupon flow is  $R_t = r_t^f D(x, t)$ , which is an adapted process, and  $D(x, t)$  only varies because of the bonds issues or repurchases by management in response to changes in  $x$ , and not because of interest-rate or default risk. When the firm maintains a constant debt ratio, we always have  $D(x, t) = LV(x, t)$ .

---

similar situation applies to endogenous default, as noted in footnote 17.

By replacing this condition in equation (4.3) we obtain

$$\frac{1}{2}\sigma^2(X, t)\frac{\partial^2 V}{\partial X^2} + g(X, t)\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + X_t(1 - \tau_t^c) = \rho_t V \quad (4.5)$$

where

$$\rho_t = r_t^z - \tau^* r_t^f L = (1 - L)r_t^z + L(1 - \tau_t^c)r_t^f \quad (4.6)$$

is the *tax-adjusted riskless cost of capital*.<sup>15</sup> It is the cost of capital under the equivalent martingale measure that incorporates the effect of interest tax shield and that does depend on the level of debt. This is a continuous-time extension of the tax-adjusted rate of return in Sick (1990). Our result is in accordance with the common practice of computing the NPV by discounting the expected free cash flows under the actual probability at the weighted average cost of capital (a risk-adjusted discount rate). Here, we use risk-neutral valuation, which allows for stochastic betas, as in many real option problems. Thus, we remove the risk premium from the discount rate and use risk-neutral expectations instead. However, we can still use the weighted average to capture interest tax shields.

Applying conditions  $V(\theta^d, X_{\theta^d}) = U(\theta^d, X_{\theta^d})$  and  $U(\theta^p, X_{\theta^p}) = 0$ , and considering that, for  $t > \theta^d$ ,  $D \equiv 0$  (since  $R \equiv 0$ ), we can use Proposition 2 to calculate the APV of the project as:

$$V(x, t) = B_t^w \hat{\mathbb{E}}_t \left[ \int_t^{\theta^d} \frac{X_s(1 - \tau_s^c)}{B_s^w} ds \right] + B_t^z \hat{\mathbb{E}}_t \left[ \frac{U(\theta^d, X_{\theta^d})}{B_{\theta^d}^z} \right], \quad (4.7)$$

for  $0 \leq t \leq \theta^d$ , where  $B_t^w = \exp \int_0^t \rho_u du$  is the time- $t$  value of one dollar accrued at the tax-adjusted riskless discount rate. Comparing Equation (4.7) to (4.4) clarifies the role of the tax-adjusted CE cost of capital,  $\rho_t$ , as the stochastic instantaneous discount rate that generates the levered asset value when applied to the unlevered cash flow process.<sup>16</sup> In

<sup>15</sup>Grinblatt and Liu (2002) have an analogous definition, although they limit the analysis to the less general setting with constant rate of returns and with  $\tau^m = 0$ .

<sup>16</sup>Grinblatt and Liu (2002) have an analogue definition of WACC, although they limit the analysis to a

particular, the randomness of  $\rho_t$  derives from the randomness of  $r_t^z$  and  $r_t^f$ , but not from the leverage ratio. Hence, equation (4.7) provides a time-consistent pricing operator for levered cash flows.

## 4.5 APV with defaultable debt

If debt is riskless, the value of the project is monotonically increasing (if  $\tau^* > 0$ ) or decreasing (if  $\tau^* < 0$ ) in the debt ratio  $L$ , so we would need some other constraints such as agency costs or laws to prevent the debt ratio from going to its limits of 0 and 1. If default can happen, then we have a tradeoff between the value of tax shields for a firm with ( $\tau^* > 0$ ) and the deadweight losses from bankruptcy, which can result in an internal optimal capital structure. When  $\tau^* < 0$ , both the tax shields and the default costs draw down firm value as debt increases, and there is no internal optimal capital structure.

We assume there is a given barrier of coupon coverage,  $x_D$ , such that default occurs when the EBIT process  $X_t$  first hits the barrier from above.<sup>17</sup> In this case, the bond-holders file for bankruptcy and receive an equity claim to the project assets, net of bankruptcy costs, which are the fraction  $\alpha$  of the unlevered asset value at the time of default.

As long as the project is solvent (i.e.,  $X_u > x_D \forall u \leq t$ ), equation (4.2) gives the total after-all-tax cash of the project. If the firm defaults at date  $t$ , the tax shield is lost over the interval  $t \leq u \leq \theta^p$ . Moreover, the deadweight bankruptcy cost  $\alpha U(t, x_D)$  is lost.

**Proposition 6.** *Define  $T_D \equiv \inf \{s \in [t, \theta^d], X_s = x_D\}$  to be the random time of default and let  $\chi_A$  be the indicator function for event  $A$ .<sup>18</sup> Denote  $x \wedge y \equiv \min\{x, y\}$ . The APV, incorporating the value of the interest tax shield, satisfies equation (4.3) with boundary*

---

less general setting with constant rate of returns and with  $\tau^m \equiv 0$ .

<sup>17</sup>The extension of our valuation approach to the case of endogenous default would entail the application of the valuation principles presented in Sections 2 and 3 to the partial differential equation for equity with a free boundary, as opposed to a known  $x_D$ . See Leland (1994) and Titman and Tsyplakov (2006) for a discussion about plausible values for  $x_D$ .

<sup>18</sup> $\chi_A(\omega) = 1$  if  $\omega \in A$ ,  $\chi_A(\omega) = 0$  otherwise.

conditions  $V(\theta^d, x) = U(\theta^d, x)$  and  $U(\theta^p, x) = 0$  and  $V(s, x_D) = (1 - \alpha)U(s, x_D)$  for all  $t \leq s \leq \theta^d \leq \theta^p$ . For  $x > x_D$ , we have

$$V(x, t) = U(x, t) + B_t^z \hat{\mathbb{E}}_t \left[ \int_t^{T_D \wedge \theta^d} \frac{\tau_s^* R_s}{B_s^z} ds \right] - \alpha B_t^z \hat{\mathbb{E}}_t \left[ \chi_{\{\exists s \in [t, \theta^d], X_s = x_D\}} \frac{U(T_D, x_D)}{B_{T_D}^z} \right]. \quad (4.8)$$

*Proof.* The APV satisfies equation (4.3) when the firm is solvent (while  $X_t > x_D$ ). The terminal date  $T_D \wedge \theta^d = \min\{T_D, \theta^d\}$  is now stochastic, so the standard Feynman-Kac solution for the PDE (4.3) has to be replaced by a generalized result where the continuation region on which the PDE holds is an open bounded set. Such a result is in Dynkin (1965, Theorem 13.14) or Lamberton and Lapeyre (1996, Theorem 5.1.9).  $\square$

The second term on the right-hand-side of equation (4.8) is the value of the interest tax shield, taking into account the risk it is lost after the project defaults on the debt. The third term is the value of the bankruptcy costs incurred at the date of default.

In the last section we also need  $D$ , the market value of corporate bonds, in the generalized Miller equilibrium economy. As above,  $R$  is the coupon payment and  $P$  the principal payment (face value) paid back at maturity  $\theta^d$ . Under the assumption of exogenous default at the threshold  $x_D$ , debt is a contingent claim on  $X$ , the EBIT of the firm. The next proposition determines the value of defaultable debt in the generalized Miller equilibrium.

**Proposition 7.** *The market value of debt satisfies*

$$\frac{1}{2} \sigma^2(X, t) \frac{\partial^2 D}{\partial X^2} + g(X, t) \frac{\partial D}{\partial X} + \frac{\partial D}{\partial t} + R_t = r^f D \quad (4.9)$$

with boundary conditions  $D(\theta^d, X_{\theta^d}) = P$ ,  $D(s, x_D) = (1 - \alpha)U(s, x_D)$  for all  $t \leq s \leq \theta^d \leq$

$\theta^p$  and, assuming  $x > x_D$ , is

$$D(x, t) = B_t^f \hat{\mathbb{E}}_t \left[ \int_t^{T_D \wedge \theta^d} \frac{R_s}{B_s^f} ds \right] + B_t^f \hat{\mathbb{E}}_t \left[ \chi_{\{\forall s \in [t, \theta^d], X_s > x_D\}} \frac{P}{B_{\theta^d}^f} \right] \\ + (1 - \alpha) B_t^f \hat{\mathbb{E}}_t \left[ \chi_{\{\exists s \in [t, \theta^d], X_s = x_D\}} \frac{U(T_D, x_D)}{B_{T_D}^f} \right] \quad (4.10)$$

where  $T_D \leq \theta^d$  is the first time  $X_t = x_D$ .

*Proof.* The valuation PDE (4.9) for  $D$  comes from the same argument we used in the proof of Proposition 2. That is, the expected after-all-tax payments to bondholders under the actual probability measure  $\mathbb{P}$  satisfies:

$$\left( \frac{1}{2} \sigma^2(X, t) \frac{\partial^2 D}{\partial X^2} + g(X, t) \frac{\partial D}{\partial X} + \frac{\partial D}{\partial t} + R_t \right) (1 - \tau^b). \quad (4.11)$$

In the after-tax consumption CAPM equilibrium, this must equal the riskless rate after tax plus the risk premium

$$\left( r^f + \lambda \frac{\partial D}{\partial X} \frac{\sigma(X, t)}{D} \right) (1 - \tau^b) D. \quad (4.12)$$

Equating (4.11) and (4.12), cancelling the common factor and simplifying yields equation (4.9).

Using the same extension to the Feynman-Kac result used in the proof of Proposition 6, the solution of this PDE subject to the boundary conditions is (4.10).  $\square$

Note that the market price of volatility risk  $\lambda$  is the same in the debt market as it is in the equity market, because we derived both from an after-all-tax equilibrium perspective where debt and equity cash flows are on an equal tax footing. Thus, the risk neutral probability distribution is the same for debt and equity markets, so we can use the same

expectation  $\hat{\mathbb{E}}$  to value risky debt flows and risky equity flows even at the market level before personal tax is considered.

## 5 Valuation of real options with debt financing and taxes

This section addresses real options valuation under the general framework introduced in Section 2, assuming that the tax shield may be valued according to the analysis of Section 4. In particular, we will differentiate our analysis according to the cases of default-free debt and defaultable debt.

Here we restrict ourselves to the basic real option to delay an investment decision, although our approach can be applied to a broader class of real options.<sup>19</sup> We then assume that we have the opportunity to delay investment in a project, under the EMM. The project has life  $\theta^p$  (possibly,  $\theta^p = \infty$ ), so that the project starts from the date the option is exercised,  $T_I$ , and ends at  $T_I + \theta^p$ . The cost to implement the project is  $I$  (an adapted process) and we have the opportunity to delay the investment until date  $\theta^o$  (possibly,  $\theta^o = \infty$ ).<sup>20</sup>

We assume that the capital expenditure to implement the project is also partially financed with incremental debt, and the incremental debt is issued if and when the option to invest is exercised.<sup>21</sup> This assumption is realistic since there would be no reason to raise capital before investment. The optimal exercise policy depends on  $X_t$ , the EBIT process,

---

<sup>19</sup>See Dixit and Pindyck (1994) and Trigeorgis (1996) for general discussions of real options.

<sup>20</sup>To simplify our analysis, we assume perfect information of shareholders and equity-holders and the absence of agency costs between shareholders and equity-holders and equity-holders and management. Hence, investment is implemented under a *first-best investment policy* (i.e., a policy aiming at maximizing the total project/firm value as opposed to a policy in the sole interest of shareholders) by the managers. The role of agency costs of debts on investment decisions have been analyzed among others by Mello and Parsons (1992), Mauer and Ott (2000) and Childs et al. (2005).

<sup>21</sup>Mauer and Ott (2000) and Childs et al. (2005) assume that the incremental investment is financed only with equity. Their assumption is included in our framework by posing that no incremental debt is issued at the time the investment decision is made.

and consequently the date when the firm will issue debt is a stopping time. Issuance of incremental debt is contingent on the decision to invest, so the financing decision is influenced by the investment decision. Conversely, the investment decision is influenced by the financing decision, since the former is made if the return from the project compensates its financing cost. The debt policy is alternatively the one in Section 4.4 (default-free) or the one in Section 4.5 (defaultable). The debt has term  $\theta^d$ , so that it is issued at  $T_I$  and is paid-back at  $T_I + \theta^d$ .

The value of the levered project, at the date it is implemented, is  $V(t, X_t)$  from equation (4.4) in the case with default-free debt or from equation (4.8) with defaultable debt. In case default is possible, given the above assumption that debt is issued conditional on the investment decision, we assume that default can happen only after the investment date. Let  $\Pi$  denote the payoff of the option at the exercise date,

$$\Pi(t, X_t) = \max\{V(t, X_t) - I, 0\}, \quad (5.1)$$

and let  $F(t, X_t)$  denote the value of the investment project including the time-value of the option to postpone the decision.

**Proposition 8.** *The value of the option to delay investment satisfies equation*

$$\frac{1}{2}\sigma^2(X_t, t)\frac{\partial^2 F}{\partial X^2} + g(X_t, t)\frac{\partial F}{\partial X} + \frac{\partial F}{\partial t} = r^z F. \quad (5.2)$$

with boundary condition  $F(T_I, X_{T_I}) = \Pi(T_I, X_{T_I})$ , and is

$$F(t, x) = B_t^z \sup_{T_I} \hat{\mathbb{E}}_t \left[ \frac{\Pi(T_I, X_{T_I})}{B_{T_I}^z} \right], \quad (5.3)$$

where  $T_I \leq \theta^o$  is the investment (stopping) time, i.e., the first time  $F(t, X_t) = \Pi(t, X_t)$ .

*Proof.* The proof that  $F$  satisfies equation (5.2) is the same as for the proof of Proposition 5. The solution in (5.3) is derived using standard results (see Duffie (2001, pp. 182–186)).  $\square$

From equation (5.3) it is worth stressing that, while the option to delay investment is still alive, the appropriate CE discount factor to be applied to its expected payoff is the one for equity flows,  $B^z$ . Obviously, this CE discount factor is independent of both the *current* capital structure, in case the investment option is owned by an ongoing firm, and the capital structure *after* the project is implemented.

In addition, equation (5.3) suggests that the option to invest in a marginal project (i.e., a project with corporate tax rate,  $\tau^c$ , equal to the marginal tax rate,  $\tau_m$ , and no tax shield  $\tau^* = 0$ ) is evaluated according to Black, Scholes, and Merton's formula, but using  $r_z$  instead of  $r_f$ . Note that in our setting, since a project cannot be all-debt financed,  $r_f$  can only be used in the special case where  $\tau^e = \tau^b$  (which implies  $\tau_m = 0$ ).

## 6 Concluding Remarks

We have shown how to implement the value of interest tax shields in a real options setting and investigated the implications of these on the valuation of an investment option.

First, we had to study interest tax shields more rigourously than is common in the literature because the differential taxation of debt and equity income at the personal level has the effect of making interest tax shields less valuable than is commonly believed. Indeed, for firms with low tax rates, interest tax shields can have negative value. In a generalized Miller tax equilibrium, there will be marginal firms and investors who are indifferent between the tax implications of debt and equity. Their tax rates can be used to determine the differential rate of return on riskless debt and riskless equity that is needed to sustain such a tax equilibrium.

This allowed us to characterize interest tax shields in terms of the difference between a firm's tax rate and the marginal firm's tax rate.

Having cash flows at a pre-corporate tax level, after-corporate tax level and an after-all-tax level leads to natural questions of the level at which cash flows should be valued. The most appropriate valuation occurs at the after-all-tax level for the marginal investor. Unfortunately, such cash flows are not readily observable. Fortunately, we have been able to show that if there is linear taxation at the personal level, a linear valuation operator after all taxes leads to a linear valuation operator after corporate tax but prior to personal tax. Moreover, the certainty-equivalent operators or risk-neutral expectation operators of these pretax and after-tax cash flows are the same. The only difference in valuation is the choice of discount rates: a riskless equity rate is correctly used to discount risk-neutral expected equity flows, a riskless debt rate is correctly used to discount risk-neutral expected debt flows and an after-all-tax rate is used to discount risk-neutral expected flows after all tax.

In terms of the fundamental pde for valuing risky payoffs or interest tax shields, this means that the only change needed to reflect that the flows go to equity or debt investors is to change the riskless discount rate in the risk-neutral required return. The coefficients corresponding to risk adjustments are the same for the equity and debt pde, despite the differential taxation.

This allows us to value interest tax shields with and without default risk on the debt and extend these results to the real option to invest. We showed that the riskless equity rate is the appropriate tax-adjusted discount rate when computing the value of the company (APV) from total cashflows. In particular, it is inappropriate to discount the interest tax shield at a debt rate. We showed that real options should be evaluated according to Black, Scholes, and Merton's formula but applying the riskless equity rate as discount factor. Note that using the general formula where the riskless debt rate is used to discount the certainty

equivalent payoff would also be inappropriate.

In our view, there is considerable opportunity for additional work along the lines of the framework we have presented here. First, other neutral taxation schemes could be analyzed. Second, our approach can be applied to a broader class of real options. Finally, other related issues, such as credit spreads of corporate bonds, can also be studied in the context of our framework.

## References

- Auerbach, A. J. (1991). Retrospective capital gain taxation. *American Economic Review*, 81(1):167–178.
- Auerbach, A. J. and Bradford, D. (2001). Generalized cash flow taxation. Technical Report 8122, National Bureau of Economic Research.
- Black, F. (1972). Capital market equilibrium with restricted borrowing. *Journal of Business*, 45:444–455.
- Brennan, M. and Schwartz, E. (1978). Corporate income taxes, valuation, and the problem of optimal capital structure. *Journal of Business*, 51(1):103–114.
- Childs, P. D., Mauer, D. C., and Ott, S. H. (2005). Interactions of corporate financing and investment decisions: The effects of agency conflicts. *Journal of Financial Economics*, 76:667–690. presented at the 4<sup>th</sup> Annual Conference on Real Options, University of Cambridge as Interaction of Corporate Financing and Investment Decisions: The Effect of Growth Options to Exchange or Expand.
- Christensen, P. O., Flor, C. R., Lando, D., and Miltersen, K. R. (2002). Dynamic capital structure with callable debt and debt renegotiations. Technical report, Norwegian School of Economics and Business Administration.
- Constantinides, G. M. (1978). Market risk adjustment in project valuation. *Journal of Finance*, 33(2):603–616.
- Constantinides, G. M. (1983). Capital market equilibrium with personal tax. *Econometrica*, 51:611–636.

- Cooper, I. and Nyborg, K. (2004). Tax-adjusted discount rates with investor taxes and risky debt. Technical report, LBS Institute of Finance - Working Paper No. 429.
- Cooper, I. and Nyborg, K. (2006). The value of tax shields IS equal to the present value of tax shields. *Journal of Financial Economics*, 81(1):215–225.
- Dangl, T. and Zechner, J. (2003). Credit risk and dynamic capital structure choice. Technical Report 4132, CEPR.
- Dixit, A. and Pindyck, R. (1994). *Investment Under Uncertainty*. Princeton University Press, Princeton, NJ.
- Duffie, D. (1988). *Security Markets: Stochastic Models*. Academic Press, London.
- Duffie, D. (2001). *Dynamic Asset Pricing Theory*. Princeton University Press, Princeton - NJ, third edition.
- Dynkin, E. B. (1965). *Markov Processes*, volume II. Academic Press.
- Fernandez, P. (2004). The value of tax shields is NOT equal to the present value of tax shields. *Journal of Financial Economics*, 73:145–165.
- Fernandez, P. (2005). Reply to “Comment on the value of tax shields is NOT equal to the present value of tax shields”. *Quarterly Review of Economics and Finance*, 45(1):188–192.
- Fieten, P., Kruschwitz, L., Laitenberger, J., Loffler, A., Tham, J., Velez-Pareja, I., and Wonder, N. (2005). Comment on “The value of tax shields is NOT equal to the present value of tax shields”. *Quarterly Review of Economics and Finance*, 45(1):184–187.
- Fisher, E. O., Heinkel, R., and Zechner, J. (1989). Dynamic capital structure choice: Theory and tests. *Journal of Finance*, 44:19–40.

- Goldstein, R., Ju, N., and Leland, H. (2001). An EBIT-based model of dynamic capital structure. *Journal of Business*, 74(4):483–512.
- Grinblatt, M. and Liu, J. (2002). Debt policy, corporate taxes, and discount rates. Technical report, Anderson School at UCLA.
- Hennessy, C. A. and Whited, T. M. (2005). Debt dynamics. *Journal of Finance*, 60(3):1129–1165.
- Hennessy, C. A. and Whited, T. M. (2006). How costly is external financing? evidence from a structural estimation. *Journal of Finance*, forthcoming.
- Jensen, B. A. (2003). On valuation before and after tax in no arbitrage models: tax neutrality in the discrete time model. Technical report, Dept. of Finance - Copenhagen Business School.
- Kane, A., Marcus, A. J., and McDonald, R. L. (1984). How big is the tax advantage to debt? *The Journal of Finance*, 39(3):841–853.
- Lamberton, D. and Lapeyre, B. (1996). *Introduction to Stochastic Calculus Applied to Finance*. Chapman & Hall, London, UK.
- Leland, H. E. (1994). Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 49(4):1213–1252.
- Leland, H. E. and Toft, K. B. (1996). Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *Journal of Finance*, 51(3):987–1019.
- Mauer, D. C. and Ott, S. H. (2000). Agency costs, underinvestment, and optimal capital structure. In Brennan, M. J. and Trigeorgis, L., editors, *Project flexibility, Agency, and Competition*, pages 151–179. Oxford University Press, New York, NY.

- Mauer, D. C. and Triantis, A. J. (1994). Interaction of corporate financing and investment decisions: a dynamic framework. *Journal of Finance*, 49:1253–1277.
- Mello, S. A. and Parsons, J. E. (1992). Measuring the agency cost of debt. *Journal of Finance*, 47(5):1887–1904.
- Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29:449–470.
- Miles, J. A. and Ezzel, J. R. (1980). The weighted average cost of capital, perfect capital markets, and project life: a clarification. *Journal of Financial and Quantitative Analysis*, 15(3):719–730.
- Miles, J. A. and Ezzel, J. R. (1985). Reformulating tax shield valuation: a note. *Journal of Finance*, 40(5):1485–1492.
- Miller, M. H. (1977). Debt and taxes. *Journal of Finance*, 32:261–275.
- Modigliani, F. and Miller, M. H. (1958). The cost of capital, corporation finance and the theory of investment. *American Economic Review*, 48:261–297.
- Modigliani, F. and Miller, M. H. (1963). Corporate income taxes and the cost of capital: a correction. *American Economic Review*, 53:433–443.
- Myers, S. C. and Majluf, N. (1984). Corporate financing and investment decisions when firms have information that investors do not have. *Journal of Financial Economics*, 5:187–221.
- Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 13:341–360.

- Ross, S. A. (1987). Arbitrage and martingales with taxation. *Journal of Political Economy*, 95:371–393.
- Ross, S. A. (2005). Capital structure and the cost of capital. *Journal of Applied Finance*, 15(1).
- Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under condition of risk. *Journal of Finance*, 19:425–442.
- Sick, G. (1990). Tax-adjusted discount rates. *Management Science*, 36:1432–1450.
- Strebulaev, I. A. (2006). Do tests of capital structure theory mean what they say? *Journal of Finance*, forthcoming.
- Taggart, R. A. (1991). Consistent valuation and cost of capital with corporate and personal taxes. *Financial Management*, 20(3):8–20.
- Titman, S. and Tsyplakov, S. (2006). A dynamic model of optimal capital structure. Technical Report FIN-03-06, McCombs School of Business - University of Texas at Austin.
- Trigeorgis, L. (1996). *Real Options: managerial flexibility and strategy in resource allocation*. MIT Press, Cambridge, MA.