Analysis of Sand Production in Unconsolidated Oil Sand Using a Coupled Erosional-Stress-Deformation Model

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Abstract
The paper presents a sand production model for a deforming oil sand matrix. The deformability of the oil sand matrix is an important issue given that in situ and well pumping induced stresses impact on the susceptibility of the oil sand to produce sand. The sand production model is formulated in a consistent manner within the framework of mixture theory with porosity as one of the main field state variables. The latter is split into two parts: one related to volume changes as a result of erosion in the oil sand matrix, and the other one due to deformations in the matrix subjected to a stress field. The coupling of the erosion model to a stress model is made through bulk volumetric strains. Also, the erosion constitutive law is intimately tied with material strengths that enter the stress model. Finally, some numerical examples of sand production restricted to a rigid matrix are given to illustrate the theory, pending the completion of the finite element implementation of a coupled stress-erosional model.

Introduction
Sand production during hydrocarbon production in oil wells is both a costly and prevalent phenomenon that is not well understood. Consequently, sand production and control has been a research topic for more than five decades. From a mechanistic viewpoint, sand production emanates from the progressive disaggregation of the poorly consolidated formation due to many factors: notably, stresses, fluid flow, thermal, solution gas drive, and reservoir heterogeneity in porosity. Once initiated, sand production can be progressive and may strike at varying degrees of severity, ranging from erosion and plugging of pumps, valves, and pipes to the development of large cavities or wormhole-like structures in the formation resulting in damage and casing collapse.

A review of sand production issues, together with the development of a model describing the erosion of sand grains in a rigid oil sand matrix, were presented in Wan and Wang(1). In this paper, the model is further developed with the constraint of the rigid matrix removed so that both stress and strength characteristics enter formally into the formulation. It is well recognized that in situ stresses, as well as those induced during drilling, have an impact on sand production. In this paper, a framework that considers the rigorous coupling between stress/deformations and erosion is proposed within the continuum theory of mixtures(2). Such rigorous coupling is scarce in the current literature, except for the work of Stavropoulou et al.(3) in which the coupling between skeleton deformations and fluid transport was formulated, but was not done in a consistent manner.

Model Description
Figure 1 illustrates the mechanics of sand production around a wellbore with sand grains being dislodged from the oil sand matrix. From a mechanistic point of view, sand production emerges as a result of an instability occurring in a viscous fluid saturated porous medium that undergoes mechanical deformation in the presence of fluid fluxes. This condition can be described within a three-phase system in which solid (s), fluid (f), and fluidized fluid (ff) interact. Despite the non-homogeneous structure of the porous medium, it is still possible to invoke a continuum theory of mixtures such that all three phases are simultaneously present everywhere to occupy a chosen representative elementary volume (REV), as shown in Figure 1. The size of the REV can be of the order of several hundreds or thousands of pore spaces. In order to keep the formulations tractable, the following assumptions are made: (i) solid skeleton is deformable and described within infinitesimal strain theory; (ii) fluidized particles are particles in suspension that move together with fluid; (iii) the densities of both solid and fluidized solid phases are equal and constant, \( \rho_{fs} = \rho_s \); (iv) fluid is incompressible so that its density \( \rho_f \) is constant; and, (v) at any instant, both fluid and fluidized particles have the same velocity so that \( \mathbf{u}_f = \mathbf{u}_{ff} \). Also, the model does not address the issue of gas solution drive on sand production mechanics.

It is pertinent to recall the definition of controlling field variables such as porosity \( \phi \), fluidized solid concentration \( c \), and flux vector \( \mathbf{v}_f \) that affect both the behaviour and interaction of each phase. These field variables are continuous and were defined in Wan and Wang(1). For instance, porosity is the volume of void space over total volume, fluidized solid concentration is the ratio of fluidized solid volume to that of void space, and flux is the volume of flow (fluidized solid and fluid) per averaged cross section area per unit time.
To obtain the governing equations for the mechanics of fully saturated porous media, balance equations are written for each phase factored with its corresponding volume fraction so as to account for its discontinuous distribution. Note that the volume fraction is defined as the volume of each phase per unit volume of the material.

**Governing Equations**

In writing balance equations of moving phases, material time derivatives must be invoked, i.e., given a function \( F(x, t) \), its material time derivative is defined by:

\[
\frac{\partial F}{\partial t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \mathbf{u}
\]

where \( \mathbf{u} \) is the velocity of the moving phase boundary. It follows that the material time derivative of a volume integral is:

\[
\frac{D}{Dt} \int_A d\Omega = \int_A \left( \frac{\partial F}{\partial t} + \nabla \cdot (\mathbf{A} \mathbf{u}) \right) d\Omega
\]

where \( A(x, t) \) denotes a property of the continuum, the boundary of the volume \( \Omega \) being a moving surface, and the integral being evaluated at time \( t \).

**Mass Conservation Equations**

**Solid Phase (s)**

The mass of the solid phase averaged out over a REV of volume \( \Omega \) can be written as:

\[
M_s = \int_\Omega (1 - \phi) \rho_s \, d\Omega
\]

where \( \phi \) is the porosity and \( \rho_s \) is the density of the solid phase. The mass conservation requires that:

\[
\frac{\partial (1 - \phi) \rho_s}{\partial t} + \nabla \cdot \left[ (1 - \phi) \rho_s \mathbf{u}_s \right] = -m
\]

where \( \mathbf{u}_s \) is the absolute velocity of the solid phase and \( -m \) is the local rate of solid loss per unit volume due to erosion phenomenon. Considering the arbitrariness of \( \Omega \), the local mass balance equation becomes:

\[
\frac{\partial (1 - \phi) \rho_s}{\partial t} + \nabla \cdot \left[ (1 - \phi) \rho_s \mathbf{u}_s \right] = -m
\]

**Fluidized Solid Phase (fs)**

The mass of fluidized solid averaged over a given REV is obtained by invoking fluidized solid concentration \( c \), porosity \( \phi \), and density of the fluidized solid phase \( \rho_{fs} \), i.e.:

\[
M_{fs} = \int_\Omega c \rho_{fs} \, d\Omega
\]

The mass conservation of fluidized solids reduces to the following local balance equation:

\[
\frac{\partial (c \rho_{fs})}{\partial t} + \nabla \cdot \left[ c \rho_{fs} \mathbf{u}_{fs} \right] = m
\]

where \( \mathbf{u}_{fs} \) is the absolute velocity of the fluidized solid phase and \( m \) is the rate of mass gain.

**Fluid Phase (ff)**

Finally, the averaged mass of fluid automatically ensues as:

\[
M_{ff} = \int_\Omega (1 - c) \phi \rho_{ff} \, d\Omega
\]

with local mass balance for this phase being:

\[
\frac{\partial (1 - c) \phi \rho_{ff}}{\partial t} + \nabla \cdot \left[ (1 - c) \phi \rho_{ff} \mathbf{u}_{ff} \right] = 0
\]

where \( \rho_{ff} \) is fluid density and \( \mathbf{u}_{ff} \) is absolute velocity of fluid phase.

In anticipation to the description of fluid flow through a porous medium, we define a volume averaged discharge velocity \( \mathbf{v}_f \) of fluid mixture relative to the solid matrix, i.e.:

\[
\mathbf{v}_f = \phi (\mathbf{u}_{ff} - \mathbf{u}_s); \quad \mathbf{u}_{ff} = \mathbf{u}_f = \mathbf{v}_f / \phi + \mathbf{u}_s
\]

The discharge velocity \( \mathbf{v}_f \) corresponds to the velocity of the fluid and fluidized solid mixture such that:

\[
\mathbf{v}_f = \mathbf{v}_{ff} + \mathbf{v}_s = (1 - c) \mathbf{v}_f + c \mathbf{v}_s
\]

where \( \mathbf{v}_f \) and \( \mathbf{v}_s \) are the discharge velocities of the fluid and fluidized solid phases, respectively.

Equations (5), (7), and (9) represent local mass balance equations for each individual phase. Successively combining Equations (5), (7), and (9) together, and eliminating fluidized solid and fluid velocities with the aid of Equation (10), we obtain the following three governing equations, i.e.:

\[
\frac{\partial (1 - \phi) \rho_s}{\partial t} + \nabla \cdot \left[ (1 - \phi) \rho_s \mathbf{u}_s \right] = -m
\]

\[
\frac{\partial (1 - c) \phi \rho_{fs}}{\partial t} + \nabla \cdot \left[ (1 - c) \phi \rho_{fs} \mathbf{u}_{fs} \right] = 0
\]

\[
\nabla \cdot \mathbf{v}_f = -\nabla \cdot \mathbf{u}_s = -\frac{\partial \rho_s}{\partial t}
\]

In Equation (14), due to incompressibility of the fluid, mass balance implies that the volume change of the matrix \( \epsilon \), due to deformations corresponds to the net rate of change of fluid fluxes. Moreover, the porosity term in Equations (12) and (13) refers to total volume changes that arise from: (i) solid skeleton deformations as a result of grain rearrangements under stresses; and, (ii) erosion as grains are dislodged from the matrix and enter the fluid phase. Thus, ideally the total porosity can be decomposed into stress induced (\( \phi_{er} \)) and erosion based (\( \phi_{er} \)) porosities such that:

\[
\phi = \phi_{er} + \phi_{er}
\]

and

\[
\frac{\partial \rho_s}{\partial t} = \frac{\partial \phi_{er}}{\partial t}
\]

Equations (12) to (14) represent a system composed of three equations with nine principal unknowns \( c, \phi, \mathbf{m}, \mathbf{v}_f \), and \( \mathbf{u}_s \) in the three dimensional case. These equations must be supplemented with an erosion law for characterizing mass generation, Darcy’s law for fluid flow, and equilibrium equations for stresses and deformations.
Constitutive Law for Mass Generation

Intuitively, there must be a critical fluid velocity at which sand production is initiated. Once initiated, the rate at which sand grains are eroded will depend principally on grain contact strength, local fluid drag forces, fluid pressures, and availability of solids. Furthermore, it is clear that the erosion process is more intense in intact regions where porosity \( \phi \) is small. Based on phenomenology, a possible functional form of mass generation can be written as:

\[
\dot{m}_s = \lambda (1-\phi) \dot{c} \frac{v_f}{\mu} \left[ \sqrt{v_f} - \sqrt{v_f^0} \right] = 0 \quad \sqrt{v_f} < \sqrt{v_f^0} \quad \text{........................................(17)}
\]

where \( \lambda \), having the dimension of inverse of length, has to be determined experimentally. Basically, \( \lambda \) provides a length scale that can be related to the grain contact strength and grain size. Once the fluid velocity exceeds a critical value also based on strength, rate of erosion follows the fluidized solid velocity \( v_f \). Thus, solid erosion is suppressed and will not be triggered unless a finite concentration \( c \) is present.

Constitutive Law for Fluid Flow

Due to the complexity of flow in porous media, semi-empirical Darcy’s law is used rather than the equations of fluid momentum balance. Darcy’s law establishes the relation between pressure gradient \( \nabla p \) and volumetric fluid mixture flux per unit area, \( v_f \). Thus:

\[
\dot{v}_f = -\frac{k}{\mu} \nabla p \quad \text{........................................(18)}
\]

where \( k \) is the effective permeability tensor that can be related to porosity via Carman-Kozeny equation, i.e.:

\[
k = k_0 \phi^3 \left(1-\phi\right)^2 I \quad \text{........................................(19)}
\]

and \( k_0 \) is a constant and \( I \) is the Kronecker delta tensor such that \( I_{ij} = \delta_{ij} \).

Equation (19) describes how the permeability changes in the eroded regions as a function of porosity. Furthermore, the parameter \( \mu \) in Equation (18) refers to the viscosity of the fluidized sand and fluid mixture. It can be related to the kinematic viscosity \( \eta \) of the fluid using averaging of the mixture of phases, i.e.:

\[
\dot{\mu} = \bar{\mu}(c) \eta; \quad \bar{\mu}(c) = (1-c) \rho_f + c \rho_s \quad \text{........................................(20)}
\]

in which \( \bar{\mu}(c) \) is the density of the fluidized sand and fluid mixture.

Equilibrium and Stress Decomposition

When considering the deforming oil sand under a stress field \( \sigma \), overall force equilibrium must be satisfied, i.e.:

\[
\nabla \cdot \sigma + b = 0 \quad \text{........................................(21)}
\]

where \( b \) are body forces per unit volume. In order to describe hydromechanical phenomena (i.e., the impact of interstitial fluids on deformations), the framework of effective stress in porous media is invoked in which the stress \( \sigma \) can be decomposed into a so-called effective stress \( \sigma^{\text{eff}} \) in the oil sand skeleton and a fluid pressure component \( p \) in the pores. Thus:

\[
\sigma = \sigma^{\text{eff}} - \beta p I \quad \text{........................................(22)}
\]

where \( \beta \) is a parameter accounting for the compressibility of the sand grains. The sign convention adopted is negative, stresses are compressive, and fluid pressures are always positive. It is instructive to have further insight into the nature of \( \sigma^{\text{eff}} \) based on stress averaging and decomposition, i.e.:

\[
\sigma^{\text{eff}} = \bar{\sigma}_s (1-\phi) - \phi p I \quad \text{........................................(23)}
\]

where \( \bar{\sigma}_s \) is the average intergranular stress. By comparing Equation (22) with Equation (23) and assuming \( \beta = 1 \), the relation between effective stress and intergranular stresses emerges as:

\[
\sigma^{\text{eff}} = (\bar{\sigma}_s - p I) (1-\phi) \quad \text{........................................(24)}
\]

and it is the effective stresses that are used when calculating a constitutive response.

Poroelectricity

In calculating deformations in a linear isotropic poroelectricity, the effective stress tensor is related to the strain tensor through a fourth order elastic tensor, \( C^* \), with two parameters, i.e., elastic modulus \( E \) and Poisson ratio \( \nu \). Thus:

\[
\sigma^{\text{eff}} = C^* : e \quad \text{........................................(25)}
\]

where,

\[
C^* = K I + 2G \left(I - \frac{1}{3} I \otimes I \right) \quad \text{........................................(26)}
\]

with,

\[
K_{ijkl} = \frac{1}{2} \left( \delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj} \right) \quad \text{........................................(27)}
\]

The symbol \( \otimes \) represents the tensor product operator in the sense that \( (1 \otimes 1)_{ijkl} \) implies \( \delta_{ij} \delta_{kl} \). Elastic moduli \( K \) and \( G \) are bulk and shear moduli expressed in terms of \( E \) and \( \nu \), i.e.:

\[
K = \frac{E}{3(1-2
u)}; \quad G = \frac{E}{2(1+\nu)} \quad \text{........................................(28)}
\]

An improvement over simple elasticity can be made by considering nonlinear elasticity with a variable elastic modulus \( E' \). Equation (24) suggests that the effective stresses borne by the solid phase in dry conditions is \( \sigma^{\text{eff}} = (1-\phi) \bar{\sigma}_s \). Thus, this means that effective elastic modulus \( E' \) takes the form of:

\[
E' = E(1-\phi) \quad \text{........................................(29)}
\]

with \( \phi \) acting as a damage parameter applied to the elastic modulus \( E \) of the solid material.

Poroplasticity

The assumption of simple elastic behaviour for the solid phase is not a good one considering that sand behaviour is mainly dissipative and dominated by grain slippage, rearrangement, dilation, and destructuration. Hence, a more adequate constitutive law based on plasticity and incorporating stress dilatancy aspects must be used. As the oil sand is sheared under the action of drilling for example, an increase in volume (dilation) has to occur for deformations to mobilize. This behavioural aspect has indeed an impact on sand production and has to be addressed within elasto-plasticity theory which invokes a yield criterion combined with a plastic flow rule to describe yield condition and plastic strains respectively. A yield function \( F(\sigma) \) based on Mohr-Coulomb is considered adequate, while a plastic potential function \( G \) must be introduced to calculate plastic strains. A stress-dilatancy equation developed by Wan & Guo(4) is used to derive an appropriate form of \( G \).
Equations (33) to (38) can be discretized so that finite element equations are readily obtained using Galerkin’s method of weighted residuals over the domain \( \Omega \) and standard finite difference formula for time derivatives. The same approach as in Wan and Wang\(^{(1)} \) can be used in order to arrive at a system of non-linear equations. Cast within a Newton-Raphson framework, the incremental form of the equations for a 4-node bilinear element emerges as:

\[
\begin{bmatrix}
[A_1]_{4\times4} & [A_2]_{4\times4} & [A_3]_{4\times4} & [A_4]_{4\times4} \\
[B_1]_{4\times4} & [B_2]_{4\times4} & [B_3]_{4\times4} & [B_4]_{4\times4} \\
[C_1]_{4\times4} & [C_2]_{4\times4} & [C_3]_{4\times4} & [C_4]_{4\times4} \\
0 & [D_1]_{8\times4} & [D_2]_{8\times4} & [D_3]_{8\times4} \\
\end{bmatrix}
\begin{bmatrix}
\Delta \epsilon_{x+1}^k \\
\Delta \epsilon_{y+1}^k \\
\Delta \phi_{x+1}^k \\
\Delta \phi_{y+1}^k \\
\end{bmatrix}_{20\times1}
= 
\begin{bmatrix}
[X_{t+1}^k]_{20\times1} \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

where \( k \) is the iteration number, \( n \) is the time step number, \( X \) is a vector containing known variables at previous iterate, and \( A_1, A_2, \ldots, B_1, B_2, \ldots \) are sub-matrices pertinent to fluid, solid, fluidized solid, and stress-deformation properties.

**Numerical Examples**

In the following simulations, a rigid matrix was considered as the finite element implementation of the theory presented in the previous sections is still in progress. As a matter of fact, the rigid matrix case involves suppressing skeleton deformations \((u = 0, \epsilon = 0)\) and eliminating Equations (37) and (38).

**Sand Production Around a Perforated Wellbore**

Figure 2 shows a close-up of the finite element grid representing one quarter of a section of a well of inner radius \( r_0 = 0.1 \) m. The outer boundary of the well extends to 5 m at which a pressure of 10 MPa is imposed, while the inner boundary at wellbore is kept at 0 MPa so as to induce drawdown. The entire finite element grid is comprised of 3,320 nodes and 3,198 elements with each node having 3 degrees of freedom, i.e., \( c, \phi, \) and \( p \).

The initial fluidized sand concentration and porosity are chosen as 0.001 and 0.33, respectively. The sand erosion rate parameter \( \lambda \) is 5 m\(^{-1}\) with \( \rho_s = 2.65 \) g/cm\(^3\) and \( \rho_f = 0.9854 \) g/cm\(^3\). The perforations \((P_1, P_2, P_3)\) at the wellbore are also shown in Figure 2 and are represented by elements having a porosity of 0.4, which is higher than the rest of the elements.

At first, isotropic permeability conditions are considered with \( k_0 = k_0 = 15 \) Darcy and \( \mu = 10^4 \) cp. Figures 3 and 4 show contours of porosity at different times \((t = 0.3 \text{ and } 1.6 \) days\) so as to illustrate the growth of loose zones that emanate from the perforations. In Figure 4, three distinct zones of very high porosity \((\phi \approx 0.9)\) propagate quite extensively into the formation with lesser erosion activity occurring in between perforations. While the model is continuum based, it is still possible to predict “wormhole” formation indicated by regions of very high porosities approaching one. Since isotropic permeability conditions were considered, the three “wormholes” shown in Figure 4 are geometrically similar. Also, the porosity contours reveal a zone of localized depletion \((\phi = 0.7 \text{ to } 0.9)\) within approximately 20 cm around the perforations. Beyond this zone, erosion proceeds in a homogeneous manner, as shown by the uniform porosity contour lines.

Next, anisotropic permeability conditions are investigated and \( k_0 = 1.5 k_0 \), with \( k_0 = 15 \) Darcy and \( \mu = 10^4 \) cp. Figure 5 shows the initiation of the “wormhole” at the first perforation \((P_1)\) with a general bias of propagation in the \( x \) direction where the permeability is the greatest. Figure 6 shows the further progression of the “wormhole” localizing along the \( x \)-direction of the well with limited erosion occurring at other perforations. There is a general redistribution of fluxes around the “wormhole” that impedes erosion at other sites than perforation \( P_1 \). Hence, this
suggests that “wormholes” can be directionally formed based on anisotropy in permeability. In general, erosion phenomenon follows the direction of highest fluxes guided by changes in permeability, fluid pressure, and porosity.

Figure 7 shows the depletion of fluid pressures at final time \( t = 1.6 \) days, while Figure 8 gives the total flux vectors close to the wellbore. As expected, the fluxes are the highest at the first perforation.

Finally, Figures 9 and 10 show the evolution of porosity and fluidized sand concentration near the first perforation (P1) for both the isotropic and anisotropic cases. It is seen that the erosion process is faster for the anisotropic case since porosity reaches one more rapidly than in the isotropic case (see Figure 9). As far as fluidized sand concentration evolution is concerned, there is not too much of a difference between isotropic and anisotropic cases as shown in Figure 10. This is because the anisotropic case was chosen such that the horizontal permeability was kept the same as in the isotropic one, but the permeability in the vertical direction was set to a lower coefficient.

Conclusions

A framework has been proposed in which coupled stress-deformation-erosion phenomena can be modelled by linking skeleton volume changes to fluid transport in a consistent manner via theory of mixtures. Hence, total porosity has two components: one emanating from volume changes due to eroding sand grains, and the other one due to sand skeleton deformations under stresses.

Numerical results presented in the second part of the paper were confined to a rigid sand matrix by suppressing all skeleton displacements in the model. Under these constraints, the model still gave valuable insights in sand production mechanisms. For example, “wormhole” propagation was captured around a perforated wellbore with localization of erosion in the vicinity of the perforations where high fluxes existed. It was also found that anisotropy of permeability dictates the wormhole propagation which follows the direction of greatest permeability. However, this observation will need to be verified when the deformation of the sand matrix will be fully incorporated into the model. This will be the subject of forthcoming publications.
Acknowledgements
The authors wish to express their sincere gratitude for funding provided by the Alberta Department of Innovation and Science.

NOMENCLATURE
Bold characters refer to vectors and tensors. Repeated indices refer to summation over the dummy index.

- \( f, fs, s \) = fluid, fluidized sand, and sand
- \( c \) = fluidized sand concentration
- \( m, m' \) = mass and mass rate
- \( k \) = effective permeability tensor
- \( p \) = fluid pressure
- \( v_i \) = volume rate discharge in direction \( i \)
- \( t \) = time
- \( u \) = displacement vector
- \( \phi \) = porosity
- \( \lambda \) = sand rate parameter; inverse of a length unit
- \( \mu \) = viscosity in cp (centipoise)
- \( \rho \) = density
- \( \Omega \) = volume of interest
- \( \otimes \) = tensor product operator
- \( \nabla \) = gradient operator
- \( D \)
- \( \frac{D}{Dt} \) = material derivative operator
- \( \mathbf{1} \) = Kronecker delta tensor

REFERENCES

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Jin Wang obtained his B.Sc. in civil engineering from Xi’an University of Technology, China in 1993, and his M.Sc. degree in civil engineering from Katholieke Universiteit Leuven, Belgium in 1998. His doctoral study began in 1999, with research work in the area of petroleum geomechanics, including well instability and sand production. His expertise is in numerical modelling and geomechanics-related reservoir problems. He has been awarded several scholarships, and became one of six finalists for the 15th annual Robert J. Melosh Medal competition at Duke University in 2003. After completing his Ph.D. at the University of Calgary in 2003, he joined Taurus Reservoir Solutions Ltd. as a consulting engineer and has been involved in a wide range of projects such as SAGD, reservoir compaction, and sand production. Based on his outstanding Ph.D. work on the prediction of sand production and control, he was awarded an Alberta Ingenuity Fund Fellowship to continue his research in this field.