Levy Processes-From Probability to Finance

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Outline

- Introduction: Probability and Stochastic Processes
- Main Contributors to the Theory of Levy Processes: P. Levy, A. Khintchine, K. Ito
- The Structure of Levy Processes
- Applications to Finance

Introduction: Probability

- *Theory of Probability:* aims to model and to measure the 'Chance'
- *The tools*: Kolmogorov's theory of probability axioms (1930s)
- *Probability* can be rigorously founded on *measure theory*

Introduction: Stochastic Processes

- *Theory of Stochastic Processes*: aims to model the interaction of 'Chance' and 'Time'
- *Stochastic Processes*: a family of random variables (**X**(t), t=>0) defined on a probability space (**Omega, F, P**) and taking values in a measurable space (**E,G**)
- **X**(t) is a (**E,G**) measurable mapping from **Omega** to **E:** a random observation made on **E** at time t

Importance of Stochastic Processes

- Not only *mathematically rich objects*
- *Applications*: physics, engineering, ecology, economics, **finance**, etc.
- *Examples*: random walks, Markov processes, semimartingales, measure-valued diffusions, **Levy Processes**, etc.

Importance of Levy Processes

- *There are many **important examples**: Brownian motion, Poisson Process, stable processes, subordinators, etc.*
- ***Generalization** of random walks to continuous time*
- *The simplest classes of **jump-diffusion** processes*
- *A natural models of noise to **build stochastic integrals** and to drive SDE*
- *Their structure is **mathematically robust and** generalizes from Euclidean space to Banach and Hilbert spaces, Lie groups, quantum groups*
- *Their structure contains many features that **generalize** naturally to much **wider classes of processes**, such as semimartingales, Feller-Markov processes, etc.*

Levy Processes in Mathematical Finance

- They can describe the observed reality of financial markets in a more accurate way than models based on Brownian motion

Levy Processes in Mathematical Finance I

- In the ‘real’ world, we observe that asset price processes have **jumps or spices** (see Figure 1.1) and risk-managers have to take them into account

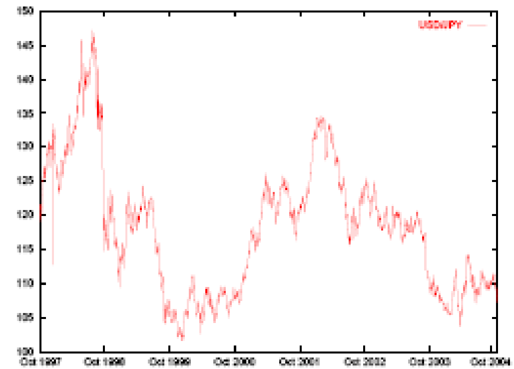


FIGURE 1.1. USD/JPY foreign exchange rate, October 1997-October 2004.

Levy Processes in Mathematical Finance II

- The empirical distribution of asset returns exhibits **fat tails and skewness**, behaviour that deviates from normality (see Figure 1.2)
- That models are essential for the estimation of **profit and loss (P&L) distributions**

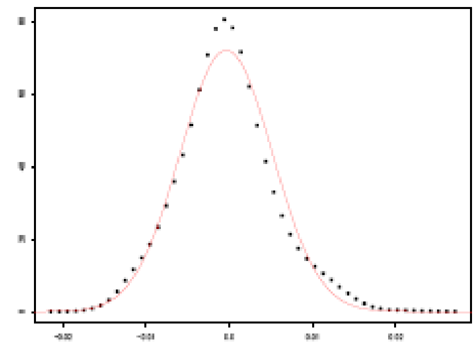


FIGURE 1.2. Empirical distribution of daily log-returns for the GBP/USD exchange rate and fitted Normal distribution

Levy Processes in Mathematical Finance III

- In the ‘risk-neutral’ world, we observe that implied volatility are constant neither across strike nor across maturities as stipulated by the classical Black-Scholes (1973) (see Figure 1.3)

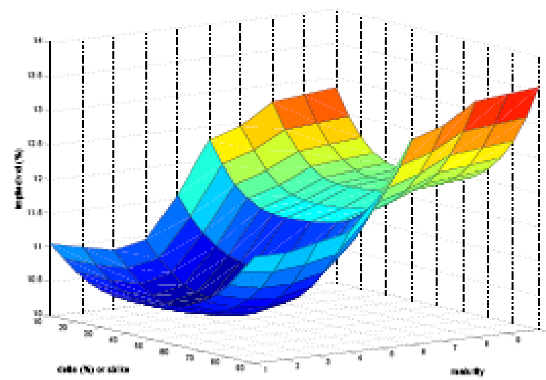


FIGURE 1.3. Implied volatilities of vanilla options on the EUR/USD exchange rate on November 5, 2001.

Levy Processes in Mathematical Finance IV

- **Levy Processes** provide us with the appropriate framework to adequately describe all these observations, both in the ‘real’ world and in the ‘risk-neutral’ world

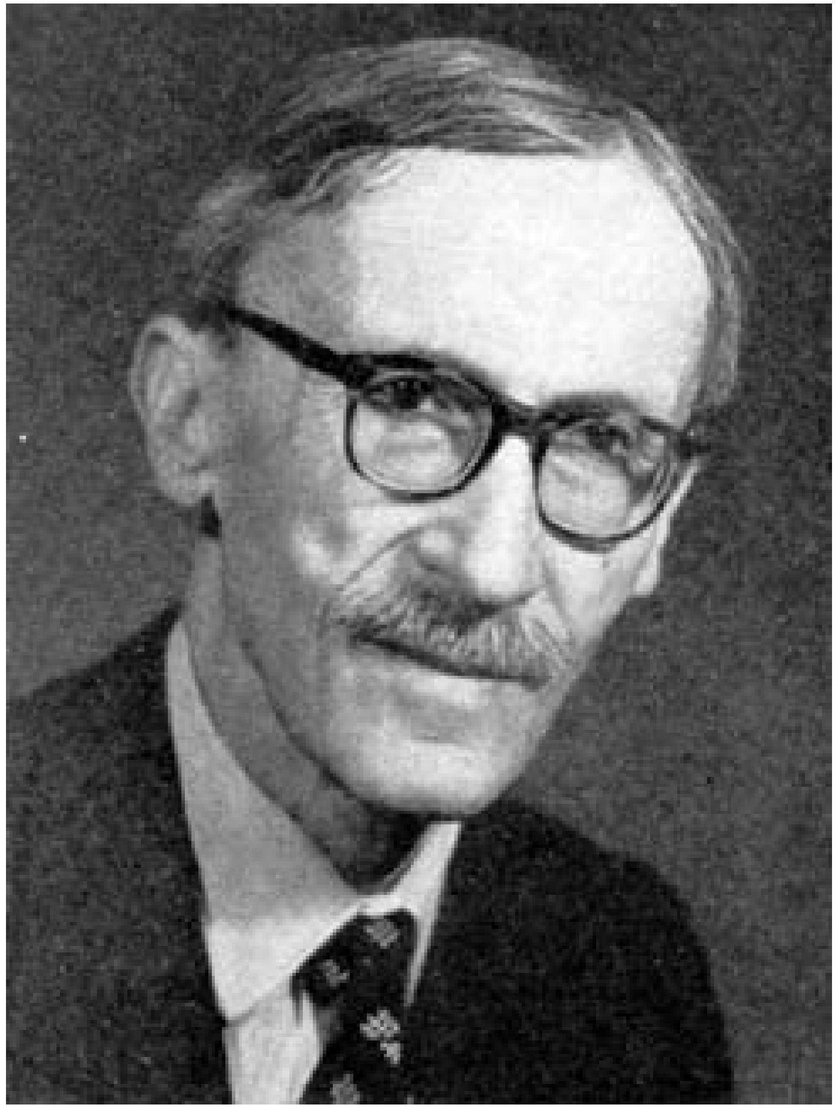
Main Original Contributors to the Theory of Levy Processes: 1930s-1940s

- Paul **Levy** (France)
- Alexander **Khintchine** (Russia)
- Kiyosi **Ito** (Japan)

Main Original Papers

- **Levy P.** *Sur les integrales dont les elements sont des variables aleatoires independentes*, Ann. R. Scuola Norm. Super. Pisa, Sei. Fis. e Mat., Ser. 2 (1934), v. III, 337-366; Ser. 4 (1935), 217-218
- **Khintchine A.** A n*ew derivation of one formula by Levy P*., Bull. Moscow State Univ., 1937, v. I, No 1, 1-5
- **Ito K.** *On stochastic processes*, Japan J. Math. 18 (1942), 261-301

Paul Levy (1886-1971)



P. Levy’s Contribution

Lévy contributed to the

- theory of probability,
- functional analysis, and other analysis problems,
- principally partial differential equations and series
- He also studied geometry

P. Levy’s Contribution I

- One of the founding fathers of the theory of **stochastic processes**
- Made major contributions to the field of **probability theory**
- Contributed to the study of **Gaussian variables and processes, law of large numbers, the central limit theorem, stable laws, infinitely divisible laws and** pioneered the study of **processes with independent and stationary increments**

S. J. Taylor writes in 1975 (BLMS):

- *At that time there was no mathematical theory of probability - only a collection of small computational problems. Now it is a fully-fledged branch of mathematics using techniques from all branches of modern analysis and making its own contribution of ideas, problems, results and useful machinery to be applied elsewhere. If there is one person who has influenced the establishment and growth of probability theory more than any other, that person must be* ***Paul Lévy.***

M. Loève, in 1971 (Annals of Probability)

- *Paul Lévy was a painter in the probabilistic world. Like the very great painting geniuses, his palette was his own and his paintings transmuted forever our vision of reality. ... His three main, somewhat overlapping, periods were:*
- ***the limit laws period***
- ***the great period of additive processes and of martingales painted in pathtime colours***
- ***the Brownian pathfinder period***

Levy’s Major Works

- *Leçons d'analyse fonctionnelle* (**1922, 2nd ed., 1951**; “**Lessons in Functional Analysis**”);
- *Calcul des probabilités* (**1925; “Calculus of Probabilities**”);
- *Théorie de l'addition des variables aléatoires* (**1937–54; “The Theory of Addition of Multiple Variables**”);
- *Processus stochastiques et mouvement brownien* (**1948; “Stochastic Processes and Brownian Motion**”).

Aleksandr Yakovlevich Khinchine

(**1894 - 1959**)

A. Khintchine’s Contributions

- **Aleksandr Yakovlevich Khinchin** (or Khintchine) is best known as a mathematician in the fields of **number theory** and **probability theory**. He is responsible for Khinchin's constant and the Khinchin-Levy constant. These are both constants used in the calculation of fraction or decimal expansions. Several constants have been subsequently calculated from the Khinchin constant including those of Robinson in 1971 looking at **nonstandard analysis**.

A. Khintchine’s works

- Khinchin's early works focused on real analysis. Later he used methods of the metric theory of functions in probability theory and number theory. He became one of the founders of the modern probability theory, discovering law of iterated logarithm in 1924, achieving important results in the field of limit theorems, giving a definition of a stationary process and laying a foundation for the theory of such processes. In number theory Khinchin made significant contributions to the metric theory of Diophantine approximation and established an important result for real continued function, discovering their property, known as Khintchin’s constant. He also published several important works on statistical physics, where he used methods of probability theory, on information theory, queuing theory and analysis.

Kiyoshi Ito
(Born: 1915 in Hokusei-cho, Mie Prefecture, Japan)

K. Ito's Contributions

- **Kiyoshi Itō** (伊藤 清 *Itō Kiyoshi*) is a Japanese mathematician whose work is now called Ito calculus. The basic concept of this calculus is the Ito integral, and the most basic among important results is Ito's lemma. It facilitates mathematical understanding of random events. His theory is widely applied, for instance in financial mathematics.

Ito and Stochastic Analysis

- A monograph entitled *Ito's Stochastic Calculus and Probability Theory* (1996), dedicated to Ito on the occasion of his eightieth birthday, contains papers which deal with recent developments of Ito's ideas:-
- *“Professor Kiyosi Ito is well known as the creator of the modern theory of* ***stochastic analysis****. Although Ito first proposed his theory, now known as* ***Ito's stochastic analysis*** *or* ***Ito's stochastic calculus****, about fifty years ago, its value in both pure and applied mathematics is becoming greater and greater. For almost all modern theories at the forefront of probability and related fields, Ito's analysis is indispensable as an essential instrument, and it will remain so in the future. For example, a basic formula, called the* ***Ito formula****, is well known and widely used in fields as diverse as physics and economics”.*

Applications of Ito’s Theory

- *Calculation using the "**Ito calculus**" is common not only to scientists in physics, population genetics, stochastic control theory, and other natural sciences, but also to mathematical finance in economics. In fact, experts in financial affairs refer to Ito calculus as "Ito's formula." Dr. Ito is the father of the modern stochastic analysis that has been systematically developing during the twentieth century.*

Short History of Modelling of Financial Markets

Typical Path of a Stock Price

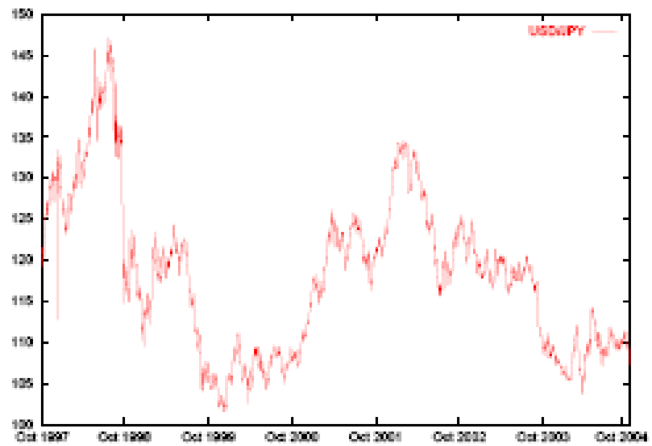


FIGURE 1.1. USD/JPY foreign exchange rate, October 1997-October 2004.

Typical Path of Brownian Motion

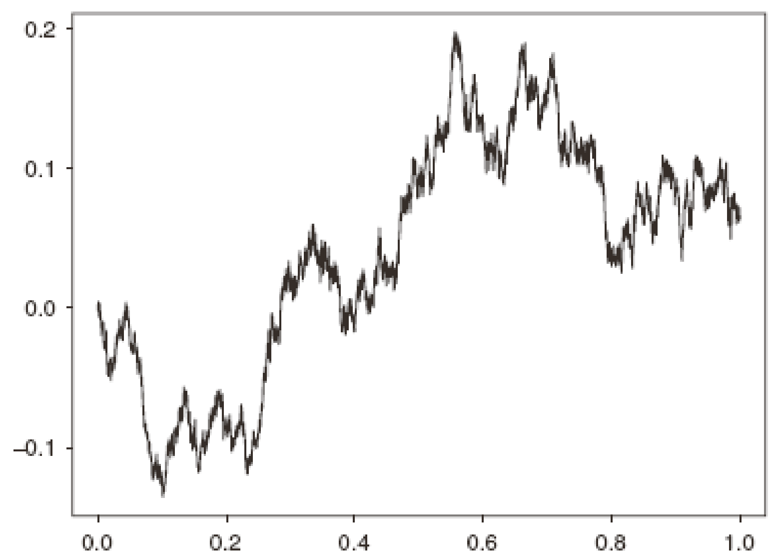


Figure 3: A sample path of a Brownian motion; $\Psi(\theta) = \theta^2/2$.

Comparison (Similar)

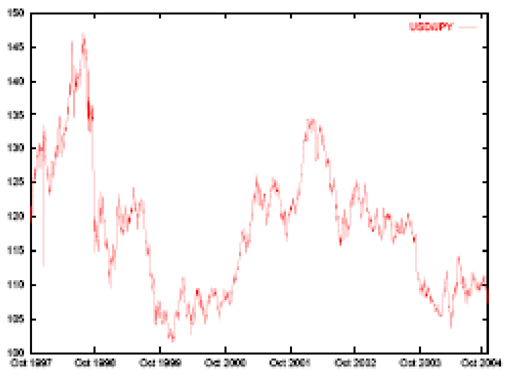


FIGURE 1.1. USD/JPY foreign exchange rate, October 1997-October 2004.

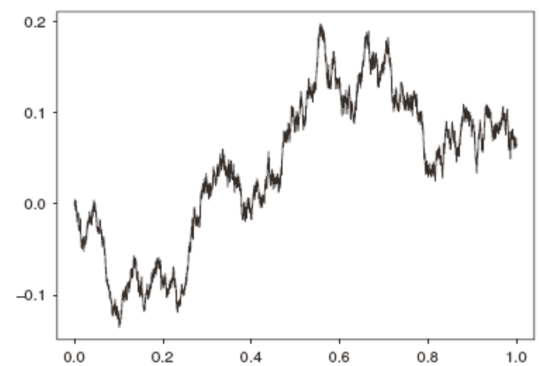


Figure 3: A sample path of a Brownian motion; $\Psi(\theta) = \theta^2/2$.

Short History of Modelling of Financial Markets

- Bachelier (1900):
  Modelled stocks as a Brownian motion with drift
- S_t-stock price at time t
- Disadvantage: S_t can take negative values

The Bachelier model:

$$S_t = S_0 + B_t, \quad t \leq T,$$

where $(B_t)_{t \leq T}$ is a Brownian motion.

Short History of Modelling of Financial Markets (cont'd)

- Black-Scholes (1973), Samuelson(1965):
  geometric Brownian motion;
  Merton(1973): geometric Brownian motion with Jumps

The Black-Scholes-Samuelson model:

$$S_t = S_0 e^{\mu t + \sigma B_t}, \quad t \leq T,$$

where $(B_t)_{t \leq T}$ is a Brownian motion and $\mu, \sigma \in \mathbb{R}$.

There are two main disadvantages of this model:

- the increments of $\ln S$ are Gaussian;
- the increments of $\ln S$ over disjoint intervals are independent.

Drawbacks of the Latest Models

- The latest option-pricing model is inconsistent with option data
- Implied volatility models can do better
- To improve on the performance of the Black-Scholes model, **Levy models** were proposed in the late 1980s and early 1990s

Some Desirable Properties of a Financial Model

Below is a list of some desirable properties of a financial model:

1. The marginal distributions of the increments of $\ln S$ are skewed.
2. The marginal distributions of the increments of $\ln S$ have heavy tails.
3. The increments of $\ln S$ are stationary in time.
4. The increments of $\ln S$ over disjoint intervals are not correlated.
5. The absolute values of the increments of $\ln S$ over disjoint intervals are positively correlated (the effect of “clustering”, “volatility persistence”).
6. The model is arbitrage free.
7. The model depends on a small number of parameters.

Levy Models in Mathematical Finance

Exponential Levy Model (ELM)

- The first disadvantage of the latest model is overcome by exponential Levy model

Exponential Levy Model

The exponential Lévy model:

$$S_t = S_0 e^{X_t}, \quad t \le T, \tag{2.1}$$

where $(X_t)_{t \le T}$ is a Lévy process.

The main disadvantage of model (2.1) is that the increments of $\ln S$ over disjoint intervals are independent. In other words, this model does not tackle the phenomenon of the persistence of volatility (clustering). This disadvantage is overcome by

The Time-Changed Exponential Levy Model

The time-changed exponential Lévy model (Carr, Geman, Madan, Yor [3]):

$$S_t = S_0 e^{(X \circ \tau)_t}, \quad t \leq T, \tag{2.2}$$

where $(X_t)_{t \geq 0}$ is a Lévy process and $(\tau_t)_{t \leq T}$ is an increasing càdlàg process that is independent of X. Here,

$$(X \circ \tau)_t := X_{\tau_t}, \quad t \leq T.$$

Whole Class of Model

Actually (2.2) is a whole class of models. It includes, in particular, the following models with a finite number of parameters:

- X is a variance gamma (VG) process;
- X is a Carr-Geman-Madan-Yor ($CGMY$) process;
- X is a normal inverse Gaussian (NIG) process;
- X is a hyperbolic (HYP) process;
- τ is a Cox-Ingersoll-Ross (CIR) process, i.e.

$$\tau_t = \int_0^t y_s ds, \quad t \leq T,$$

where y is a solution of the stochastic differential equation

$$dy_t = \theta(\eta - y_t)dt + \sigma\sqrt{y_t}dB_t.$$

The Exponential Levy Model with the Stochastic Integral

Another modification of model (2.1) that tackles the phenomenon of the volatility persistence is

The exponential Lévy model with the stochastic integrals (Eberlein, Kallsen, Kirsten [8]):

$$S_t = S_0 e^{(\sigma \bullet X)_t}, \quad t \leq T, \tag{2.3}$$

where $(X_t)_{t \leq T}$ is a Lévy process and $(\sigma_t)_{t \leq T}$ is an X-integrable process that is independent of X. Here,

$$(\sigma \bullet X)_t := \int_0^t \sigma_s dX_s, \quad t \leq T.$$

Comparison of Different Models

A model	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$S_t = S_0 e^{\mu t + \sigma B_t}$	$-$	$-$	$+$	$+$	$-$	$+$	2
$S_t = S_0 e^{X_t}$, X is VG, $CGMY$, NIG, or HYP	$+$	$+$	$+$	$+$	$-$	$+$	3,4
$S_t = S_0 e^{(X \circ \tau)_t}$, X is VG, $CGMY$, NIG, or HYP, τ is CIR	$+$	$+$	$+$	$+$	$+$	$+$	6,7
$S_t = S_0 e^{(\sigma \bullet X)_t}$, X is VG, $CGMY$, NIG, or HYP, σ is CIR	$+$	$+$	$+$	$+$	$+$	$+$	6,7

Table 1: Comparison of different models

Levy Processes

Continuous-Time Stochastic Process

- A continuous-time stochastic process assigns a **random variable** X(t) to each point $t \geq 0$ in time. In effect it is a random function of t.

Increments of Stochastic Process

- The **increments** of such a process are the differences $X(s) - X(t)$ between its values at different times $t < s$.

Independent Increments of Stochastic Process

- To call the increments of a process **independent** means that increments *X(s)* – *X(t)* and *X(u)* – *X(v)* are independent random variables whenever the two time intervals [t,s] and (v,u) do not overlap and, more generally, any finite number of increments assigned to pairwise non-overlapping time intervals are mutually (not just pairwise) independent

Stationary Increments of Stochastic Process

- To call the increments **stationary** means that the probability distribution of any increment $X(s) - X(t)$ depends only on the length $s - t$ of the time interval; increments with equally long time intervals are identically distributed.

Definition of Levy Processes X(t)

- X(t) **is a Levy Process** if
- X(t) has *independent and stationary increments*
- Each X(0)=0 w.p.1
- X(t) is *stochastically continuous*, i. e, for all a>0 and for all s=>0,

  P (|X(t)-X(s)|>a)->0

  when t->s

Characteristic Function

- “The shortest path between two truths in the real domain passes through the complex domain.”-**Jacques Hadamard**
- <u>To understand the structure of Levy processes we need characteristic function</u>

Characteristic Function

$$\phi_t(u) = \mathbb{E}(e^{iu\cdot X(t)}) = \int_{\mathbb{R}^d} e^{iu\cdot y} p_t(dy),$$

where p_t is the law (or distribution) of $X(t)$, i.e., $p_t = P \circ X(t)^{-1}$, and $\mathbb{E}$ denotes expectation. ϕ_t is continuous and positive definite; indeed, a famous theorem of Bochner asserts that all continuous positive definite mappings from $\mathbb{R}^d$ to $\mathbb{C}$ are Fourier transforms of finite measures on $\mathbb{R}^d$.

The Structure of Levy Processes: The Levy-Khintchine Formula

- If X(t) is a Levy process, then its characteristic function equals to

$$\phi(u) = e^{t\eta(u)} \qquad t \geq 0 \text{ and } u \in R^d$$

where

$$\eta(u) = ibu - \frac{1}{2}uau + \int_{R^d-(0)} [e^{iuy} - 1 - iuy\mathbf{1}_{|y|<1}(y)]\nu(dy)$$

Levy-Khintchine Triplet

A Lévy process can be seen as comprising of three components:

- drift, **b**
- diffusion component, **a**
- jump component, **v**

$$\eta(u) = ibu - \frac{1}{2}uau + \int_{R^d-(0)} [e^{iuy} - 1 - iuy1_{|y|<1}(y)]\nu(dy)$$

Levy-Khintchine Triplet

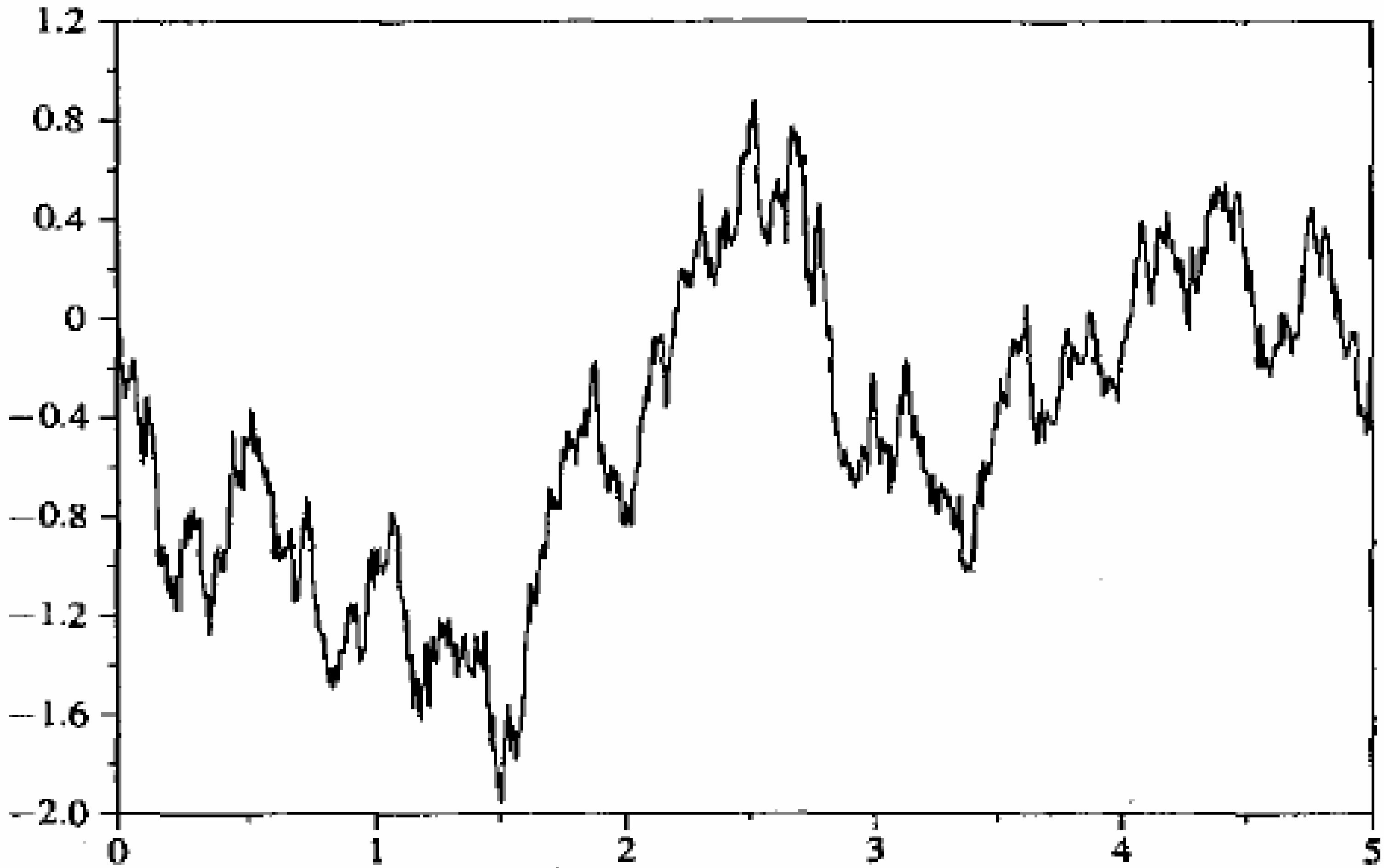
These three components, and thus the Lévy-Khintchine representation of the process, are fully determined by the **Lévy-Khintchine triplet (b,a,v)**

So one can see that a purely continuous Lévy process is a Brownian motion with drift 0: triplet for Brownian motion (0,a,0)

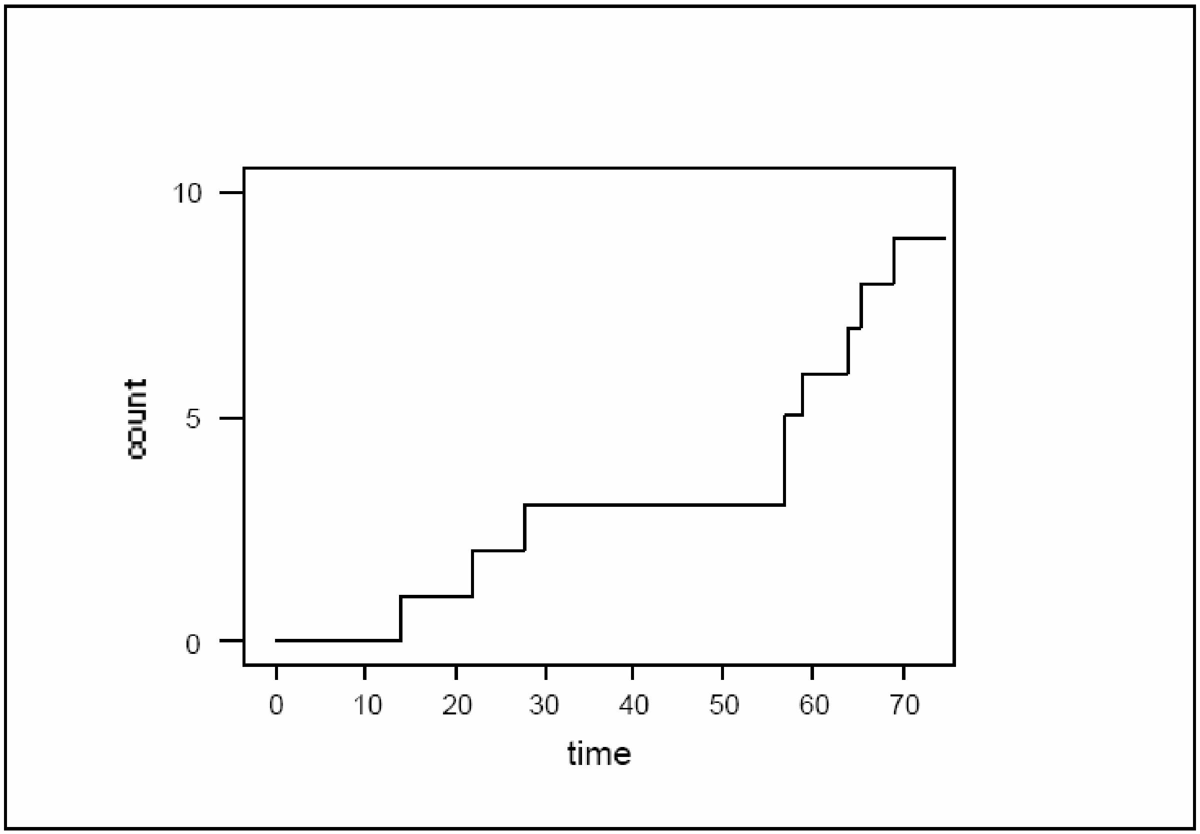
Examples of Levy Processes

- **Brownian motion**: characteristic (0,a,0)
- **Brownian motion with drift** (Gaussian processes): characteristic (b,a,0)
- **Poisson process**: characteristic (0,0,*lambda*x*delta1),* *lambda-intensity, delta1-Dirac mass concentrated at 1*
- **The compound Poisson process**
- **Interlacing processes**=Gaussian process +compound Poisson process
- **Stable processes**
- **Subordinators**
- **Relativistic processes**

Some Paths: Standard Brownian Motion



Some Paths: Standard Poisson Process



Some Paths: Compound Poisson Process

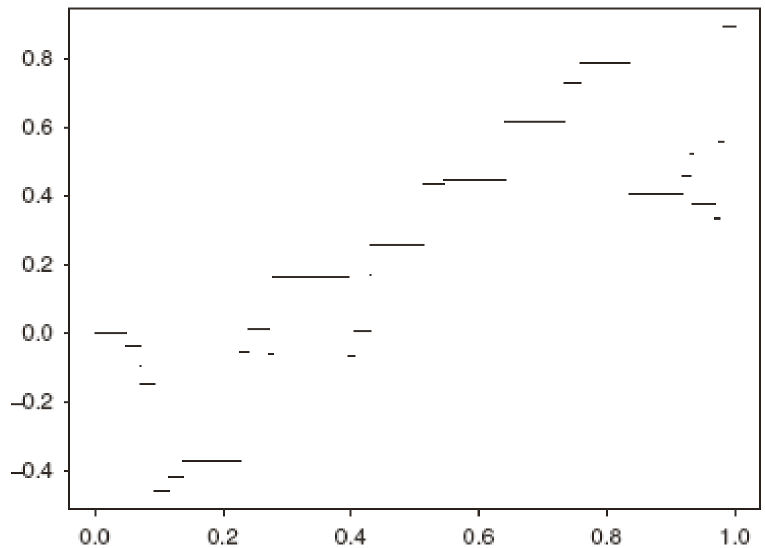


Figure 2: A sample path of a compound Poisson process; $\Psi(\theta) = \lambda \int_{\mathbb{R}} (1 - e^{i\theta x}) F(dx)$ where λ is the jump rate and F is the common distribution of the jumps.

Some Paths: Cauchy Process

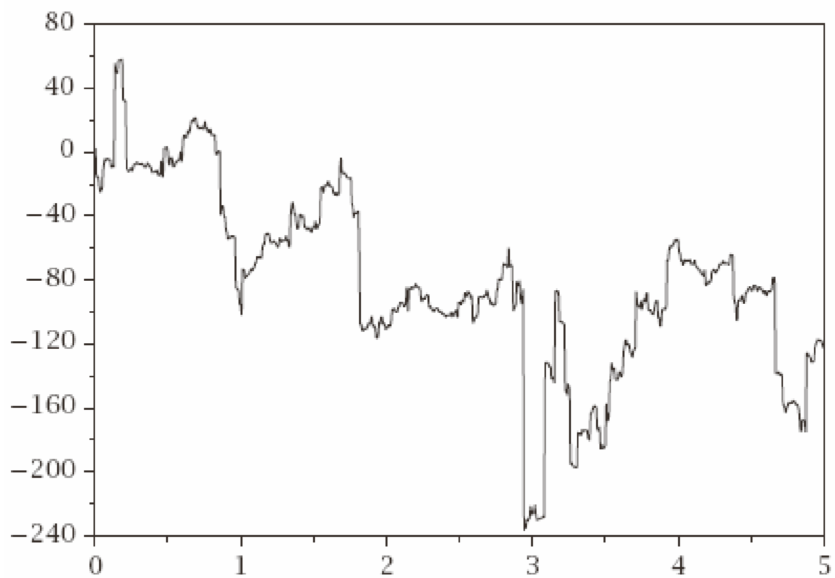


Figure 2. Simulation of the Cauchy process. The Cauchy process is stable with $\alpha = 1$. Jump discontinuities are represented by vertical lines. This process is also self-similar so the path has a fractal nature.

Some Paths: Variance Gamma

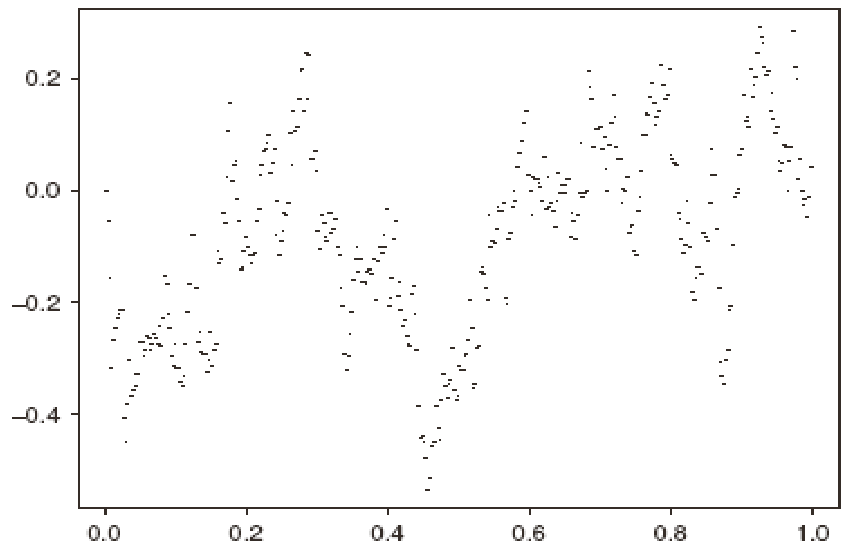


Figure 5: A sample path of a variance gamma processes. The latter has characteristic exponent given by $\Psi(\theta) = \beta \log(1 - \mathrm{i}\theta c/\alpha + \beta^2\theta^2/2\alpha)$ where $c \in \mathbb{R}$ and $\beta > 0$.

Some Paths: Normal Inverse Gaussian Process

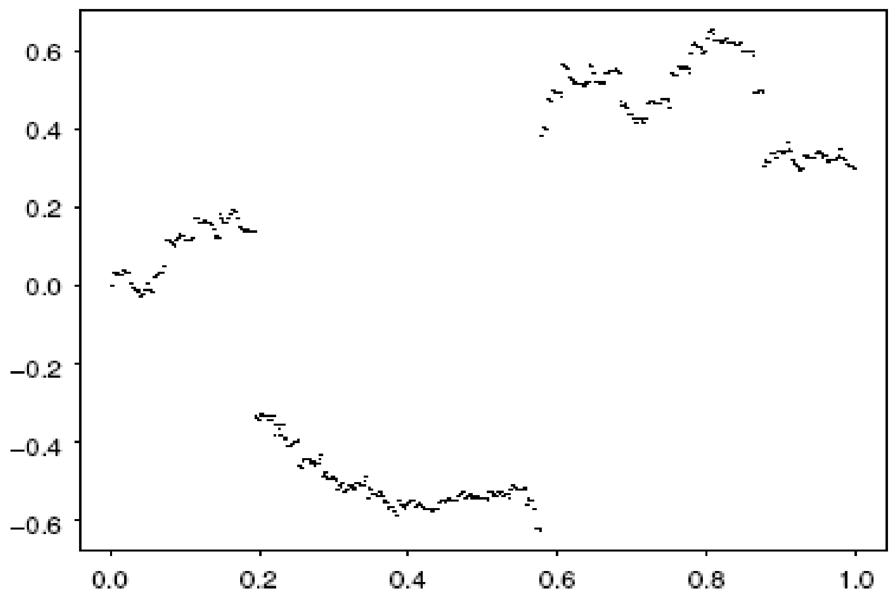


Figure 6: A sample path of a normal inverse Gaussian process; $\Psi(\theta) = \delta(\sqrt{\alpha^2 - (\beta + i\theta)^2} - \sqrt{\alpha^2 - \beta^2})$ where $\alpha, \delta > 0$, $|\beta| < \alpha$.

Some Paths: Mixed

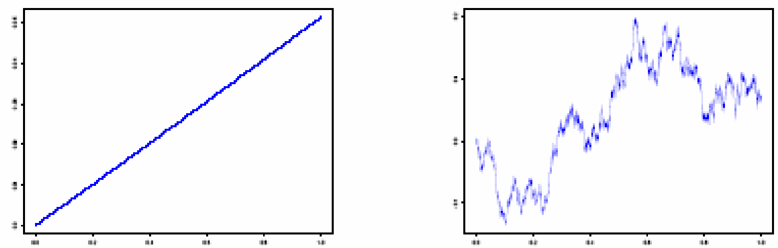


FIGURE 2.4. Examples of Lévy processes: linear drift (left) and Brownian motion

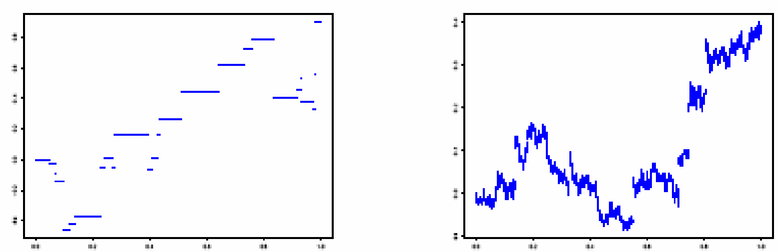


FIGURE 2.5. Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion

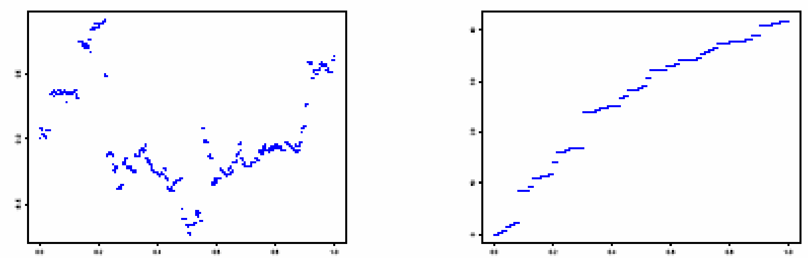


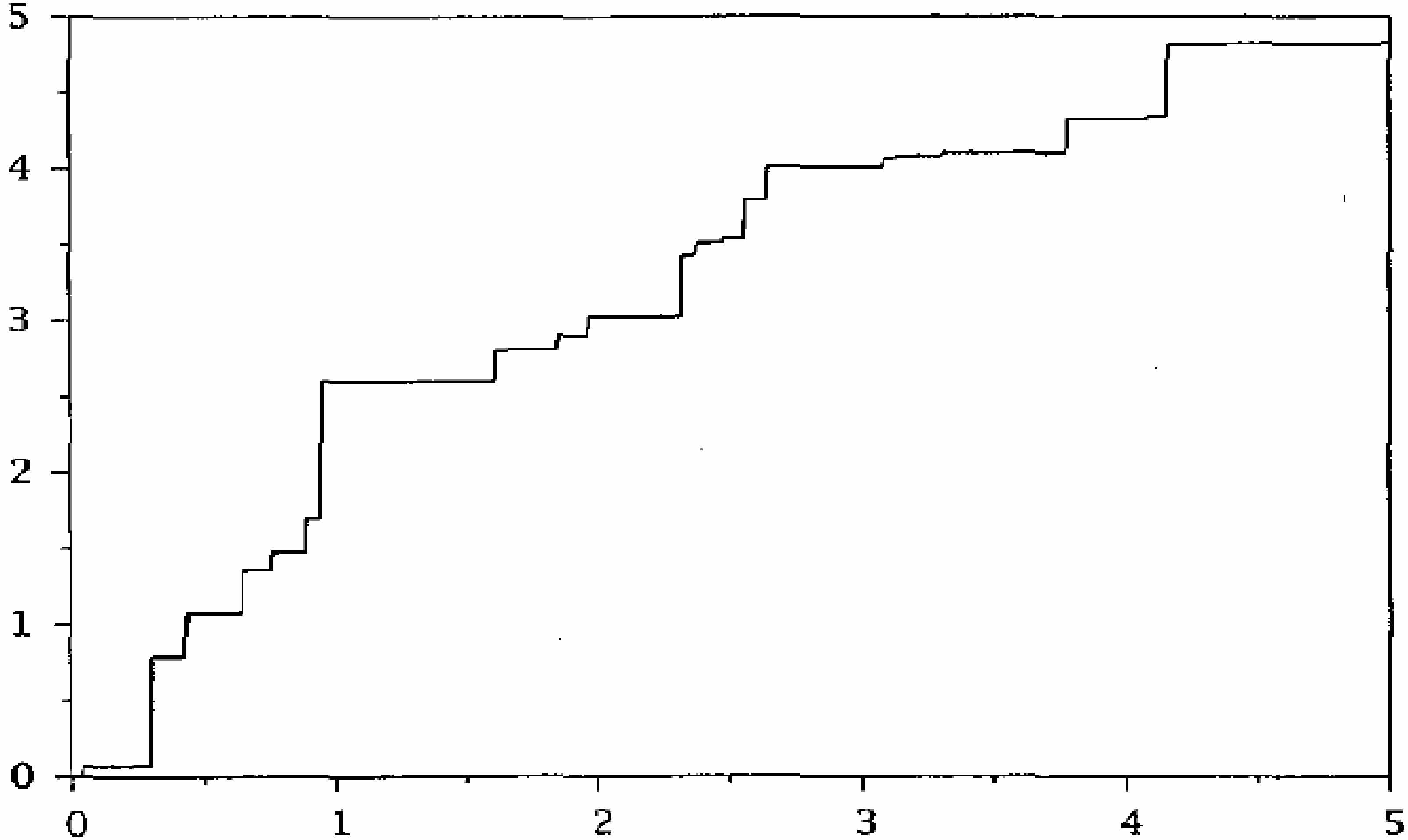
FIGURE 7.10. Simulated path of a normal inverse Gaussian (left) and an inverse Gaussian process

Subordinators

- A *subordinator* T(t) is a one-dimensional Levy process that is non-decreasing
- Important application: time change of Levy process X(t) :

Y(t):=X(T(t)) is also a new Levy process

Simulation of the Gamma Subordinator



The Levy-Ito Decomposition

Structure of the Sample Paths of Levy Processes:
The Levy-Ito Decomposition:

$$X(t) = bt + B_a(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx),$$

where $\tilde{N}(t, dx) = N(t, dx) - t\nu(dx)$ is a compensated Poisson random measure with intensity $\nu(dx)$.

Application to Finance

- Replace Brownian motion in BSM model with a more general Levy process (P. Carr, H. Geman, D. Madan and M. Yor)
- Idea:

1) *small jumps* term describes the day-to-day jitter that causes minor fluctuations in stock prices;

2) *big jumps term* describes large stock price movements caused by major market upsets arising from, e.g., earthquakes, etc.

Main Problems with Levy Processes in Finance I

- *Market is incomplete*, i.e., there may be more than one possible pricing formula
- One of the methods to overcome it: *entropy minimization*
- *Example:* hyperbolic Levy process (E. Eberlain) (with no Brownian motion part); a pricing formula have been developed that has minimum entropy

Main Problems with Levy Processes in Finance II

- Black-Scholes-Merton formula contains the *constant of volatility (standard deviation)*
- One of the methods to improve it: *stochastic volatility models (SDE for volatility)*
- *Example:* stochastic volatility is an Ornstein-Uhlenbeck process driven by a subordinator T(t) (Levy process)

Stochastic Volatility Model Using Levy Process

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dw(t)$$

where

$$\sigma^2(t) = e^{-\lambda t}\sigma^2(0) + \int_0^t e^{-\lambda(t-s)}dT(\lambda s), \quad \lambda > 0.$$

References on Levy Processes (Books)

- **D. Applebaum**, *Levy Processes and Stochastic Calculus*, Cambridge University Press, 2004
- **O.E. Barndorff-Nielsen, T. Mikosch and S. Resnick** (Eds.), *Levy Processes: Theory and Applications*, Birkhauser, 2001
- **J. Bertoin**, *Levy Processes*, Cambridge University Press, 1996
- **W. Schoutens**, *Levy Processes in Finance: Pricing Financial Derivatives*, Wiley, 2003
- **R. Cont and P Tankov**, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, 2004

Mathematical Beauty by K. Ito

- K. Ito gives a wonderful description **mathematical beauty** in ‘K Ito, *My Sixty Years in Studies of Probability Theory : acceptance speech of the Kyoto Prize in Basic Sciences* (1998)’, which he then relates to the way in which he and other mathematicians have developed his fundamental ideas:-

Mathematical Beauty by K. Ito I

- *‘In precisely built mathematical structures, mathematicians find the same sort of beauty others find in enchanting pieces of music, or in magnificent architecture.’*

Mathematical Beauty by K. Ito II

- *‘There is, however, one great difference between the beauty of mathematical structures and that of great art.’*

Mathematical Beauty by K. Ito II

- *‘Music by Mozart, for instance, impresses greatly even those who do not know musical theory’*

Mozart’s Music (Mozart's D major concerto K. 314)

Mathematical Beauty by K. Ito III

- *‘The cathedral in Cologne overwhelms spectators even if they know nothing about Christianity’*

Cologne Cathedral

Mathematical Beauty by K. Ito IV

- *‘The beauty in mathematical structures, however, cannot be appreciated without understanding of a group of numerical formulae that express laws of logic. Only mathematicians can read "musical scores" containing many numerical formulae, and play that "music" in their hearts.’*

The End

- **Thank You for Your Time and Attention!**