Stochastic Volatility and Change of Time: Overview

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\[ M(t) = B(T(t)) \]
Outline

- Volatility: Types
- Stochastic Volatility (SV):1-& Multi-Factor
- Change of Time (CT)
- Relationship: SV & CT
- Numerical Example
- Problems
Volatility

- **Volatility** is the standard deviation of the change in value of a financial instrument with specific time horizon.
- It is often used to quantify the **risk** of the instrument over that time period.
- The higher volatility, the riskier the security.
Types of Volatilities

- **Historical V**: standard deviation (uses historical (daily, weekly, monthly, quarterly, yearly)) price data to empirically measure the volatility of a market or instrument in the past.

- **Implied V**: volatility implied by the market price of the option based on an option pricing model (smile and skew-varying volatility by strike).
Volatility Smile

The models by Black & Scholes (continuous-time (B,S)-security market, 1973) and Cox & Rubinstein (discrete-time (B,S)-security market (binomial tree), 1979) are unable to explain the negative skewness and leptokurticity (fat tail) commonly observed in the stock markets.

The famous implied-volatility smile would not exist under their assumptions.
Commodity: Coffee

Coffee options trade on New York's Coffee, Sugar and Cocoa Exchange (CSCE).
Coffee Call Option

CSCE May 2001 coffee call option implied volatilities as of March 12, 2001
Implied Volatility: Volatility Smile

Graph indicates implied volatilities at various strikes for the May 2001 calls based upon their March 12, 2001 settlement prices. The pattern of implied volatilities form a "smile" shape, which is called a volatility smile.
Most derivatives markets exhibit persistent patterns of volatilities varying by strike. In some markets, those patterns form a **smile**. In others, such as **equity index options** markets, it is more of a skewed curve. This has motivated the name **volatility skew**. In practice, either the term "volatility smile" or "volatility skew" (or simply **skew**) may be used to refer to the general phenomena of volatilities varying by strike.
Another dimension to the problem of volatility skew is that of volatilities varying by expiration. This is illustrated for CSCE coffee options. It indicates what is known as a volatility surface.
Types of Volatilities II

- **Jump-Diffusion Volatility**
- **Level-Dependent Volatility** (CEV or Firm Model) - *function of the spot price alone*
- **Local $V$** - *function of the spot price and time* (Dupire formulae, 1994)
- **Stochastic $V$:** volatility is not constant, but a stochastic process (explains smile and skew)
In addition to the volatility smile observable from the implied volatilities of the options, there is evidence that assumption of a pure diffusion for the stock return is not accurate.

‘Fat Tails’ have been observed away from the mean of the stock return.

This phenomenon is called leptokurtic and could be explained in many different ways.

One way to explain smile and leptocurtic is to introduce a jump-diffusion process.
Jump-Diffusion and Leverage Effect

- Jump-diffusion is not a level-dependent volatility process
- Explains leverage effect
- **Merton (1976)** was first to introduce jumps in the stock distribution
- **Kou (2000)** used the same idea to explain both existence of fat tails and the volatility smile
Level-Dependent Volatility

- Level-dependent $V$ (LDV)-function of spot price alone
- Important feature of the level-dependent volatility: represents the negative correlation between the stock price and the volatility (leverage effect)
- LDV by Firm structure model: Bensoussan, Crouhy & Galai (1995)

\[ \sigma = \sigma(S) \]
\[ \sigma(t, S) = CS_t^\gamma \]
Local Volatility

- Local Volatility: \( \text{V-function of the spot price and time} \)
- Volatility smile was retrieved from the option prices

- Dupire (1994)-local volatility formula (V-call price)

- Derman & Kani (1994)-used the binomial (or trinomial tree) framework instead of the continuous one

\[
\sigma = \sigma(t, S)
\]

\[
\sigma^2(K, T) = \frac{\partial V}{\partial T} + rK \frac{\partial V}{\partial K} - \frac{1}{2} K^{-2} \frac{\partial^2 V}{\partial K^2}
\]
Local Volatility: Drawbacks

- The LV models are very elegant and theoretically sound.
- However, they present in practice many stability issues.
- They are ill-posed inversion problems and are extremely sensitive to the input data.
- This might introduce arbitrage opportunities and in some cases negative probabilities or variances.
Stochastic Volatility (SV)

**SV** is the main concept used in the fields of financial economics and mathematical finance to deal with the endemic time-varying volatility and co-dependence found in financial markets.

Such dependence has been known for a long time, early comments include Mandelbrot (1963) and Officer (1973).
Stochastic Volatility

The aim with a stochastic volatility model: *volatility appears not to be constant* and indeed varies, at least in part, randomly. The idea is to make the *volatility itself a stochastic process*.

*Stochastic volatility* models are useful because they explain in a self-consistent way why it is that options with different strikes and expirations have different Black-Scholes implied volatilities (the *volatility smile*)
Two Approaches to Introduce SV

- One approach: to change the clock time $t$ to a random time $T(t)$ (change of time)

- Another approach: change constant volatility into a positive stochastic process

\[ \sigma W(t) \Rightarrow W(T(t)) \]

\[ \sigma \equiv \sigma(t), \quad \int_0^t \sigma^2(s)ds < +\infty \]
Stochastic Volatilities: Continuous-Time Models

- **Ornstein-Uhlenbeck Process**
- **Hull & White (1987)** (GBM, positive)
- **Wiggins (1987)** (GBM, positive)
- **Scott (1989)** (OU, mean-reverting, positive)
- **Stein & Stein (1991)** (OU, mean-reverting, negative)
- **Heston (1993)** (mean-reverting, semi-analytical pricing formulae)

\[
\begin{align*}
    d\sigma_t &= -\alpha \sigma_t dt + \beta dZ_t \\
    \frac{d\sigma}{\sigma} &= \mu dt + \xi dW_2, \quad \rho = 0 \\
    \frac{d\sigma}{\sigma} &= \mu dt + \xi dW_2, \quad \rho \neq 0 \\
    d\ln(\sigma^2) &= (w - \zeta \ln(\sigma^2)) dt + \xi dW_2, \quad \rho \neq 0 \\
    d\sigma &= (w - \zeta \sigma) dt + \xi dW_2, \quad \rho = 0 \\
    d\sigma^2 &= (w - \zeta \sigma^2) dt + \xi \sigma dW_2, \quad \rho \neq 0
\end{align*}
\]
Stochastic Volatilities: Continuous-Time Models II

- Heston & Nandi (1997) showed that OU process corresponds to a special case of the GARCH model.

- Another popular process is the continuous-time GARCH(1,1) process, developed by Engle (1982) and Bollerslev (1986) in discrete framework.

\[ d\sigma = \theta(w - \sigma)dt + \xi\sigma dW \]
Stochastic Volatilities: Discrete-Time Models

Even though continuous time models provide the natural framework for an analysis of option pricing, **discrete time models** are ideal for the **statistical and descriptive analysis** of the patterns of daily price changes.

**Volatility Clustering**: there are periods of high and low variance (‘large changes tend to be followed by small changes’) (**Mandelbrot**) led to use of GARCH models.
Discrete-Time SV Models: Two Main Classes

- The first class, the autoregressive random variance (ARV) or stochastic variance models, is a discrete time approximation to the continuous time diffusion models that we outlined above.

- The second class is the autoregressive conditional heteroskedastic (ARCH) models introduced by Engle (1982), and its descendents (GARCH (Bolerslev (1986)), NARCH, NGARCH (Duan, 1996), LGARCH, EGARCH, GJR-GARCH, etc.)
SV With Delay: Continuous-Time GARCH Model with Delay

(Kazmerchuk, Swishchuk, Wu (2002))

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]

\[
\sigma_n^2 = \gamma V + \frac{\alpha}{l} \ln^2(S_{n-1}/S_{n-1-l}) + (1 - \alpha - \gamma)\sigma_{n-1}^2
\]
General Class of SV Models

\[ \frac{dS}{S} = \mu dt + \gamma f(\sigma) \left[ \sqrt{1 - \rho^2} dW_1 + \rho dW_2 \right] \]
\[ \frac{d\sigma}{\sigma} = \beta(\sigma) dt + g(\sigma) dW_2 \]
Specification of General SV Models

\[
\frac{dS}{S} = \mu dt + S^\gamma f(\sigma) \left[ \sqrt{1 - \rho^2} dW_1 + \rho dW_2 \right] \\
\frac{d\sigma}{\sigma} = \beta(\sigma) dt + g(\sigma) dW_2
\]

<table>
<thead>
<tr>
<th>Authors &amp; year</th>
<th>Specification</th>
<th>Remarks</th>
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| Hull-White 1987 | \( f(v) = v, \)  \\ \( \beta(v) = 0, \)  \\ \( g(v) = \sigma, \)  \\ \( \rho = 0, \gamma = 0 \) | Local variance: 
Geometric Brownian motion. 
Options priced by mixing. |
| Wiggins 1987 | \( f(v) = e^{v/2}, \)  \\ \( \beta(v) = \kappa(\theta - v)/v, \)  \\ \( g(v) = \sigma, \)  \\ \( \rho = 0, \gamma = 0 \) | Local volatility: 
Ornstein-Uhlenbeck in logarithms. |
| Stein-Stein 1991 | \( f(v) = |v|, \)  \\ \( \beta(v) = \kappa(\theta - v)/v, \)  \\ \( g(v) = \sigma/v, \)  \\ \( \rho = 0, \gamma = 0 \) | Local volatility: 
Reflected Ornstein-Uhlenbeck. |
| Heston 1993 | \( f(v) = \sqrt{v}, \)  \\ \( \beta(v) = \kappa(\theta - v)/v, \)  \\ \( g(v) = \sigma/\sqrt{v}, \)  \\ \( \rho \in [-1,1], \gamma = 0 \) | Local variance: 
CIR process. First model with correlation. Options priced by inversion of characteristic funct. |
| Romano-Touzi 1997 | \( f(v) = \sqrt{v}, \)  \\ \( \beta \) and \( g \) are free,  \\ \( \rho \in [-1,1], \gamma = 0 \) | Extension of mixing to correlation. |
| SABR 2002 | \( f(v) = v \)  \\ \( \beta(v) = 0, \)  \\ \( g(v) = \sigma, \)  \\ \( \rho \in [-1,1], \gamma \in [-1,0] \) | Level dependence in volatility. 
Options priced perturbation tech. |
Heston-Like Model
(J. Gatheral (2005), Merrill Lynch)

\[
\frac{dS}{S} = \mu_t \, dt + \sqrt{\sigma} \, dW_1 \\
\sigma = \alpha(S, \sigma) \, dt + \eta \beta(S, \sigma) \sqrt{\sigma} \, dW_2
\]
Multi-Factor SV Models

One-Factor SV Models (*all above-mentioned*):
1) incorporate the leverage between returns and volatility and
2) reproduce the ‘skew’ of implied volatility

However, it *fails to match either the high conditional kurtosis of returns (Chernov et. al. (2003)) or the full term structure of implied volatility surface (Cont&Tankov (2004))*

Adding *jump components* in returns and/or volatility process, or considering *multi-factor SV models* are two primary generalizations of one-factor SV models
Multi-Factor SV Model
(J.-P. Fouque, C.-H. Han (2005))

\[ \frac{dS}{S} = \mu dt + \sigma dW_1 \]
\[ \sigma = f(Y, Z) \]
\[ dY = \alpha c_1(Y) dt + \sqrt{\alpha} g_1(Z) dW_2 \]
\[ dZ = \delta c_2(Z) dt + \sqrt{\delta} g_2(Z) dW_3 \]
Multi-Factor SV Models

- **Chernov et al. (2003):** used efficient method of moments to obtain comparable empirical-of-fit from affine jump-diffusion mousiondels & two-factor SV family models
- **Molina et al. (2003):** used a Markov Chain Monte Carlo method to find strong evidence of two-factor SV models with well-separated time scales in foreign exchange data
- **Cont & Tankov (2004):** found that jump-diffusion models have a fairly good fit to the implied volatility surface
- **Fouque et al. (2000):** found that two-factor SV models provide a better fit to the term structure of implied volatility than one-factor SV models by capturing the behavior at short and long maturities
- **Swishchuk (2006):** introduced two-factor and three-factor SV models with delay (incorporating mean-reverting level as a random process (GBM, OU, Pilipovich or continuous-time GARCH(1,1) model))
Advantages and Disadvantages of Multi-Factor SV Models

- Multi-Factor SV models do not admit in general explicit solutions for option prices
- But have direct implications on hedges
- Comparison: class of jump-diffusion models (Bates (1996)) enjoys closed-form solutions for option prices but the complexity of hedging strategies increases due to jumps
- There is no strong empirical evidence to judge the overwhelming position between jump-diffusion models and multi-factor SV models
Other Generalization of SVM


- Multivariate models: introducing volatility clustering into traditional factor models (Diebold & Nerlove (1989))
Change of Time: Definition and Examples

- **Change of Time**: change time from $t$ to a non-negative process $T(t)$ with non-decreasing sample paths.

- **Example 1** (*Subordinator*): $X(t)$ and $T(t)>0$ are some processes, then $X(T(t))$ is subordinated to $X(t)$; $T(t)$ is change of time.

- **Example 2** (*Time-Changed Brownian Motion*): $M(t)=B(T(t))$, $B(t)$-Brownian motion.

- **Example 3** (*Product Process*):

\[
M_t = \int_0^t \sigma_s dW_s
\]

\[
T(t) = [M]_t = \int_0^t \sigma_s^2 ds
\]
Interpretation of CT

- If $M(t)$ is a **martingale** (another name- **fair game** process)
- Then $M(t)=B(T(t))$ (Dambis-Dubins-Schwartz Theorem)
- Time-change is the **quadratic variation** process $T(t)=[M(t)]$
- Then $M(t)$ can be written as a **SV process** (martingale representation theorem, Doob (1953))
- This implies that **time-changed BMs are canonical in continuous sample path price processes and SVMs are special cases of this class**

\[
T(t) = [M]_t = \int_0^t \sigma_s^2 ds \\
M_t = \int_0^t \sigma_s dW_s
\]
Time-Changed Brownian Motion by Bochner

Bochner (1949) (‘Diffusion Equation and Stochastic Process’, Proc. N.A.S. USA, v. 35)-introduced the notion of change of time (CT) (time-changed Brownian motion)

Clark (1973) (‘A Subordinated Stochastic Process Model with Fixed Variance for Speculative Prices’, Econometrica, 41, 135-156)-introduced Bochner’s (1949) time-changed Brownian motion into financial economics:

He wrote down a model for the log-price $M$ as

$$M(t)=B(T(t)),$$

where $B(t)$ is Brownian motion, $T(t)$ is time-change ($B$ and $T$ are independent)
Change of Time: Short History. I.

- **Feller (1966)** (‘An Introduction to Probability Theory’, vol. II, NY: Wiley)-introduced subordinated processes $X(T(t))$ with Markov process $X(t)$ and $T(t)$ as a process with independent increments (i.e., Poisson process); $T(t)$ was called *randomized operational time*

- **Johnson (1979)** (‘Option Pricing When the Variance Rate is Changing’, working paper, UCLA)-introduced *time-changed SVM in continuous time*

- **Johnson & Shanno (1987)** (‘Option Pricing When the Variance is Changing’, J. of Finan. & Quantit. Analysis, 22, 143-151)-studied the pricing of options using *time-changing SVM*

Barndorff-Nielsen, Nicolato & Shephard (2003) (‘Some recent development in stochastic volatility modelling’)-review and put in context some of their recent work on stochastic volatility (SV) modelling, including the relationship between subordination and SV (random time-chronometer)

Time-Changed Models and SVMs

- The probability literature has demonstrated that SVMs and their time-changed BM relatives and time-changed models are fundamentals


Change of Time: Simplest (Martingale) Case

\[ M(t) - \text{martingale}, \quad \lim_{t \to +\infty} [M](t) = +\infty \]

\[ \phi_t := \inf\{u : [M](u) > t\} \]

\[ W(t) := M(\phi_t) - \text{Brownian motion} \]

\[ M(t) = W([M](t)) - \text{martingale} \]

\[ \phi = [M]^{-1}(t), \quad \phi_t^{-1} = [M](t) \]
Change of Time:
General (Itô Integral) Case

\[ M(t) = \int_0^t \sigma(s) dW(s) - \text{martingale}, \quad [M] = \int_0^t \sigma^2(s) ds \to +\infty \]

\[ \phi_t := \inf\{u : [M](u) > t\} \]

\[ W(t) := M(\phi_t) - \text{Brownian motion} \]

\[ M(t) = W([M](t)) - \text{martingale} \]

\[ \phi_t = [M]^{-1}(t), \quad \phi_t^{-1} = [M](t) = \int_0^t \sigma^2(s) ds \]
Change of Time: SDE’s Case

\[ dX(t) = \alpha(t, X(t))dW(t) \]

\[ V(t) = X(0) + \tilde{W}(t) \]

\[ \phi_t = \int_0^t \alpha^{-2}(\phi_s, X(0) + \tilde{W}(s))ds. \]

\[ X(t) := V(\phi_t^{-1}) = X(0) + \tilde{W}(\phi_t^{-1}) \]
Geometric Brownian Motion SVM

\[ dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \]
Change of Time Method

\[ V(t) = e^{\mu t} S(t) \Rightarrow V(t) = S(0) + \tilde{W}(\phi_t^{-1}), \]

where

\[ \phi_t^{-1} = \sigma^2 \int_0^t (S(0) + \tilde{W}(\phi_s^{-1}))^2 \, ds, \]

and \( \tilde{W}(t) \) is one-dimensional Wiener process.
Solution for GBM Equation
Using Change of Time

\[ S(t) = e^{\mu t} (S(0) + \tilde{W}(\phi_t^{-1})) \]
Explicit Expression for $\tilde{W}(\phi_t^{-1})$

\[
dV(t) = \sigma V(t) dW(t) \Rightarrow V(t) = V(0) e^{\sigma W(t) - \frac{\sigma^2 t}{2}}
\]

\[
V(t) = V(0) + \tilde{W}(\phi_t^{-1}) \Rightarrow W(\phi_t^{-1}) = S(0)(e^{\sigma W(t) - \frac{\sigma^2 t}{2}} - 1).
\]
$$dS_t = a(L - S_t)dt + \sigma S_t dW_t,$$

where $W$ is a standard Wiener process, $\sigma > 0$ is the volatility, the constant $L$ is called the 'long-term mean' of the process, to which it reverts over time, and $a > 0$ measures the 'strength' of mean reversion.
\[ S_t = e^{-\alpha t}[S_0 - L + \tilde{W}(\phi_t^{-1})] + L, \] (4)

where \( \tilde{W}(t) \) is an \( \mathcal{F}_t \)-measurable standard one-dimensional Wiener process, \( \phi_t^{-1} \) is an inverse function to \( \phi_t \):

\[ \phi_t = \sigma^{-2} \int_0^t (S_0 - L + \tilde{W}(s) + e^{a\phi_s} L)^{-2} ds. \] (5)

We note that

\[ \phi_t^{-1} = \sigma^2 \int_0^t (S_0 - L + \tilde{W}(\phi_t^{-1}) + e^{a\phi_s} L)^2 ds, \] (6)
Explicit Expression for $\tilde{W}(\phi_t^{-1})$

$$\tilde{W}(\phi_t^{-1}) = S(0)(e^{\sigma W(t)} - \frac{\sigma^2 t}{2} - 1) + L(1 - e^{at}) + aLe^{\sigma W(t)} - \frac{\sigma^2 t}{2} \int_0^t e^{as} e^{-\sigma W(s) + \frac{\sigma^2 s}{2}} ds.$$
Explicit Expression for $\tilde{W}(\phi_t^{-1})$

$$\tilde{W}(\phi_t^{-1}) = m_1(t) + Lm_2(t),$$

where

$$m_1(t) := S(0)(e^{\sigma W(t)}-\frac{\sigma^2 t}{2} - 1)$$

and

$$m_2(t) = (1 - e^{at}) + a e^{\sigma W(t)}-\frac{\sigma^2 t}{2} \int_0^t e^{as} e^{-\sigma W(s)+\frac{\sigma^2 s}{2}} ds.$$
Comparison: Solution of GBM & MRM

\[ S(t) = e^{μt} (S(0) + \tilde{W}(φ_t^{-1})) \]

\[ S_t = e^{-αt} [S_0 - L + \tilde{W}(φ_t^{-1})] + L \]
Explicit Expression for $S(t)$

$$S(t) = e^{-at}[S_0 - L + \tilde{W}(\phi_t^-)] + L$$
$$= e^{-at}[S_0 - L + m_1(t) + Lm_2(t)] + L$$
$$= S(0)e^{-at}e^{\sigma W(t)-\frac{\sigma^2 t}{2}} + aLe^{-at}e^{\sigma W(t)-\frac{\sigma^2 t}{2}} \int_0^t e^{as}e^{-\sigma W(s)+\frac{\sigma^2 s}{2}} ds,$$
Heston Model (1993)

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sigma_t \, d\omega_1^t \\
\sigma_t^2 &= k(\theta^2 - \sigma_t^2) dt + \gamma \sigma_t \, d\omega_2^t
\end{align*}
\]
Explicit Solution for CIR Process: CTM

\[ d\sigma_t = k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dw_t \]

\[ \sigma_t^2 = e^{-kt}(\sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1})) + \theta^2 \]

\[ \phi_t = \gamma^{-2} \int_0^t \left\{ e^{k\phi_s} (\sigma_0^2 - \theta^2 + \tilde{w}^2(t)) + \theta^2 e^{2k\phi_s} \right\}^{-1} ds \]
Comparison: Solutions to the Three Models

- GBM

\[ S(t) = e^{\mu t} (S(0) + \tilde{W} (\phi_t^{-1})) \]

- MRM

\[ S_t = e^{-at} [S_0 - L + \tilde{W} (\phi_t^{-1})] + L \]

- Heston model

\[ \sigma_t^2 = e^{-kt} (\sigma_0^2 - \theta^2 + \tilde{w}^2 (\phi_t^{-1})) + \theta^2 \]
A stock volatility swap is a forward contract on the annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R(S) - K_{vol}),$$

where $\sigma_R(S)$ is the realized stock volatility (quoted in annual terms) over the life of contract,

$$\sigma_R(S) := \sqrt{\frac{1}{T} \int_0^T \sigma^2_s ds},$$
Why Trade Volatility (Variance)?

- Volatility Swaps allow investors to profit from the risks of an increase or decrease in future volatility of an index of securities or to hedge against these risks.
- If you think current volatility is low, for the right price you might want to take a position that profits if volatility increases.
Statistics on Log Returns of S&P Canada Index (Jan 1997-Feb 2002)

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<th>Statistics on Log Returns $S&amp;P60$ Canada Index</th>
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<td>Series:</td>
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</table>
Histograms of Log-Returns for S&P60 Canada Index
Convexity Adjustment

Convexity Adjustment (S&P60 Canada Index)

Volatility

Maturity (years)

non-adjusted volatility
adjusted volatility
S&P60 Canada Index Volatility Swap

![Chart showing the relationship between delivery price and maturity for the S&P60 Canada Index Volatility Swap, with two lines representing naive strike and adjusted strike.]
Wilmott, Javaheri & Haug (2002) Model

Wilmott, Javaheri & Haug (GARCH and Volatility Swaps, Wilmott Magazine, 2002) Result

\[ d\sigma = \theta(w - \sigma)dt + \xi \sigma dW \]

-continuous-time GARCH(1,1) model
Wilmott, Javaheri & Haug (2002) Volatility Swap

S&P500 Volatility Swap

- Naive strike
- Adjusted strike

Delivery Price vs. Maturity (years)
Comparison

Wilmott, Javaheri & Haug (2002)
Continuous-time GARCH(1,1)
S&P500

Sw (2004), Heston model,
S&P60 Canada Index

S&P500 Volatility Swap

S&P60 Canada Index Volatility Swap

Delivery Price

Maturity (years)
Comparison

Wilmott, Javaheri & Haug (2002), Convexity adjustment, S&P500

Sw (2004), Convexity adjustment, S&P60 Canada Index
Summary (SV and CT)

**GBM Model**

1. \[ dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \]

\[ S(t) = e^{\mu t}(S(0) + \tilde{W}(\phi_t^{-1})) \]

\[ \tilde{W}(\phi_t^{-1}) = S(0)(e^{\sigma W(t)} - \frac{\sigma^2 t}{2} - 1) \]

**Mean-Reverting Model**

2. \[ dS_t = a(L - S_t)dt + \sigma S_t dW_t \]

\[ S_t = e^{-at}[S_0 - L + \tilde{W}(\phi_t^{-1})] + L \]

\[ \tilde{W}(\phi_t^{-1}) = S(0)(e^{\sigma W(t)} - \frac{\sigma^2 t}{2} - 1) + L(1-e^{-at}) + aLe^{\sigma W(t) - \frac{\sigma^2 t}{2}}\int_0^t e^{as} e^{-\sigma W(s)} + \frac{\sigma^2 s}{2} ds \]

**Heston Model**

3. \[ d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma \sigma_t dW_t^2 \]

\[ \sigma_t^2 = e^{-kt} (\sigma_0^2 - \theta^2 + \tilde{\omega}^2(\phi_t^{-1})) + \theta^2 \]

\[ \tilde{\omega}^2(\phi_t^{-1}) \] -martingale

- sum of two martingales
Problems. I.

\[ d\sigma_t^2 = k(\theta_t^2 - \sigma_t^2)dt + \gamma_t \sigma_t dw_t^2 \]

\[ \sigma_t^2 = e^{-kt} (\sigma_0^2 - \theta^2 + \tilde{w}^2(\phi_t^{-1})) + \theta^2 \]

-explicit expression?

To calculate an option price for Heston model, for example

We know all the moments at this moment, though
Problems II

Continuous-Time SV Model with Delay

Solution by Change of Time Method?
The End

Thank you for your time and attention!

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