OPTIMAL CONTROL OF STOCHASTIC DIFFERENTIAL DELAY EQUATIONS WITH APPLICATION IN ECONOMICS

ANATOLI F. IVANOV AND ANATOLY V. SWISHCHUK

Abstract. The paper is devoted to the study of optimal control of stochastic differential delay equations and their applications. By using the Dynkin formula and solution of the Dirichlet-Poisson problem, the Hamilton-Jacobi-Bellman (HJB) equation and the inverse HJB equation are derived. Application is given to a stochastic model in economics.

1. Introduction

In our previous paper [6], the following controlled stochastic differential delay equation (SDDE) was introduced:

\[ x(t) = x(0) + \int_0^t a(x(s - 1), u(s))ds + \int_0^t b(x(s - 1), u(s))dw(s), \]

where \( x(t) = \phi(t), \quad t \in [-1, 0], \) is a given continuous process, \( u(t) \) is a control process and \( w(t) \) is a standard Wiener process.

We presented Dynkin formula and solutions of Dirichlet-Poisson problem for the SDDE. These results can be obtained from the results about the Dynkin formulas and boundary value problems for multiplicative operator functionals of Markov processes [12]. By using the Dynkin formula and solution of the Dirichlet-Poisson problem, the Hamilton-Jacobi-Bellman (HJB) and the inverse HJB equations have been stated.

In paper [6] we have found the stochastic optimal control and optimal performance for the SDDE. The results have been presented there without proof.

Date: August 26, 2004.


Key words and phrases. Stochastic differential delay equations, Stochastic optimal control, Hamilton-Jacobi-Bellman equation, Dynkin formula, Dirichlet-Poisson problem, Economics applications, Ramsey economics model with delay and randomness.
In this paper, we give complete proofs of two theorems from [6], Theorem 1 (HJB equation) and Theorem 2 (inverse of the HJB equation) about the stochastic optimal control. For definitions related to the stochastic optimal control and stochastic optimal performance see [10]. Application is given to a stochastic model in economics, a Ramsey model [2, 11] that takes into account the delay and randomness in the production cycle.

The model is described by the equation
\[ dK(t) = [BK(t - T) - u(K(t))C(t)]dt + \sigma(K(t - T))dw(t) \]
where \( K \) is the capital, \( C \) is the production rate, \( u \) is a control process, \( B \) is a positive constant, \( \sigma \) is a standard deviation of the "noise". The "initial capital"
\[ K(t) = \phi(t), \quad t \in [-T, 0], \]
is a continuous bounded positive function. For this stochastic economic model the optimal control is found to be \( u_{\text{min}} = K(0) \cdot C(0) \), and the optimal performance is
\[ J(K, u_{\text{min}}) = \frac{K^2(0)}{2} + \frac{K^2(0) \cdot C^2(0)}{2} + \int_{-T}^{0} \phi^2(\theta) d\theta = \frac{K^2(0)}{2} (1 + C^2(0)) + \int_{-T}^{0} \phi^2(\theta) d\theta. \]

By time rescaling, the delay \( T \) can be normalized to \( T = 1 \), which will be our assumption in the theoretical considerations that follow. The obtained results are valid however for general delay \( T > 0 \).

We note that global stability of SDDEs is considered in [7]. Surveys of the results on SDDEs and their applications, in particular in biology and finance, are presented in [5, 8].

2. Controlled stochastic differential delay equations

2.1. Assumptions and existence of solutions. Below we recall some basic notions and facts from [1, 4, 5, 6, 8] necessary for subsequent exposition in this paper. Let \( x(t), t \in [-1, \infty) \) be a stochastic process, \( F_{uw}(s) \) be a minimal \( \sigma \)-algebra with respect to which \( x(t) \) is measurable for every \( t \in [u, v] \). Let \( w(t), t \in [-1, \infty) \) be a Wiener process with \( w(0) = 0 \), and let \( F_{uw}(dw) \) be a minimal Borel \( \sigma \)-algebra such that \( w(t) - w(s) \) is measurable for all \( t, s \) with \( u \leq t \leq s \leq v \). Let \( u(t) \in U, t \in [-1, \infty) \) be a stochastic process whose values can be chosen from the given Borel set \( U \) and such that \( u(t) \) is \( F_{uw}(u) \)-adapted for all \( t \in [u, v] \).

Let \( C \) denote the metric space of all continuous functions defined on the interval \([-1, 0]\) with the standard metric \( |h| = \sup_{-1 \leq t \leq 0} |h(t)| \) [3].
Let $a(\cdot, \cdot), b(\cdot, \cdot)$ be continuous functionals defined on $C \times U$. A stochastic process $x(t)$ is called a solution of the stochastic differential delay equation

\[(1)\quad dx(t) = a(x(t - 1), u(t))dt + b(x(t - 1), u(t))dw(t), \quad t \in [-1, \infty)\]

if

\[F_{-1t}(x) \lor F_{0t}(dw) \lor F_{0t}(u)\]

is independent of $F_{t\infty}(dw)$ for every $t \in [-1, \infty)$, where $F_1 \lor F_2 \lor F_3$ stands for the minimal $\sigma$-algebra containing $F_1, F_2,$ and $F_3$, and

\[x(t) - x(s) = \int_s^t a(x(r - 1), u(r))dr + \int_s^t b(x(r - 1), u(r))dw(r),\]

where the last integral is the Ito integral.

Equation (1) is meant in the integral form

\[(2)\quad x(t) = x(0) + \int_0^t a(x(s - 1), u(s))ds + \int_0^t b(x(s - 1), u(s))dw(s)\]

with the initial condition $x(t) = \phi(t), t \in [-1, 0]$, where $\phi \in C$ is a given continuous process. Therefore, we assume that the processes $\phi(t), t \in [-1, 0], w(t)$ and $u(t), t \geq 0,$ are defined on the probability space $(\Omega, F, P)$ and $F = F_{-1t}(x) \lor F_{0t}(dw) \lor F_{0t}(u)$.

Let the following conditions be satisfied for equation (2)

A.1 $a(\phi, u)$ and $b(\phi, u)$ are continuous real-valued functionals defined on $C \times U$;

A.2 $\phi \in C$ is continuous with probability 1 in the interval $[-1, 0]$, independent of $w(s), s \geq 0$, and $E|\phi(t)|^4 < \infty$;

A.3 $\forall \phi, \psi \in C$:

\[(3)\quad |a(\phi, u) - a(\psi, u)| + |b(\phi, u) - b(\psi, u)| \leq K \int_{-1}^{0} |\phi(\theta) - \psi(\theta)|d\theta,\]

with $|a(\eta, u)| + |b(\eta, u)| \leq M$ for some $M, K > 0$ and all $\eta \in C, u \in U$.

Under assumptions A.1-A.3 the solution of the initial value problem (2) exists and is a unique stochastic continuous Markov process [1, 4, 9]. The solution $x(t)$ can be viewed at time $t \geq 0$ as an element $x_t$ of the space $C$, or as a point in $\mathbb{R}$. We shall use both interpretations in this paper, as appropriate.

2.2. Weak infinitesimal operator of $x(t)$. A real valued functional $J(\cdot)$ on $C$ is said to be in the domain of $A^u$, the weak infinitesimal operator (w.i.o.), if the limit

\[\lim_{t \to 0^+} ((E_x^u J(x_t) - J(x))/t) = q(x, u), \quad x = x_0 = \phi \in C, u \in U,\]
exists pointwise in $\mathcal{C} \times \mathcal{U}$, and $\lim_{t \to 0^+} \sup_{x,u} |E_x^u q(x_t, u) - q(x,u)| = 0$. Here $x_t := x_t(\theta) = x(t + \theta), \theta \in [-1,0]$, is in $\mathcal{C}$ and $E_x^u$ is the expectation under the conditional probability with respect to $x$ and $u$. We set $A^u J(x) := q(x,u)$.

For an open and bounded set $G \subset \mathbb{R}$, we denote by $A^u_G$ the w.i.o. of $\tilde{x}_t = x_t$ stopped at $\tau := \inf \{ t : x(t) \not\in G \}$.

Let the following condition be satisfied

\textbf{A.4} For every $\rho > 0$ and all initial functions $\psi, \phi \in \mathcal{C}$ with $|\psi| \leq \rho, |\phi| \leq \rho$ the inequality (3) is satisfied. $\tilde{A}$ is the weak infinitesimal operator of the process $\tilde{x}(t)$ satisfying equation (2) with $a,b$ replaced by $\tilde{a}, \tilde{b}$ and satisfying assumptions A.1-A.3. Let $\tilde{a} = a, \tilde{b} = b$ on the bounded open set $G$. Functions $\tilde{a}, \tilde{b}$ can be defined outside $G$ so that $|\tilde{x}_t| \leq K < \infty$.

Let $F$ be continuous and bounded on bounded sets. Then if $F \in D(\tilde{A})$ and $\tilde{A} u J = q$ is bounded on bounded sets, the restriction of $F$ to $G$ is in $D(\tilde{A}^u_G)$, and

$$\tilde{A}^u_G J(x) = L^u F(x(0)) = q(x,u) = F'_x(x(0)) a(x,u) + F''_{xx}(x(0)) \frac{1}{2} b^2(x,u)$$

where $u = u(0)$ (see[9]).

It is not simple to completely characterize the domain of the weak infinitesimal operator of either processes $\phi$ or $x(t)$. For example, in the case $J(x) = x(-1)$ the operator is not necessarily in $D(\tilde{A}_u)$, since $x(t)$ can be not differentiable.

It is possible to study functionals $J(\cdot)$ whose dependence on $\phi \in \mathcal{C}$ is in the form of an integral. For example, let the above conditions be satisfied for the functional

$$J(x) = \int_{-1}^{0} F(\phi(s), x(0)) \, ds,$$

where $F : \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is continuous. Let in addition $F(\alpha, \beta), F'_\alpha(\alpha, \beta), F''_{\alpha\beta}(\alpha, \beta)$ be continuous in $\alpha, \beta$. Then $J(x) \in D(\tilde{A}^u_{\mathcal{C}})$ and

$$\tilde{A}^u_{\mathcal{C}} J(x) = q(x) = F(x(0),x(0)) - F(x(-1),x(0)) +$$

$$+ \int_{-1}^{0} L^u F(x(s), x(0)) \, ds,$$

where the operator $L^u$ is defined by (4) and acts on $F$ as a function of $x(0)$ only [9].
2.3. Dynkin formula for SDDEs. Note that from (2) and (5) we obtain the following Ito formula for the functional $J$

$$J(x_t) = \int_{-1}^{0} F(\phi(\theta), x(t)) d\theta :$$

$$J(x_t) = J(x(0)) + \int_{0}^{t} F(x(s), x(s)) ds - \int_{0}^{t} F(x(s-1), x(s)) ds +$$

$$+ \int_{0}^{t} \int_{-1}^{0} L_u F(\phi(\theta), x(s)) d\theta ds + \int_{0}^{t} \int_{-1}^{0} \sigma(x(s-1)) F_x(\phi(\theta), x(s)) d\theta dw(s).$$

Let $\tau$ be a stopping time for the strong Markov process $x(t)$ such that $E_x|\tau| < \infty$. Then we have the following Dynkin formula [12]

$$E_x J(x_\tau) = J(x(0)) + E_x \int_{0}^{\tau} F(x(s), x(s)) ds -$$

$$- \int_{0}^{\tau} F(x(s-1), x(s)) ds + E \int_{0}^{\tau} \int_{-1}^{0} L_u F(\phi(\theta), x(s)) d\theta ds$$

$$= J(x(0)) + E_x \int_{0}^{\tau} \tilde{A}_u J(x_s) ds.$$

2.4. Solution of Dirichlet-Poisson problem for SDDEs. Let $\psi \in C(\partial G)$ be bounded and let $F(x, \phi, u) \in C(G \times C \times \mathcal{U})$ be such that

$$E_x \left[ \int_{-1}^{0} \int_{0}^{\tau_G} |F(\phi(\theta), x(s), u(s)) dsd\theta \right] < \infty \quad \text{for all} \quad x \in G.$$

Define

$$J(x, u) := E_x \left[ \int_{-1}^{0} \int_{0}^{\tau_G} F(\phi(\theta), x(s), u(s)) dsd\theta \right] + E_x [\psi(\tau_G)], x \in G.$$

Then [12]

$$\tilde{A}_u J(x, u) = - \int_{-1}^{0} F(x, \phi(\theta), u) d\theta, \quad \text{in} \ G \ \forall u \in \mathcal{U}$$

and

$$\lim_{t \to \tau_G} J(x(t), u) = \psi(x(\tau_G)) \quad \forall x \in G.$$

2.5. Statement of the Problem. We assume that the cost function is given in the form

$$(7) \quad J(x, u) := E_x \left[ \int_{-1}^{0} \int_{0}^{\tau_G} F(x(s), \phi(\theta), u(s)) dsd\theta + \psi(x(\tau_G)) \right],$$

where $\psi$ is a bounded real function, $F$ is bounded real and continuous, and $\tau_G$ is the exit time of the solution $x(t)$ from the fixed open set
$G \subset \mathbb{R}$. In particular, $\tau_G$ can be a fixed time $t_0$. We assume that $E_x|\tau| < \infty, \forall x \in G$.

The problem is as follows. For each $x \in G$ find number $J^*(x)$ and control $u^* = u^*(x, \omega)$ such that

$$J^*(x) := \inf_u \{ J(x, u) \} = J(x, u^*),$$

where the infimum is taken over all $F_t$-adapted processes $u(t) \in U$. Such a control $u^*$, if it exists, is called an optimal control and $J^*(x)$ is called the optimal performance.

3. Hamilton-Jacobi-Bellman equation for SDDEs

We consider only Markov controls $u(t) = u(x(t))$. For every $\nu \in U$ define the following operator

$$(A^\nu J)(x) = F(x(0), x(0), \nu(0)) - F(x(-1), x(0), \nu(0)) + \int_{-1}^{0} L^\nu F(\phi(\theta), x(0), \nu(0)) d\theta, \quad \nu(0) = \nu_0,$$

where operator $L^\nu$ is given by (4):

$$J(x) := \int_{-1}^{0} F(\phi(\theta), x(0)) d\theta$$

For each control $\nu$ the solution $x(t)$ of equation (2) is an Ito diffusion with the generator $(A^\nu J)(x) = (A^\nu J)(x)$.

**Theorem 1. (HJB-equation)**

Define

$$J^*(x) = \inf \{ J(x, u) : u = u(x) - \text{ Markov control } \}.$$ (9)

Suppose that $J \in C^2(G)$ and the optimal control $u^*$ exists. Then

$$\inf_{\nu \in U} \left[ \int_{-1}^{0} F(x, \phi(\theta), \nu) d\theta + (A^\nu J^*)(x) \right] = 0, \quad \forall x \in G,$$

and

$$J^*(x) = \psi(x), \quad \forall x \in \partial G,$$

where functions $F$ and $\psi$ are given by (7) and operator $A^\nu$ is given by (8), $\partial G$ is the boundary of set $G$ for $x(t)$.

The infimum in (9) is achieved when $\nu = u^*(x)$, where $u^*$ is optimal. In other words,

$$\int_{-1}^{0} F(x, \phi(\theta), u^*) d\theta + (A^{u^*} J^*)(x) = 0, \quad \forall x \in G,$$

which is equation (10).
Proof. Now we proceed to prove (10). Fix \( x \in G \) and choose a Markov process \( u \). Let \( \alpha \leq \tau_G \) be a stopping time. By using the strong Markov property of \( x(t) \) we obtain for \( J(x, u) \):

\[
E_x[J(x(\alpha), u)] = E_x \left[ E_x(\alpha) \left[ \int_{-1}^{\tau_G} F(x(s), \phi(\theta), u(s)) \, ds \, d\theta + \psi(x(\tau_G)) \right] \right] =
\]

\[
= E_x \left[ S_\alpha \left( \int_{-1}^{\tau_G} F(x(s), \phi(\theta), u(s)) \, ds \, d\theta + \psi(x(\tau_G)) \right) / F_\alpha \right] =
\]

\[
= E_x \left[ \int_{-1}^{\tau_G} F(x(s), \phi(\theta), u(s)) \, ds \, d\theta + \psi(x(\tau_G)) - \int_{-1}^{\alpha} F(x(s), \phi(\theta), u(s)) \, ds \, d\theta \right] =
\]

\[
= J(x, u) - E_x \left[ \int_{-1}^{\alpha} F(x(s), \phi(\theta), u(s)) \, ds \, d\theta \right],
\]

where \( S_\alpha \) is a shift operator. Therefore

(12)

\[ J(x, u) = E_x \left[ \int_{-1}^{\alpha} F(x(s), \phi(\theta), u(s)) \, ds \, d\theta \right] + E_x[J(x(\alpha), u(\alpha))]. \]

Now let \( V \subset G \) be of the form \( V := \{ y \in G : |y - x| < \epsilon \} \). Let \( \alpha = \tau_V \) be the first exit time of the solution \( x(t) \) from \( V \).

Suppose the optimal control \( u^* \) exists. For every \( \nu \in \mathcal{U} \) choose:

(13)

\[
u(x), \quad \text{if } x \in V
\]

\[
u(x), \quad \text{if } x \in G \setminus V.
\]

Then \( J^*(x(\alpha)) = J(u^*, x(\alpha)) \), and by combining (12) and (13), we obtain

\[ J^*(x) \leq J(x, \nu) = E_x \left[ \int_{-1}^{\alpha} F(x(r), \phi(\theta), \nu(r)) \, dr \, d\theta \right] + E_x[J(x(\alpha), \nu)]. \]

By Dynkin formula (6) we have

\[ E_x[J(x(\alpha), \nu)] = J(x) + E_x \left[ \int_0^\alpha A^\nu J(x(r), \nu) \, dr \right], \]
where $A^\nu$ is defined by (5). By substituting the latter into the previous inequality we obtain

$$J^*(x) \leq E_x \left[ \int_{-1}^0 \int_0^\alpha F(x(s), \phi(\theta), \nu(s)) \, ds \right] + J(x) + E_x \left[ \int_0^\alpha A^\nu J(x(r), \nu) \, dr \right],$$

or

$$E_x \left[ \int_{-1}^0 \int_0^\alpha F(x(r), \phi(\theta), \nu(r)) \, dr \, d\theta + \int_0^\alpha A^\nu J(x(r), \nu) \, dr \right] \geq 0.$$

Therefore,

$$E_x \left[ \int_{-1}^0 \int_0^\alpha F(x(r), \phi(\theta), \nu(r)) \, d\theta \, dr + \int_0^\alpha (A^\nu J)(x(r), \nu) \, dr \right] \geq 0.$$

By letting $\epsilon \to 0$ we derive

$$\int_{-1}^0 F(x, \phi(\theta), \nu) \, d\theta + (A^\nu J)(x, \nu) \geq 0,$$

which combined with (11) gives (10).

**Theorem 2. (Converse of the HJB-equation)**

Let $g$ be a bounded function in $C^2(G) \cap C(\overline{G})$. Suppose that for all $u \in U$ the inequality

$$\int_{-1}^0 F(x, \phi(\theta), u) \, d\theta + (A^u g)(x) \geq 0, \quad x \in G$$

and the boundary condition

(14) \quad $g(x) = \psi(x), \quad x \in \partial G$

are satisfied. Then $g(x) \leq J(x, u)$ for all Markov controls $u \in U$ and for all $x \in G$.

Moreover, if for every $x \in G$ there exists $u^0$ such that

(15) \quad $\int_{-1}^0 F(x, \phi(\theta), u^0) \, d\theta + (A^{u^0} g)(x) = 0$,

then $u^0$ is a Markov control, $g(x) = J(x, u^0) = J^*(x)$, and therefore $u^0$ is an optimal control.

**Proof.** Assume that $g$ satisfies hypotheses (14) and (15). Let $u$ be a Markov control. Then $A^u J \geq - \int_{-1}^0 F(x, \phi(\theta), u) \, d\theta$ for $u$ in $G$, and
we have by Dynkin formula (6)

\[ E_x [g(x(\tau_r))] = g(x) + E_x \left[ \int_0^{\tau_r} (A^u g)(x_s) \, ds \right] \geq g(x) - E_x \left[ \int_{-1}^0 \int_0^{\tau_r} F(x_s, \phi(\theta), u_s) \, d\theta ds \right]. \]

This gives

\[ g(x) \leq E_x \left[ \int_{-1}^0 \int_0^{\tau_r} F(x_s, \phi(\theta), u_s) \, d\theta ds + g(x_{\tau_r}) \right] \rightarrow_{r \to +\infty} \]

\[ = E_x \left[ \int_{-1}^0 \int_0^{\tau_G} F(x_s, \phi(\theta), u_s) \, d\theta ds + \psi(x_{\tau_G}) \right] = J(x, u), \]

which proves the first assertion of the theorem, where \( \tau_r := \min \{ r, \tau_G, \inf \{ t > 0 : |x_t| \geq r \} \} \), \( \forall r < +\infty \). If \( u^0 \) is such that (15) holds, then the above calculation gives the equality. This completes the proof.

4. Economics Model and Its Optimization

4.1. Description of the model. In 1928 F.R. Ramsey introduced an economics model describing the rate of change of capital \( K \) and labor \( L \) in a market by a system of ordinary differential equations [11]. With \( P \) and \( C \) being the production and consumption rates, respectively, the model is given by

\[ \frac{dK(t)}{dt} = P(t) - C(t), \quad \frac{dL(t)}{dt} = a(t)L(t), \]

where \( a(t) \) is the rate of growth of labor (population).

The production, capital and labor are related by the Cobb-Douglas formula, \( P(t) = AK^\alpha(t)L^\beta(t) \), where \( A, \alpha, \beta \) are some positive constants [2]. In certain circumstances the dependence of \( P \) on \( K \) and \( L \) is linear, i.e. \( \alpha = \beta = 1 \), which will be our assumption throughout this section. We shall also assume that the labor is constant, \( L(t) = L_0 \), which is true for certain markets or relatively short time intervals of several years. Therefore, the production rate and the capital are related by \( P(t) = BK(t) \), where \( B = AL_0 \). Another important assumption we make is that the production rate is subject to small random disturbances, i.e. \( P(t) = BK(t) + \text{"noise"} \). System (16) then results in

\[ \frac{dK(t)}{dt} = BK(t) + \text{"noise"} - C(t), \]

which can be rewritten in the differential form as

\[ dK(t) = [BK(t) - C(t)] \, dt + \sigma(K(t)) \, dw(t), \]
where \( w(t) \) is a standard Wiener process, \( \sigma(K) \) is a given (small) real function, characteristic of the noise.

The original model of Ramsey is based on the assumption of instant transformation of the investments. This can be accepted as satisfactory in only very rough models. In the reality the transformation of the invested capital cannot be accomplished instantly. A certain essential period of time is normally required for this transformation, such as the length of the production cycle in many economical situations. Therefore, a more accurate assumption is that the rate of change of capital \( K \) at present time \( t \) depends on the investment that was made at time \( t - T \), where \( T \) is the cycle duration required for the creation of working capital. This leads to the following delay differential equation

\[
dK(t) = [BK(t - T) - C(t)] \, dt + \sigma(K(t - T)) \, dw(t).
\]

Our next assumption is that the consumption rate \( C \) can be controlled by the available amount of the capital, i.e. it is of the form \( C(t) \, u(K(t)) \), where \( u(\cdot) \) is a control. By normalizing the delay to \( T = 1 \) (by time rescaling) one arrives at the equation

\[
\text{(17)} \quad dK(t) = [BK(t - 1) - u(K(t)) \, C(t)] \, dt + \sigma(K(t - 1)) \, dw(t).
\]

The initial investment of the capital \( K \) is naturally represented in equation (17) by a given initial function \( \phi \)

\[
\text{(18)} \quad K(t) = \phi(t), \quad t \in [-1, 0].
\]

Therefore, we propose to study a modified Ramsey model with delay and random perturbations given by the system (17)-(18).

4.2. Optimization calculation. Usually one wants to minimize the investment capital under the assumption of labor being constant. Let us choose the following cost function

\[
J(K, u) = \frac{K^2(0)}{2} + \int_{-1}^{0} \phi^2(\theta) \, d\theta + \frac{u^2(0)}{2}.
\]

The operator \( A^uJ \) has the following form

\[
A^uJ = \frac{K^2(0)}{2} + \phi^2(0) + \frac{u^2(0)}{2} - \left[ \frac{K^2(0)}{2} + \phi^2(-1) + \frac{u^2(0)}{2} \right] \\
+ \left[ K(0) \cdot (B \cdot K(0) - u(0) \cdot C(0)) + \frac{1}{2} \sigma^2(K(0)) \right],
\]

\[10\]
since
\[ F(K(0), K(0), u(0)) = \frac{K^2(0)}{2} + \phi^2(0) + \frac{u^2(0)}{2}, \]
\[ F(K(0), K(-1), u(0)) = \frac{K^2(0)}{2} + \phi^2(-1) + \frac{u^2(0)}{2}, \]
\[ L^u J(x, u) = K(0) \left( B \cdot K(0) - u(0) \cdot C(0) + \frac{1}{2} \sigma^2(K(0)) \right). \]

From (8) we obtain the following HJB-equation
\[
\inf_u \left[ \frac{K^2(0)}{2} + \int_{-1}^{0} \phi^2(\theta) \, d\theta + \frac{u^2(0)}{2} + \phi^2(0) - \phi^2(-1) + B \cdot K^2(0) \\
+ u(0) \cdot K(0) \cdot C(0) + \frac{1}{2} \sigma^2(K(0)) \right] = 0,
\]
or equivalently,
\[
\inf_u \left[ u^2(0) - 2K(0)C(0)u(0) + (2\phi^2(0) - 2\phi^2(-1) + 2 \int_{-1}^{0} \phi^2(\theta) \, d\theta \\
+ K^2(0)(1 + 2B) + \sigma^2(K(0))) \right] = 0.
\]
Let
\[
4K^2(0)C^2(0) \geq 4(2\phi^2(0) - 2\phi^2(-1) + 2 \int_{-1}^{0} \phi^2(\theta) \, d\theta \\
+ K^2(0)(1 + 2B) + \sigma^2(K(0))),
\]
or
\[
K^2(0) \cdot (C^2(0) - 3 - 2B) \geq 2 \int_{-1}^{0} \phi^2(\theta) \, d\theta - 2\phi^2(-1) + \sigma^2(K(0)),
\]
since \( K(0) = \phi(0). \) Hence, the infimum is achieved when
\[
u = -\left( -\frac{2K(0) \cdot C(0)}{2} \right) = K(0) \cdot C(0).
\]
Therefore \( u_{\text{min}} = K(0) \cdot C(0) \) and
\[
J(K, u_{\text{min}}) = \frac{K^2(0)}{2} + \frac{K^2(0) \cdot C^2(0)}{2} + \int_{-1}^{0} \phi^2(\theta) \, d\theta = \\
= \frac{K^2(0)}{2} + \frac{1}{2} \sigma^2(K(0)) + \int_{-1}^{0} \phi^2(\theta) \, d\theta.
\]
Note that in the case of general delay \( T > 0 \) in model (17)-(18) the last expression for \( J \) remains valid with the integration range \([-T, 0]\).

**Acknowledgment.** The first author was supported in part by the National Science Foundation (Grant no. INT-0203702) and by the
AvH Stiftung (in the final stage of preparation). Part of this work was done during the second author’s visit to the Pennsylvania State University in 2003.

References


Anatoli F. Ivanov
Department of Mathematics
Pennsylvania State University
P.O. Box PSU, Lehman, PA 18627, USA
E-mail address: afi1@psu.edu

Anatoly V. Swishchuk
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta T2N 1N4, Canada
E-mail address: aswish@math.ucalgary.ca