Second Edition

Elementary Applied Partial Differential Equations
with Fourier Series and Boundary Value Problems

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2.2 LINEARITY

As in the study of ordinary differential equations, the concept of linearity will be very important for us. A linear operator \( L \) by definition satisfies

\[
L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)
\]  

(2.2.1)

for any two functions \( u_1 \) and \( u_2 \), where \( c_1 \) and \( c_2 \) are arbitrary constants. \( \partial / \partial t \) and \( \partial^2 / \partial x^2 \) are examples of linear operators since they satisfy (2.2.1):

\[
\frac{\partial}{\partial t}(c_1u_1 + c_2u_2) = c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t}
\]

\[
\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) = c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2}
\]

It can be shown (see Exercise 2.2.1) that any linear combination of linear operators is a linear operator. Thus, the heat operator

\[
\frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2}
\]

is also a linear operator.

A linear equation for \( u \) is of the form

\[
L(u) = f,
\]

(2.2.2)

where \( L \) is a linear operator and \( f \) is known. Examples of linear partial differential equations are

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + f(x, t)
\]

(2.2.3a)

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + f(x, t)
\]

(2.2.3b)

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

(2.2.3c)

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u.
\]

(2.2.3d)

Examples of nonlinear partial differential equations are

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u^4
\]

(2.2.3e)

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}.
\]

(2.2.3f)

That \( u^4 \) and \( u \partial u / \partial x \) terms are nonlinear; they do not satisfy (2.2.1).

If \( f = 0 \), then (2.2.2) becomes \( L(u) = 0 \), called a linear homogeneous equation. Examples of linear homogeneous partial differential equations include the heat equation,

\[
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0,
\]

(2.2.4)
as well as (2.2.3c) and (2.2.3d). From (2.2.1) it follows that \( L(0) = 0 \) (let \( c_1 = c_2 = 0 \)). Therefore, \( u = 0 \) is always a solution of a linear homogeneous equation. For example, \( u = 0 \) satisfies the heat equation (2.2.4). We call \( u = 0 \) the trivial solution of a linear homogeneous equation. The simplest way to test whether an equation is homogeneous is to substitute the function \( u \) identically equal to zero. If \( u = 0 \) satisfies a linear equation, then it must be that \( f = 0 \) and hence the linear equation is homogeneous. Otherwise, the equation is said to be non-homogeneous [e.g., (2.2.3a) and (2.2.3b)].

The fundamental property of linear operators (2.2.1) allows solutions of linear equations to be added together in the following sense:

**Principle of Superposition**

If \( u_1 \) and \( u_2 \) satisfy a linear homogeneous equation, then an arbitrary linear combination of them, \( c_1u_1 + c_2u_2 \), also satisfies the same linear homogeneous equation.

The proof of this depends on the definition of a linear operator. Suppose that \( u_1 \) and \( u_2 \) are two solutions of a linear homogeneous equation. That means that \( L(u_1) = 0 \) and \( L(u_2) = 0 \). Let us calculate \( L(c_1u_1 + c_2u_2) \). From the definition of a linear operator,

\[
L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2).
\]

Since \( u_1 \) and \( u_2 \) are homogeneous solutions, it follows that \( L(c_1u_1 + c_2u_2) = 0 \). This means that \( c_1u_1 + c_2u_2 \) satisfies the linear homogeneous equation \( L(u) = 0 \) if \( u_1 \) and \( u_2 \) satisfy the same linear homogeneous equation.

The concepts of linearity and homogeneity also apply to boundary conditions, in which case the variables are evaluated at specific points. Examples of linear boundary conditions are the conditions we have discussed:

\[
\begin{align*}
\frac{\partial u}{\partial x}(L, t) &= f(t) \\
\frac{\partial u}{\partial x}(0, t) &= 0 \\
K\frac{\partial u}{\partial x}(L, t) &= h(u(L, t) - g(t))
\end{align*}
\]

A nonlinear boundary condition, for example, would be

\[
\frac{\partial u}{\partial x}(L, t) = u^2(L, t).
\]

Only (2.2.5c) is satisfied by \( u = 0 \) (of the linear conditions) and hence is homogeneous. It is not necessary that a boundary condition be \( u(0, t) = 0 \) for \( u = 0 \) to satisfy it.

**EXERCISES 2.2**

2.2.1. Show that any linear combination of linear operators is a linear operator.

2.2.2. (a) Show that \( L(u) = \frac{\partial}{\partial x} K\frac{\partial u}{\partial x} \) is a linear operator.

(b) Show that usually \( L(u) = \frac{\partial}{\partial x} \left[ K(x, u)\frac{\partial u}{\partial x} \right] \) is not a linear operator.

2.2.3. Show that \( \frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} + Q(u, x, t) \) is linear if \( Q = \alpha(x, t)u + \beta(x, t) \) and in addition homogeneous if \( \beta(x, t) = 0 \).

2.2.4. In this exercise we derive superposition principles for nonhomogeneous problems.

(a) Consider \( L(u) = f \). If \( u_\phi \) is a particular solution, \( L(u_\phi) = f \), and if \( u_1 \) and \( u_2 \) are homogeneous solutions, \( L(u_1) = 0 \), show that \( u = u_\phi + c_1u_1 + c_2u_2 \) is another particular solution.

(b) If \( L(u) = f_1 + f_2 \), where \( u_\phi \) is a particular solution corresponding to \( f_1 \), what is a particular solution for \( u_2 \)?

2.2.5. If \( L \) is a linear operator, show that \( L(\Sigma_{i=1}^{N} c_iu_i) = \Sigma_{i=1}^{N} c_i L(u_i) \). Use this result to show that the principle of superposition may be extended to any finite number of homogeneous solutions.

**2.3 HEAT EQUATION WITH ZERO TEMPERATURES AT FINITE ENDS**

**2.3.1 Introduction**

Partial differential equation (2.1.1) is linear, but it is homogeneous only if there are no sources, \( Q(x, t) = 0 \). The boundary conditions (2.1.3) are also linear, and they too are homogeneous only if \( T_1(t) = 0 \) and \( T_2(t) = 0 \). Thus we first propose to study

\[
\begin{align*}
PDE: \quad \frac{\partial u}{\partial t} &= \frac{k}{\partial x^2} u \quad &0 < x < L \\
BC: \quad u(0, t) &= 0 \quad t > 0 \\
IC: \quad u(x, 0) &= f(x)
\end{align*}
\]

The problem consists of a linear homogeneous partial differential equation with linear homogeneous boundary conditions. There are two reasons for our investigating this type of problem, (2.3.1)–(2.3.3), besides the fact that we claim it can be solved by the method of separation of variables. First, this problem is a relevant physical problem corresponding to a one-dimensional rod (0 < x
with no sources and both ends immersed in a 0° temperature bath. We are very interested in predicting how the initial thermal energy (represented by the initial condition) changes in this relatively simple physical situation. Second, it will turn out that in order to solve the nonhomogeneous problem (2.1.1)--(2.1.3), we will need to know how to solve the homogeneous problem, (2.3.1)--(2.3.3).

### 2.3.2 Separation of Variables

In the method of separation of variables, we attempt to determine solutions in the form

\[
    u(x, t) = \phi(x)G(t)
\]

(2.3.4)

where \( \phi(x) \) is only a function of \( x \) and \( G(t) \) only a function of \( t \). Equation (2.3.4) must satisfy the linear homogeneous partial differential equation (2.3.1) and boundary conditions (2.3.2), but for the moment we set aside (ignore) the initial condition. The product solution, (2.3.4), does not satisfy the initial conditions. Later we will explain how to satisfy the initial conditions.

Let us be clear from the beginning—we do not give any reasons why we choose the form (2.3.4). (Daniel Bernoulli invented this technique in the 1700s. IT WORKS, as we shall see.) We substitute the assumed product form, (2.3.4), into the partial differential equation (2.3.1):

\[
    \frac{\partial u}{\partial t} = k \frac{\partial^2 G}{\partial x^2},
\]

\[
    \frac{\partial^2 u}{\partial x^2} = \frac{d^2 \phi}{dx^2} G(t),
\]

and consequently the heat equation (2.3.1) implies that

\[
    \frac{dG}{dt} = \frac{k}{\phi} \frac{d^2 \phi}{dx^2} G(t).
\]

(2.3.5)

We note that we can "separate variables" by dividing both sides of (2.3.5) by \( \phi(x)G(t) \):

\[
    \frac{1}{G} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2}.
\]

(2.3.6)

Now the variables have been "separated" in the sense that the left-hand side is only a function of \( t \) and the right-hand side only a function of \( x \). We can continue in this way, but it is convenient (i.e., not necessary) also to divide by the constant \( k \), and thus

\[
    \frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2}.
\]

(2.3.7)

where \( \lambda \) is an arbitrary constant known as the separation constant.* We will explain momentarily the mysterious minus sign, which was introduced only for convenience.

Equation (2.3.7) yields two ordinary differential equations, one for \( G(t) \) and one for \( \phi(x) \):

\[
    \frac{d^2 \phi}{dx^2} = -\lambda \phi
\]

(2.3.8)

\[
    \frac{dG}{dt} = -\lambda kG.
\]

(2.3.9)

We reiterate that \( \lambda \) is a constant and it is the same constant that appears in both (2.3.8) and (2.3.9). The product solutions, \( u(x, t) = \phi(x)G(t) \), must also satisfy the two homogeneous boundary conditions. For example, \( u(0, t) = 0 \) implies that \( \phi(0)G(t) = 0 \). There are two possibilities. Either \( G(t) = 0 \) (the meaning of \( = \) is identically zero, for all \( t \)) or \( \phi(0) = 0 \). If \( G(t) = 0 \), then from (2.3.4), the assumed product solution is identically zero, \( u(x, t) = 0 \). This is not very interesting. \( u(x, t) = 0 \) is called the trivial solution since \( u(x, t) = 0 \) automatically satisfies any homogeneous PDE and any homogeneous BC. Instead, we look for nontrivial solutions. For nontrivial solutions, we must have

\[
    \phi(0) = 0.
\]

(2.3.10)

By applying the other boundary condition, \( u(L, t) = 0 \), we obtain in a similar way that

\[
    \phi(L) = 0.
\]

(2.3.11)

Product solutions, in addition to satisfying two ordinary differential equations, (2.3.8) and (2.3.9), must also satisfy boundary conditions (2.3.10) and (2.3.11).

* As further explanation for the constant in (2.3.7), let us say the following. Suppose that the left-hand side of (2.3.7) is some function of \( t \), \( (1/kG) dG/dt = w(t) \). If we differentiate with respect to \( x \), we get zero; \( \theta = d/dx (1/\phi) d^2 \phi/ dx^2 \). Since \( 1/\phi d^2 \phi/dx^2 \) is only a function of \( x \), this implies that \( 1/\phi d^2 \phi/dx^2 \) must be a constant, its derivative equaling zero. In this way (2.3.7) follows.
2.3.3 Time-Dependent Equation

The advantage of the product method is that it transforms a partial differential equation, which we do not know how to solve, into two ordinary differential equations. The boundary conditions impose two conditions on the \( x \)-dependent ordinary differential equation (ODE). The time-dependent equation has no additional conditions, just

\[
\frac{d\phi}{dt} = -\lambda \phi.
\]

(2.3.12)

Let us solve (2.3.12) first before we discuss solving the \( x \)-dependent ODE with its two homogeneous boundary conditions. Equation (2.3.12) is a first-order linear homogeneous differential equation with constant coefficients. We can obtain its general solution quite easily. Nearly all constant-coefficient (linear and homogeneous) ODEs can be solved by seeking exponential solutions, \( G = e^{rt} \), where in this case by substitution the characteristic polynomial is \( r = -\lambda k \). Therefore, the general solution of (2.3.12) is

\[
G(t) = ce^{-\lambda t}.
\]

(2.3.13)

We have remembered that for linear homogeneous equations, if \( e^{-\lambda t} \) is a solution, then \( ce^{-\lambda t} \) is a solution (for any arbitrary multiplicative constant \( c \)). The time-dependent solution is a simple exponential. Recall that \( \lambda \) is the separation constant, which for the moment is arbitrary. However, eventually we will discover that only certain values of \( \lambda \) are allowable. If \( \lambda > 0 \), the solution exponentially decays as \( t \) increases (since \( k > 0 \)). If \( \lambda < 0 \), the solution exponentially increases, and if \( \lambda = 0 \), the solution remains constant in time. Since this is a heat conduction problem and the temperature \( u(x, t) \) is proportional to \( G(t) \), we do not expect the solution to grow exponentially in time. Thus, we expect \( \lambda \geq 0 \); we have not proved that statement, but will do so later. When we see any parameter we often automatically assume that it is positive, even though we shouldn't. Thus, it is rather convenient that we have discovered that we expect \( \lambda \geq 0 \). In fact, that is why we introduced the expression \(-\lambda \) when we separated variables (see (2.3.7)). If we had introduced \( \mu \) (instead of \(-\lambda \)), then our previous arguments would have suggested that \( \mu \leq 0 \). In summary, when separating variables in (2.3.7), we mentally solve the time-dependent equation and see that \( G(t) \) does not exponentially grow only if the separation constant was \( \leq 0 \). We then introduce \(-\lambda \) for convenience, since we would now expect \( \lambda \geq 0 \). We next show how we actually determine all allowable separation constants. We will verify mathematically that \( \lambda \geq 0 \), as we expect by the physical arguments presented above.

2.3.4 Boundary Value Problem

The \( x \)-dependent part of the assumed product solution, \( \phi(x) \), satisfies a second-order ODE with two homogeneous boundary conditions:

\[
\frac{d^2\phi}{dx^2} = -\lambda \phi
\]

\[
\phi(0) = 0
\]

\[
\phi(L) = 0.
\]

(2.3.14)

We call (2.3.14) a boundary value problem for ordinary differential equations. In the usual first course in ordinary differential equations, only initial value problems are specified. For example (think of Newton's law of motion for a particle), we solve second-order differential equations (in \( \frac{dy}{dt^2} = F \)) subject to initial conditions \((y(0) \text{ and } \frac{dy}{dt}(0) \text{ given}) \) both at the same time. Initial value problems are quite nice, as usually there exist unique solutions to initial value problems. However, (2.3.14) is quite different. It is a boundary value problem, since the two conditions are not given at the same place (e.g., \( x = 0 \)) but at two different places, \( x = 0 \) and \( x = L \). There is no simple theory which guarantees that the solution exists or is unique to this type of problem. In particular, we note that \( \phi(x) = 0 \) satisfies the ODE and both homogeneous boundary conditions, no matter what the separation constant \( \lambda \) is, even if \( \lambda < 0 \); it is referred to as the trivial solution of the boundary value problem. It corresponds to \( u(x, t) = 0 \), since \( u(x, t) = \phi(x)G(t) \). If solutions of (2.3.14) had been unique, then \( \phi(x) = 0 \) would be the only solution; we would not be able to obtain nontrivial solutions of a linear homogeneous PDE by the product (separation of variables) method. Fortunately, there are other solutions of (2.3.14). However, there do not exist nontrivial solutions of (2.3.14) for all values of \( \lambda \). Instead, we will show that there are certain special values of \( \lambda \), called eigenvalues of the boundary value problem (2.3.14), for which there are nontrivial solutions, \( \phi(x) \). A nontrivial \( \phi(x) \), which exists only for certain values of \( \lambda \), is called an eigenfunction corresponding to the eigenvalue \( \lambda \).

Let us try to determine the eigenvalues \( \lambda \). In other words, for what values of \( \lambda \) are there nontrivial solutions of (2.3.14). We solve (2.3.14) directly. The second-order ODE is linear and homogeneous with constant coefficients; two independent solutions are usually obtained in the form of exponentials, \( \phi = e^{rt} \). Substituting this exponential into the differential equation yields the characteristic polynomial \( r^2 = -\lambda \). The solutions corresponding to the two roots have significantly different properties depending on the value of \( \lambda \). There are four cases:

1. \( \lambda > 0 \), in which the two roots are purely imaginary and are complex conjugates of each other, \( r = \pm i \sqrt{\lambda} \).
2. \( \lambda = 0 \), in which the two roots coalesce and are equal, \( r = 0 \).
3. \( \lambda < 0 \), in which the two roots are real and unequal, \( r = \pm \sqrt{-\lambda} \), one positive and one negative. (Note that in this case \(-\lambda \) is positive, so that the square root operation is well defined.)
4. \( \lambda \) itself complex.

* The word eigenvalue comes from the German word *eigenwert*, meaning characteristic value.
We will ignore the last case (as most of you would have done anyway) since we will later (Chapter 5) prove that $\lambda$ is real in order for a nontrivial solution of the boundary value problem (2.3.14) to exist. From the time-dependent solution, using physical reasoning, we expect that $\lambda \gg 0$; perhaps then it will be unnecessary to analyze case 3. However, we will demonstrate a mathematical reason for the omission of this case.

**Eigenvalues and eigenfunctions ($\lambda > 0$).** Let us first consider the case in which $\lambda > 0$. The boundary value problem is

$$
\frac{d^2\phi}{dx^2} = -\lambda \phi \quad (2.3.15)
$$

$$
\phi(0) = 0 \quad (2.3.16a)
$$

$$
\phi(L) = 0. \quad (2.3.16b)
$$

If $\lambda > 0$, exponential solutions have imaginary exponents, $e^{\pm \sqrt{\lambda}x}$. In this case, the solutions oscillate. If we desire real independent solutions, the choices $\cos \sqrt{\lambda} x$ and $\sin \sqrt{\lambda} x$ are usually made ($\cos \sqrt{\lambda} x$ and $\sin \sqrt{\lambda} x$ are each a linear combination of $e^{\pm \sqrt{\lambda}x}$). Thus, the general solution of (2.3.15) is

$$
\phi = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x, \quad (2.3.17)
$$

an arbitrary linear combination of two independent solutions. (The linear combination may be chosen from any two independent solutions.) $\cos \sqrt{\lambda} x$ and $\sin \sqrt{\lambda} x$ are usually the most convenient, but $e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$ can be used. In some examples, other independent solutions are chosen. For example, Exercise 2.3.2(1) illustrates the advantage of sometimes choosing $\cos \sqrt{\lambda} (x - a)$ and $\sin \sqrt{\lambda} (x - a)$ as independent solutions.

We now apply the boundary conditions. $\phi(0) = 0$ implies that

$$
0 = c_1.
$$

The cosine term vanishes, since the solution must be zero at $x = 0$. Thus, $\phi(x) = c_2 \sin \sqrt{\lambda} x$. Only the boundary condition at $x = L$ has not been satisfied. $\phi(L) = 0$ implies that

$$
0 = c_2 \sin \sqrt{\lambda} L.
$$

Either $c_2 = 0$ or $\sin \sqrt{\lambda} L = 0$. If $c_2 = 0$, then $\phi(x) = 0$ since we already determined that $c_1 = 0$. This is the trivial solution, and we are searching for those values of $\lambda$ that have nontrivial solutions. The eigenvalues $\lambda$ must satisfy

$$
\sin \sqrt{\lambda} L = 0. \quad (2.3.18)
$$

$\sqrt{\lambda} L$ must be a zero of the sine function. A sketch of $\sin z$ (see Fig. 2.3.1) or our knowledge of the sine function shows that $\sqrt{\lambda} L = n\pi$. $\sqrt{\lambda} L$ must equal an integral multiple of $\pi$, where $n$ is a positive integer since $\sqrt{\lambda} > 0$ ($n = 0$ is not appropriate since we assumed that $\lambda > 0$ in this derivation). The eigenvalues $\lambda$ are

$$
\lambda = \left( \frac{n\pi}{L} \right)^2, \quad n = 1, 2, 3, \ldots \quad (2.3.19)
$$

![Figure 2.3.1 Zeros of sin z.](image)

The eigenfunction corresponding to the eigenvalue $\lambda = (n\pi/L)^2$ is

$$
\phi(x) = c_2 \sin \sqrt{\lambda} x = c_2 \sin \frac{n\pi x}{L}, \quad (2.3.20)
$$

where $c_2$ is an arbitrary multiplicative constant. Often we pick a convenient value for $c_2$, for example $c_2 = 1$. We should remember, though, that any specific eigenfunction can always be multiplied by an arbitrary constant, since the PDE and BCs are linear and homogeneous.

**Eigenvalue ($\lambda = 0$).** Now we will determine if $\lambda = 0$ is an eigenvalue for (2.3.15) subject to the boundary conditions (2.3.16). $\lambda = 0$ is a special case. If $\lambda = 0$, (2.3.15) implies that

$$
\phi = c_1 x + c_2 x,
$$

corresponding to the double-zero roots, $r = 0, 0$, of the characteristic polynomial.* To determine whether $\lambda = 0$ is an eigenvalue, the homogeneous boundary conditions must be applied. $\phi(0) = 0$ implies that $0 = c_1$, and thus $\phi = c_2 x$. In addition, $\phi(L) = 0$ implies that $0 = c_2 L$. Since the length $L$ of the rod is positive ($\neq 0$), $c_2 = 0$ and thus $\phi(x) = 0$. This is the trivial solution, so we say that $\lambda = 0$ is not an eigenvalue, for this problem [2.3.15] and (2.3.16)]. Be wary, though; $\lambda = 0$ is an eigenvalue for other problems and should be looked at individually for any new problem you may encounter.

**Eigenvalues ($\lambda < 0$).** Are there any negative eigenvalues? If $\lambda < 0$, the solution of

$$
\frac{d^2\phi}{dx^2} = -\lambda \phi \quad (2.3.21)
$$

is not difficult, but you may have to be careful. The roots of the characteristic polynomial are $r = \pm \sqrt{-\lambda}$, so solutions are $e^{\sqrt{-\lambda}x}$ and $e^{-\sqrt{-\lambda}x}$. If you do not like the notation $\sqrt{-\lambda}$, you may prefer what is equivalent (if $\lambda < 0$), namely $\sqrt{|\lambda|}$. However, $\sqrt{|\lambda|} \neq \sqrt{\lambda}$, since $\lambda < 0$. It is convenient to let

$$
\lambda = -s,
$$

* Please do not say that $\phi = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$ is the general solution for $\lambda = 0$. If you do that, you find for $\lambda = 0$ that the general solution is an arbitrary constant. Although an arbitrary constant solves (2.3.15) when $\lambda = 0$, (2.3.15) is still a linear second-order differential equation; its general solution must be a linear combination of two independent solutions. It is possible to choose $\sin \sqrt{\lambda} x/\sqrt{\lambda}$ as a second independent solution so that as $\lambda \to 0$ it agrees with the solution $x$. However, this involves too much work. It is better just to consider $\lambda = 0$ as a separate case.
in the case in which $\lambda < 0$. Then $s > 0$, and the differential equation (2.3.21) becomes

$$\frac{d^2 \phi}{ds^2} = s \phi.$$  \hspace{1cm} (2.3.22)

Two independent solutions are $e^{+\sqrt{s}s}$ and $e^{-\sqrt{s}s}$, since $s > 0$. The general solution is

$$\phi = c_1 e^{\sqrt{s}s} + c_2 e^{-\sqrt{s}s}.$$  \hspace{1cm} (2.3.23)

Frequently, we instead use the hyperbolic functions. As a review, the definitions of the hyperbolic functions are

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2},$$

simple linear combinations of exponentials. These are sketched in Fig. 2.3.2. Note that $\sinh 0 = 0$ and $\cosh 0 = 1$ (the results analogous to those for trigonometric functions). Also note that $d/dz \cosh z = \sinh z$ and $d/dz \sinh z = \cosh z$, quite similar to trigonometric functions, but easier to remember because of the lack of the annoying appearance of any minus signs in the differentiation formulas. If hyperbolic functions are used instead of exponentials, the general solution of (2.3.22) can be written as

$$\phi = c_1 \cosh \sqrt{s} x + c_2 \sinh \sqrt{s} x,$$  \hspace{1cm} (2.3.24)

a form equivalent to (2.3.23). To determine if there are any negative eigenvalues ($\lambda < 0$, but $s > 0$ since $\lambda = -s$), we again apply the boundary conditions. Either form (2.3.23) or (2.3.24) can be used; the same answer is obtained either way. From (2.3.24), $\phi(0) = 0$ implies that $c_2 = 0$, and hence $\phi = c_1 \sinh \sqrt{s} x$. The other boundary condition, $\phi(L) = 0$, implies that $c_1 \sinh \sqrt{s} L = 0$. Since $\sqrt{s} L > 0$ and since sinh is never zero for a positive argument (see Fig. 2.3.2), it follows that $c_1 = 0$. Thus, $\phi(x) = 0$. The only solution of (2.3.22) for $\lambda < 0$ that solves the homogeneous boundary conditions is the trivial solution. Thus, there are no negative eigenvalues. For this example, the existence of negative eigenvalues would have corresponded to exponential growth in time. We did not expect such solutions on physical grounds, and here we have verified mathematically in an explicit manner that there cannot be any negative eigenvalues for this problem. In some other problems there can be negative eigenvalues. Later (Sec. 5.3) we will formulate a theory, involving the Rayleigh quotient, in which we will know before we start many problems that

there cannot be negative eigenvalues. This will at times eliminate calculations such as the ones just performed.

**Eigenfunctions—summary.** We summarize our results for the boundary value problem resulting from separation of variables:

$$\frac{d^2 \phi}{ds^2} + \lambda \phi = 0$$

$$\phi(0) = 0$$

$$\phi(L) = 0.$$

This boundary value problem will arise many times in the text. It is helpful to nearly memorize the result that the eigenvalues $\lambda$ are all positive (not zero or negative),

$$\lambda = \left( \frac{n\pi}{L} \right)^2,$$

where $n$ is any positive integer, $n = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are

$$\phi(x) = \sin \frac{n\pi x}{L}.$$

If we introduce the notation $\lambda_1$ for the first (or lowest) eigenvalue, $\lambda_2$ for the next, and so on, we see that $\lambda_n = (n\pi/L)^2, n = 1, 2, \ldots$. The corresponding eigenfunctions are sometimes denoted $\phi_n(x)$, the first few of which are sketched in Fig. 2.3.3. All eigenfunctions are (of course) zero at both $x = 0$ and $x = L$. Notice that $\phi_1(x) = \sin \pi x/L$ has no zeros for $0 < x < L$, and $\phi_2(x) = \sin 2\pi x/L$ has one zero for $0 < x < L$. In fact, $\phi_n(x) = \sin n\pi x/L$ has $n - 1$ zeros for $0 < x < L$. We will claim later (see Sec. 5.3) that, remarkably, this is a general property of eigenfunctions.

![Figure 2.3.3 Eigenfunctions sin nπx/L and their zeros.](image-url)
Spring–mass analog. We have obtained solutions of $d^2\phi/dx^2 = -\lambda\phi$. Here we present the analog of this to a spring–mass system, which some of you may find helpful. A spring–mass system subject to Hooke’s law satisfies $md^2y/dt^2 = -ky$, where $k > 0$ is the spring constant. Thus, if $\lambda > 0$, the ODE (2.3.15) may be thought of as a spring–mass system with a restoring force. Thus, if $\lambda > 0$, the solution should oscillate. It should not be surprising that the BCs (2.3.16) can be satisfied for $\lambda > 0$; a nontrivial solution of the ODE, which is zero at $x = 0$, has a chance of being zero again at $x = L$ since there is a restoring force and the solution of the ODE oscillates. We have shown that this can happen for specific values of $\lambda > 0$. However, if $\lambda < 0$, then the force is not restoring. It would seem less likely that a nontrivial solution which is zero at $x = 0$ could possibly be zero again at $x = L$. We must not always trust our intuition entirely, so we have verified these facts mathematically.

2.3.5 Product Solutions and the Principle of Superposition

In summary, we obtained product solutions of the heat equation, $\partial u/\partial t = k \partial^2 u/\partial x^2$, satisfying the specific homogeneous boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$ only corresponding to $\lambda > 0$. These solutions, $u(x, t) = \phi(x)G(t)$, have $G(t) = ce^{-kt}$ and $\phi(x) = c_n \sin \sqrt{k}x$, where we determined from the boundary conditions $\{\phi(0) = 0$ and $\phi(L) = 0\}$ the allowable values of the separation constant $\lambda, \lambda = (n\pi/L)^2$. Here $n$ is a positive integer. Thus, product solutions of the heat equation are

$$u(x, t) = B \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2t}, \quad n = 1, 2, \ldots, (2.3.25)$$

where $B$ is an arbitrary constant ($B = cc_2$). This is a different solution for each $n$. Note that as $t$ increases, these special solutions exponentially decay. In particular, for these solutions, $\lim_{t \to \infty} u(x, t) = 0$. In addition, $u(x, t)$ satisfies a special initial condition, $u(x, 0) = B \sin n\pi x/L$.

Initial value problems. We can use the simple product solutions, (2.3.25), to satisfy an initial value problem if the initial condition happens to be just right. For example, suppose that we wish to solve the following initial value problem:

PDE: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

BC: $u(0, t) = 0$

IC: $u(x, 0) = 4 \sin \frac{3\pi x}{L}$

Our product solution $u(x, t) = B \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2t}$ satisfies the initial condition $u(x, 0) = B \sin n\pi x/L$. Thus, by picking $n = 3$ and $B = 4$, we will have satisfied the initial condition. Our solution of this example is thus

$$u(x, t) = 4 \sin \frac{3\pi x}{L} e^{-k(3\pi/L)^2t}.$$  

It can be proved that this physical problem (as well as most we consider) has a unique solution. Thus, it does not matter what procedure we used to obtain the solution.

Principle of superposition. The product solutions appear to be very special, since they may be used directly only if the initial condition happens to be of the appropriate form. However, we wish to show that these solutions are useful in many other situations; in fact, in all situations. Consider the same PDE and BCs, but instead subject to the initial condition

$$u(x, 0) = 4 \sin \frac{3\pi x}{L} + 7 \sin \frac{8\pi x}{L}.$$  

The solution of this problem can be obtained by adding together two simpler solutions obtained by the product method:

$$u(x, t) = 4 \sin \frac{3\pi x}{L} e^{-k(3\pi/L)^2t} + 7 \sin \frac{8\pi x}{L} e^{-k(8\pi/L)^2t}.$$  

We immediately see that this solves the initial condition (substitute $t = 0$) as well as the boundary conditions (substitute $x = 0$ and $x = L$). Only slightly more work shows that the partial differential equation has been satisfied. This is an illustration of the principle of superposition.

Superposition (extended). The principle of superposition can be extended to show that if $u_1, u_2, u_3, \ldots, u_M$ are solutions of a linear homogeneous problem, then any linear combination of these is also a solution. Let $u = u_1 + c_1u_2 + c_2u_3 + \cdots + c_Mu_M = \sum_{n=1}^M c_n u_n$, where $c_n$ are arbitrary constants. Since we know from the method of separation of variables that $\sin n\pi x/L e^{-k(n\pi/L)^2t}$ is a solution of the heat equation (solving zero boundary conditions) for all positive $n$, it follows that any linear combination of these solutions is also a solution of the linear homogeneous heat equation. Thus,

$$u(x, t) = \sum_{n=1}^M B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2t} \quad (2.3.26)$$

solves the heat equation (with zero boundary conditions) for any finite $M$. We have added solutions to the heat equation, keeping in mind that the "amplitude" $B$ could be different for each solution, yielding the subscript $B_n$. Equation (2.3.26) shows that we can solve the heat equation if initially

$$u(x, 0) = f(x) = \sum_{n=1}^M B_n \sin \frac{n\pi x}{L}, \quad (2.3.27)$$

that is, if the initial condition equals a finite sum of the appropriate sine functions. What should we do in the usual situation in which $f(x)$ is not a finite linear combination of the appropriate sine functions? We claim that the theory of Fourier series (to be described with considerable detail in Chapter 3) states that:
1. Any function \( f(x) \) (with certain very reasonable restrictions, to be discussed later) can be approximated (in some sense) by a finite linear combination of \( \sin \frac{n \pi x}{L} \).

2. The approximation may not be very good for small \( M \), but gets to be a better and better approximation as \( M \) is increased (see Sec. 5.10).

3. Furthermore, if we consider the limit as \( M \to \infty \), then not only is (2.3.27) the best approximation to \( f(x) \) using combinations of the eigenfunctions, but (again in some sense) the resulting infinite series will converge to \( f(x) \) [with some restrictions on \( f(x) \), to be discussed].

We thus claim (and clarify and make precise in Chapter 3) that “any” initial condition \( f(x) \) can be written as an infinite linear combination of \( \sin \frac{n \pi x}{L} \), known as a type of Fourier series:

\[
 f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{L}.
\]  

(2.3.28)

What is more important is that we also claim that the corresponding infinite series is the solution of our heat conduction problem:

\[
 u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{L} e^{-k(n \pi/L)^2 t}.
\]  

(2.3.29)

Analyzing infinite series such as (2.3.28) and (2.3.29) is not easy. We must discuss the convergence of these series as well as briefly discuss the validity of an infinite series solution of our entire problem. For the moment, let us ignore these somewhat theoretical issues and concentrate on the construction of these infinite series solutions.

### 2.3.6 Orthogonality of Sines

One very important practical point has been neglected. Equation (2.3.29) is our solution with the coefficients \( B_n \) satisfying (2.3.28) (from the initial conditions), but how do we determine the coefficients \( B_n \)? We assume it is possible that

\[
 f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{L},
\]  

(2.3.30)

where this is to hold over the region of the one-dimensional rod, \( 0 \leq x \leq L \). We will assume that standard mathematical operations are also valid for infinite series. Equation (2.3.30) represents one equation in an infinite number of unknowns, but it should be valid at every value of \( x \). If we substitute a thousand different values of \( x \) into (2.3.30), each of the thousand equations would hold, but there would still be an infinite number of unknowns. This is not an efficient way to determine the \( B_n \). Instead, we frequently will employ an extremely important technique based on noticing (perhaps from a table of integrals) that the eigenfunctions \( \sin \frac{n \pi x}{L} \) satisfy the following integral property:

\[
 \int_0^L \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ L/2 & \text{if } m = n. \end{cases}
\]  

(2.3.31a)

(2.3.31b)

where \( m \) and \( n \) are positive integers.

To use these conditions, (2.3.31), to determine \( B_m \), we multiply both sides of (2.3.30) by \( \sin \frac{n \pi x}{L} \) (for any fixed integer \( m \), independent of the “dummy” index \( n \)):

\[
 f(x) \sin \frac{n \pi x}{L} = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L}.
\]  

(2.3.32)

Next we integrate (2.3.32) from \( x = 0 \) to \( x = L \):

\[
 \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L} \, dx.
\]  

(2.3.33)

For finite series, the integral of a sum of terms equals the sum of the integrals. We assume that this is valid for this infinite series. Now we evaluate the infinite sum. From the integral property (2.3.31), we see that each term of the sum is zero whenever \( n \neq m \). In summing over \( n \), eventually \( n \) equals \( m \). It is only for that one value of \( n \), i.e., \( n = m \), that there is a contribution to the infinite sum. The only term that appears on the right-hand side of (2.3.33) occurs when \( n \) is replaced by \( m \):

\[
 \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx = B_m \int_0^L \sin^2 \frac{n \pi x}{L} \, dx.
\]

Since the integral on the right equals \( L/2 \), we can solve for \( B_m \):

\[
 B_m = \frac{1}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx = \frac{2}{L} \int_0^L f(x) \sin \frac{m \pi x}{L} \, dx.
\]  

(2.3.34)

This result is very important and so is the method by which it was obtained. Try to learn both. The integral in (2.3.34) is considered to be known since \( f(x) \) is the given initial condition. The integral cannot usually be evaluated, in which case numerical integrations (on a computer) may need to be performed to get explicit numbers for \( B_m \), \( m = 1, 2, 3, \ldots \).

You will find that the formula (2.3.31b), \( \int_0^L \sin^2 \frac{n \pi x}{L} \, dx = L/2 \), is quite useful in many different circumstances, including applications having nothing to do with the material of this text. One reason for its applicability is that there are many periodic phenomena in nature (sin \( \omega t \)), and usually energy or power is proportional to the square (\( \sin^2 \omega t \)). The average energy is then proportional.
to \( f_{w}^{2\pi/w} \sin^2 \omega t \, dt \) divided by the period \( 2\pi/w \). It is worthwhile to memorize that the average over a full period of sine or cosine squared is \( \frac{1}{2} \). Thus, the integral over any number of complete periods of the square of a sine or cosine is one-half the length of the interval. In this way \( f_{w}^{2\pi/w} \sin^2 \frac{n\pi x}{L} \, dx = L/2 \), since the interval 0 to \( L \) is either a complete or a half period of \( \sin n\pi x/L \).

**Orthogonality.** Whenever \( f_{w}^{2\pi/w} A(x)B(x) \, dx = 0 \) we say that the functions \( A(x) \) and \( B(x) \) are orthogonal over the interval \( 0 \leq x \leq L \). We borrow the terminology “orthogonal” from perpendicular vectors because \( f_{w}^{2\pi/w} A(x)B(x) \, dx = 0 \) is analogous to a zero dot product, as is explained further in the appendix to this section. A set of functions each member of which is orthogonal to every other member is called an orthogonal set of functions. An example is that of the functions \( \sin n\pi x/L \), the eigenfunctions of the boundary value problem

\[
\frac{d^2\phi}{dx^2} + \lambda \phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.
\]

They are mutually orthogonal because of (2.3.31a). Therefore, we call (2.3.31) an orthogonality condition.

In fact, we will discover that for most other boundary value problems, the eigenfunctions will form an orthogonal set of functions (with certain modifications discussed in Chapter 5 with respect to Sturm-Liouville eigenvalue problems).

**2.3.7 Formulation, Solution, and Interpretation of an Example**

As an example, let us analyze our solution in the case in which the initial temperature is constant, 100°C. This corresponds to a physical problem that is easy to reproduce in the laboratory. Take a one-dimensional rod and place the entire rod in a large tub of boiling water (100°C). Let it sit there for a long time. After a while (we expect) the rod will be at 100°C throughout. Now insulate the lateral sides (if that had not been done earlier) and suddenly (at \( t = 0 \)) immerse the two ends in large well-stirred baths of ice water, 0°C. The mathematical problem is

**PDE:** \[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad t > 0, \quad 0 < x < L \quad (2.3.35a) \]

**BC:**
- \( u(0, t) = 0 \)
- \( u(L, t) = 0 \) \quad (2.3.35b)

**IC:** \( u(x, 0) = 100 \) \quad \( 0 < x < L \). \quad (2.3.35c)

According to (2.3.29) and (2.3.34), the solution is

\[
u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t},
\]

where

\[
B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
\]

and \( f(x) = 100 \). Recall that the coefficient \( B_n \) was determined by having (2.3.36) satisfy the initial condition,

\[
u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}
\]

We calculate the coefficients \( B_n \) from (2.3.37):

\[
B_n = \frac{2}{L} \int_{0}^{L} \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} \, dx = \frac{200}{L} \left( \frac{-L}{n\pi} \cos \frac{n\pi x}{L} \right) \bigg|_{0}^{L}
\]

\[
= \frac{200}{n\pi} (1 - \cos n\pi) = \begin{cases} 0 & \text{n even} \\ \frac{400}{n\pi} & \text{n odd} \end{cases} \quad (2.3.39)
\]

since \( \cos n\pi = (-1)^n \) which equals 1 for \( n \) even and \(-1\) for \( n \) odd. The series (2.3.38) will be studied further in Chapter 3. In particular, we must explain the intriguing situation that the initial temperature equals 100 everywhere, but the series (2.3.38) equals 0 at \( x = 0 \) and \( x = L \) (due to the boundary conditions).

**Approximations to the initial value problem.** We have now obtained the solution to the initial value problem (2.3.35) for the heat equation with zero boundary conditions \( (x = 0 \) and \( x = L \) and initial temperature distribution equaling 100. The solution is (2.3.36), with \( B_n \) given by (2.3.39). The solution is quite complicated, involving an infinite series. What can we say about it? First, we notice that \( \lim_{t \to \infty} u(x, t) = 0 \). The temperature distribution approaches a steady state, \( u(x, t) = 0 \). This is not surprising physically since both ends are at 0; we expect all the initial heat energy contained in the rod to flow out the ends. The equilibrium problem, \( dU/dx = 0 \) with \( u(0) = 0 \) and \( u(L) = 0 \), has a unique solution, \( u(x) \), agreeing with the limit as \( t \) tends to infinity of the time-dependent problem.

One question of importance that we can answer is the manner in which the solution approaches steady state. If \( t \) is large, what is the approximate temperature distribution, and how does it differ from the steady state? We note that each term in (2.3.36) decays at a different rate. The more oscillations in space, the faster the decay. If \( t \) is such that \( k(t)/L^2 \) is large, then each succeeding term is much smaller than the first. We can then approximate the infinite series by only the first term:

\[
u(x, t) = \frac{400}{\pi} \sin \frac{\pi x}{L} e^{-k(t)/L^2 t}.
\]

The larger \( t \) is, the better this is as an approximation. Even if \( k(t)/L^2 = \frac{1}{4} \), this is not a bad approximation since

\[
\frac{e^{-k(t)/L^2 t}}{e^{-k(t)/L^2 t}} = e^{-k(t)/L^2 t} = e^{-4} = 0.018 \ldots
\]

Thus, if \( k(t)/L^2 \geq \frac{1}{2} \), we can use the simple approximation. We see that for these times the spatial dependence of the temperature is just the simple rise and fall of \( \sin \pi x/L \), as illustrated in Fig. 2.3.4. The peak amplitude, occurring in
the middle $x = L/2$, decays exponentially in time. For $k(x/\pi)^2$ less than $\frac{1}{2}$, the spatial dependence cannot be approximated by one simple sinusoidal function; more terms are necessary in the series. The solution can be easily computed, using a finite number of terms. In some cases many terms may be necessary, and there would be better ways to calculate $u(x, t)$.

### 2.3.8 Summary

Let us summarize the method of separation of variables as it appears for the one example:

**PDE:**

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

**BC:**

$u(0, t) = 0$

$u(L, t) = 0$

**IC:**

$u(x, 0) = f(x)$.

1. Make sure that you have a linear and homogeneous PDE with linear and homogeneous BC.
2. Temporarily ignore the nonzero IC.
3. Separate variables (determine differential equations implied by the assumption of product solutions) and introduce a separation constant.
4. Determine separation constants as the eigenvalues of a boundary value problem.
5. Solve other differential equations. Record all product solutions of the PDE obtainable by this method.
6. Apply the principle of superposition (form a linear combination of all product solutions).
7. Attempt to satisfy the initial condition.
8. Determine coefficients using the orthogonality of the eigenfunctions.

These steps should be understood, not memorized. It is important to note that:

1. The principle of superposition applies to solutions of the PDE (do not add up solutions of various different ordinary differential equations).
2. Do not apply the initial condition $u(x, 0) = f(x)$ until after the principle of superposition.

### EXERCISES 2.3

#### 2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

*\(a\)\] $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

*\(b\)\] $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}$$

*\(c\)\] $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

*\(d\)\] $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial r^2}$$

*\(e\)\] $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

*\(f\)\] $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

#### 2.3.2. Consider the differential equation

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0.$$

Determine the eigenvalues $\lambda$ (and corresponding eigenfunctions) if $\phi$ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0$, $\lambda = 0$, $\lambda < 0$). You may assume that the eigenvalues are real.

*\(a\)\] $\phi(0) = 0$ and $\phi(\pi) = 0$

*\(b\)\] $\phi(0) = 0$ and $\phi(\pi) = 0$

*\(c\)\] $\phi(0) = 0$ and $\phi(\pi) = 0$ (If necessary, see Sec. 2.4.1.)

*\(d\)\] $\phi(0) = 0$ and $\phi(\pi) = 0$ (If necessary, see Sec. 5.8.)

*\(e\)\] $\phi(0) = 0$ and $\phi(\pi) = 0$ (You may assume that $\lambda > 0$.)

*\(f\)\] $\phi(0) = 0$ and $\phi(\pi) = 0$ (If necessary, see Sec. 2.4.1.)

#### 2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$u(0, t) = 0$

$u(L, t) = 0$.

Solve the initial value problem if the temperature is initially

*\(a\)\] $u(x, 0) = 6 \sin \frac{9\pi x}{L}$

*\(b\)\] $u(x, 0) = 3 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L}$

*\(c\)\] $u(x, 0) = 2 \cos \frac{3\pi x}{L}$

*\(d\)\] $u(x, 0) = \begin{cases} 1 & 0 < x < L/2 \\ 2 & L/2 < x < L \end{cases}$

(Your answer in part (c) may involve certain integrals that do not need to be evaluated.)

#### 2.3.4. Consider

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to $u(0, t) = 0$, $u(L, t) = 0$, and $u(x, 0) = f(x)$. 

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The first difficulty that arises is that we claim (3.1.1) will not be valid for all functions \( f(x) \). However, (3.1.1) will hold for some kinds of functions and will need only a small modification for other kinds of functions. In order to communicate various concepts easily, we will discuss only functions \( f(x) \) that are piecewise smooth. A function \( f(x) \) is piecewise smooth (on some interval) if the interval can be broken up into pieces (or sections) such that in each piece the function \( f(x) \) is continuous* and its derivative \( df/dx \) is also continuous. The function \( f(x) \) may not be continuous, but the only kind of discontinuity allowed is a finite number of jump discontinuities. A function \( f(x) \) has a jump discontinuity at a point \( x = x_0 \) if the limit from the left \( f(x_0^-) \) and the limit from the right \( f(x_0^+) \) both exist (and are unequal), as illustrated in Fig. 3.1.1. An example of a piecewise smooth function is sketched in Fig. 3.1.2. Note that \( f(x) \) has two jump discontinuities, at \( x = x_1 \) and at \( x = x_2 \). Also, \( f(x) \) is continuous for \( x_1 \leq x \leq x_2 \), but \( df/dx \) is not continuous for \( x_1 < x < x_2 \). Instead, \( df/dx \) is continuous for \( x_1 \leq x < x_2 \) and \( x_2 < x \leq x_3 \). The interval can be broken up into pieces in which both \( f(x) \) and \( df/dx \) are continuous. In this case there are four pieces, \( x \leq x_1 \), \( x_1 \leq x \leq x_2 \), \( x_2 \leq x \leq x_3 \), and \( x_3 \leq x \). Almost all functions occurring in practice (and certainly most that we discuss in this book) will be piecewise smooth. Let us briefly give an example of a function that is not piecewise smooth. Consider \( f(x) = x^{1/2} \), as sketched in Fig. 3.1.3. It is not.

* We do not give a definition of a continuous function here. However, one known useful fact is that if a function approaches \( \infty \) at some point, then it is not continuous in any interval including that point.

![Figure 3.1.1 Jump discontinuity at x = x₀.](image)

![Figure 3.1.2 Example of a piecewise smooth function.](image)

![Figure 3.1.3 Example of a function that is not piecewise smooth.](image)

3.2 STATEMENT OF CONVERGENCE THEOREM

Definitions of Fourier coefficients and a Fourier series. We will be forced to distinguish carefully between a function \( f(x) \) and its Fourier series over the interval \( -L \leq x \leq L \):

\[
\text{Fourier series} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.
\]

(3.2.1)

The infinite series may not even converge, and if it converges, it may not converge to \( f(x) \). However, if the series converges, we learned in Chapter 2 how to determine the Fourier coefficients \( a_0 \), \( a_n \), \( b_n \) using certain orthogonality integrals, (2.3.31). We will use those results as the definition of the Fourier coefficients:

\[
\begin{align*}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.
\end{align*}
\]

(3.2.2)
The Fourier series of \( f(x) \) over the interval \( -L \leq x \leq L \) is defined to be the infinite series (3.2.1) where the Fourier coefficients are given by (3.2.2). We immediately note that a Fourier series does not exist unless for example \( a_0 \) exists [i.e., unless \( \int_{-L}^{L} f(x) \, dx = \infty \)]. This eliminates certain functions from our consideration. For example, we do not ask what the Fourier series of \( f(x) = 1/x^2 \) is.

Even in situations in which \( \int_{-L}^{L} f(x) \, dx \) exists, the infinite series may not converge; furthermore, if it converges, it may not converge to \( f(x) \). We use the notation

\[
f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},
\]

where \( \sim \) means that \( f(x) \) is on the left-hand side and the Fourier series of \( f(x) \) (on the interval \( -L \leq x \leq L \)) is on the right-hand side (even if the series diverges), but the two functions may be completely different. The symbol \( \sim \) is read as “has the Fourier series (on a given interval).”

**Convergence theorem for Fourier series.** At first we state a theorem summarizing certain properties of Fourier series:

1. If \( f(x) \) is piecewise smooth on the interval \( -L \leq x \leq L \), then the Fourier series of \( f(x) \) converges
   1. to the periodic extension of \( f(x) \), where the periodic extension is continuous;
   2. to the average of the two limits, usually
      \[
      \frac{1}{2} [f(x+) + f(x-)],
      \]
      where the periodic extension has a jump discontinuity.

We refer to this as Fourier's theorem. It is proved in many of the references listed in the bibliography.

Mathematically, if \( f(x) \) is piecewise smooth, then for \( -L < x < L \) (excluding the end points),

\[
f(x+) + f(x-) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},
\]

where the Fourier coefficients are given by (3.2.2). At points where \( f(x) \) is continuous, \( f(x+) = f(x-) \) and hence (3.2.4) implies that for \( -L < x < L \),

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.
\]

The Fourier series actually converges to \( f(x) \) at points between \( -L \) and \( +L \), where \( f(x) \) is continuous. At the end points, \( x = L \) or \( x = -L \), the infinite series converges to the average of the two values of the periodic extension.

Outside the range \( -L \leq x \leq L \), the Fourier series converges to a value easily determined using the known periodicity (with period \( 2L \)) of the Fourier series.

**Sketching Fourier series.** Now we are ready to apply Fourier's theorem. To sketch the Fourier series of \( f(x) \) (on the interval \( -L \leq x \leq L \)), we:

1. Sketch \( f(x) \) (preferably for \( -L \leq x \leq L \) only).
2. Sketch the periodic extension of \( f(x) \).

According to Fourier's theorem, the Fourier series converges (here converge means equals) to the periodic extension, where the periodic extension is continuous (which will be almost everywhere). However, at points of jump discontinuity of the periodic extension, the Fourier series converges to the average. Therefore, there is a third step:

3. Mark an “x” at the average of the two values at any jump discontinuity of the periodic extension.

**Example.** Consider

\[
f(x) = \begin{cases} 
0 & x < L \\
1 & x > L
\end{cases}
\]

We would like to determine the Fourier series of \( f(x) \) on \( -L \leq x \leq L \). We begin by sketching \( f(x) \) for all \( x \) in Fig. 3.2.1 (although we only need the sketch for \( -L \leq x \leq L \) ) Note that \( f(x) \) is piecewise smooth, so we can apply Fourier's theorem. The periodic extension of \( f(x) \) is sketched in Fig. 3.2.2. Often the
understanding of the process is made clearer by sketching at least three full periods, \(-3L \leq x \leq 3L\), even though in the applications to partial differential equations only the interval \(-L \leq x \leq L\) is absolutely needed. The Fourier series of \(f(x)\) equals the periodic extension of \(f(x)\), wherever the periodic extension is continuous (i.e., at all \(x\) except the points of jump discontinuity, which are \(x = L/2, L/2 + 2L, -L, L/2 - 2L, \text{ etc.}\)). According to Fourier’s theorem, at these points of jump discontinuity, the Fourier series of \(f(x)\) must converge to the average. These should be marked, perhaps with an \(\times\), as in Fig. 3.2.2. At \(x = L/2\) and \(x = L\) (as well as \(x = L/2 \pm 2nL\) and \(x = L \pm 2nL\)), the Fourier series converges to the average, \(\bar{f}\). In summary for this example,

\[
\sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \begin{cases} \frac{1}{2} & x = -L \\ 0 & -L < x < L/2 \\ \frac{1}{4} & x = L/2 \\ \frac{1}{4} & L/2 < x < L \\ \frac{1}{2} & x = L \end{cases}
\]

Fourier series can converge to rather strange functions, but they are not so different from the original function.

Fourier coefficients. For a given \(f(x)\), it is not necessary to calculate the Fourier coefficients in order to sketch the Fourier series of \(f(x)\). However, it is important to know how to calculate the Fourier coefficients, given by (3.2.2). The calculation of Fourier coefficients can be an algebraically involved process. Sometimes it is an exercise in the method of integration by parts. Often, calculations can be simplified by judiciously using integral tables. In any event, we can always use a computer to approximate the coefficients numerically. As an overly simple example but one that illuminates some important points, consider \(f(x)\) given by (3.2.5). From (3.2.2), the coefficients are

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = \frac{1}{2L} \int_{-L/2}^{L/2} dx = \frac{1}{4} \\
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx = \frac{1}{L} \int_{-L/2}^{L/2} \cos \frac{n\pi x}{L} dx = \frac{1}{n\pi} \sin \frac{n\pi}{L} \bigg|_{L/2}^{L} \\
= \frac{1}{n\pi} \left( \sin \frac{n\pi}{2} - \sin \frac{n\pi}{2} \right) \\
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = \frac{1}{L} \int_{-L/2}^{L/2} \sin \frac{n\pi x}{L} dx = \frac{-1}{n\pi} \cos \frac{n\pi}{L} \bigg|_{L/2}^{L} \\
= \frac{-1}{n\pi} \left( \cos \frac{n\pi}{2} - \cos \frac{n\pi}{2} \right)
\]

We omit simplifications that arise by noting that \(\sin n\pi = 0\), \(\cos n\pi = (-1)^n\), and so on.

EXERCISES 3.2

3.2.1. For the following functions, sketch the Fourier series of \(f(x)\) (on the interval \(-L \leq x \leq L\)). Compare \(f(x)\) to its Fourier series:

(a) \(f(x) = 1\)  
(b) \(f(x) = x^2\)  
(c) \(f(x) = 1 + x\)  
(d) \(f(x) = e^x\)  
(e) \(f(x) = \begin{cases} x & x < 0 \\ 2x & x > 0 \end{cases}\)  
(f) \(f(x) = \begin{cases} 0 & x < 0 \\ 1 + x & x > 0 \end{cases}\)  
(g) \(f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}\)

3.2.2. For the following functions, sketch the Fourier series of \(f(x)\) (on the interval \(-L \leq x \leq L\)) and determine the Fourier coefficients:

(a) \(f(x) = x\)  
(b) \(f(x) = e^{-x}\)  
(c) \(f(x) = \sin \frac{\pi x}{L}\)  
(d) \(f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}\)  
(e) \(f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}\)  
(f) \(f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}\)  
(g) \(f(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}\)

3.2.3. Show that the Fourier series operation is linear; that is, show that the Fourier series of \(c_1 f(x) + c_2 g(x)\) is the sum of \(c_1\) times the Fourier series of \(f(x)\) and \(c_2\) times the Fourier series of \(g(x)\).

3.2.4. Suppose that \(f(x)\) is piecewise smooth. What value does the Fourier series of \(f(x)\) converge to at the end point \(x = -L\) or \(x = L\)?

3.3 FOURIER COSINE AND SINE SERIES

In this section we show that the series of sines only (and the series of cosines only) are special cases of a Fourier series.

3.3.1 Fourier Sine Series

Odd functions. An odd function is a function with the property \(f(-x) = -f(x)\). The sketch of an odd function for \(x < 0\) will be minus the mirror image of \(f(x)\) for \(x > 0\), as illustrated in Fig. 3.3.1. Examples of odd functions are \(f(x) = x^3\) (in fact, any odd power) and \(f(x) = \sin 4x\). The integral of an
odd function over a symmetric interval is zero (any contribution from \(x > 0\) will be canceled by a contribution from \(x < 0\)).

**Fourier series of odd functions.** Let us calculate the Fourier coefficients of an odd function:

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = 0
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx = 0.
\]

Both are zero because the integrand, \(f(x) \cos \frac{n\pi x}{L}\), is odd (being the product of an even function \(\cos \frac{n\pi x}{L}\) and an odd function \(f(x)\)). Since \(a_n = 0\), all the cosine functions (which are even) will not appear in the Fourier series of an odd function. The Fourier series of an odd function is an infinite series of odd functions (sines):

\[
f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},
\]

(3.3.1)

if \(f(x)\) is odd. In this case formulas for the Fourier coefficients \(b_n\) may be simplified:

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx,
\]

(3.3.2)

since the integral of an even function over the symmetric interval \(-L\) to \(+L\) is twice the integral from 0 to \(L\). For odd functions information about \(f(x)\) is needed only for \(0 \leq x \leq L\).

**Fourier sine series.** However, only occasionally are we given an odd function and asked to compute its Fourier series. Instead, frequently series of only sines arise in the context of separation of variables. Recall that the temperature in a one-dimensional rod \(0 < x < L\) with zero temperature ends \(u(0, t) = u(L, t) = 0\) satisfies

\[
u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\alpha n^2 \pi^2 / L},
\]

(3.3.3)

where the initial condition \(u(x, 0) = f(x)\) is satisfied if

\[
f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.
\]

(3.3.4)

\(f(x)\) must be represented as a series of sines; (3.3.4) appears in the same form as (3.3.1). However, there is a significant difference. In (3.3.1) \(f(x)\) is given as an odd function and defined for \(-L \leq x \leq L\). In (3.3.4) \(f(x)\) is only defined for \(0 \leq x \leq L\) (it is just the initial temperature distribution); \(f(x)\) is certainly not necessarily odd. If \(f(x)\) is only given for \(0 \leq x \leq L\), then it can be extended as an odd function; see Fig. 3.3.2, called the odd extension of \(f(x)\). The odd extension of \(f(x)\) is defined for \(-L \leq x \leq L\). Fourier’s theorem will apply [if

the odd extension of \(f(x)\) is piecewise smooth, which just requires that \(f(x)\) is piecewise smooth for \(0 \leq x \leq L\)]. Moreover, since the odd extension of \(f(x)\) is certainly odd, its Fourier series only involves sines:

\[
f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad -L \leq x \leq L,
\]

where \(B_n\) are given by (3.3.2). However, we are only interested in what happens between \(x = 0\) and \(x = L\). In that region \(f(x)\) is identical to its odd extension:

\[
f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 \leq x \leq L,
\]

(3.3.5)

where

\[
B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.
\]

(3.3.6)

We call this the Fourier sine series of \(f(x)\) (on the interval \(0 \leq x \leq L\)). This series (3.3.5) is nothing but an example of a Fourier series. As such, we can simply apply Fourier’s theorem; just remember that \(f(x)\) is only defined for \(0 \leq x \leq L\). We may think of \(f(x)\) as being odd (although it is not necessarily) by extending \(f(x)\) as an odd function. Formula (3.3.6) is very important, but does not need to be memorized. It can be derived from the formulas for a Fourier series simply by assuming that \(f(x)\) is odd. (It is more accurate to say that we consider the odd extension of \(f(x)\).) Formula (3.3.6) is a factor of 2 larger than the Fourier series coefficients since the integrand is even. In (3.3.6) the integrals are only from \(x = 0\) to \(x = L\).

According to Fourier’s theorem, sketching the Fourier sine series of \(f(x)\) is easy:

1. Sketch \(f(x)\) (for \(0 < x < L\)).
2. Sketch the odd extension of \(f(x)\).
3. Extend as a periodic function (with period \(2L\)).
4. Mark an \(\times\) at the average at points where the odd periodic extension of \(f(x)\) has a jump discontinuity.
Example. As an example, we show how to sketch the Fourier sine series of \( f(x) = 100 \). We consider \( f(x) = 100 \) only for \( 0 \leq x \leq L \). We begin by sketching in Fig. 3.3.3 its odd extension. The Fourier sine series of \( f(x) \) equals the Fourier series of the odd extension of \( f(x) \). In Fig. 3.3.4 we repeat periodically the odd extension (with period \( 2L \)). At points of discontinuity, the average is marked with an \( \times \). According to Fourier’s theorem (as illustrated in Fig. 3.3.4), the Fourier sine series of 100 actually equals 100 for \( 0 < x < L \), but the infinite series does not equal 100 at \( x = 0 \) and \( x = L \):

\[
100 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.
\]

At \( x = 0 \), Fig. 3.3.4 shows that the Fourier sine series converges to 0, because at \( x = 0 \) the odd property of the sine series yields the average of 100 and \(-100\), which is 0. For similar reasons, the Fourier sine series also converges to 0 at \( x = L \). These observations agree with the result of substituting \( x = 0 \) (and \( x = L \)) into the infinite series of sines. The Fourier coefficients are determined from (3.3.6) as before [see (2.3.39)]:

\[
b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = \frac{200}{L} \int_{0}^{L} \sin \frac{n\pi x}{L} \, dx = \begin{cases} 0 & n \text{ even} \\ \frac{400}{n\pi} & n \text{ odd}. \end{cases}
\]

We recall from Sec. 2.3 that the method of separation of variables implied that

\[
u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-\left(\frac{n\pi t}{L}\right)^2}.
\]

The initial conditions are satisfied if

\[
100 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.
\]

This may be interpreted as the Fourier sine series of \( f(x) = 100 \) [see (3.3.8)]. Equivalently, \( b_n \) may be determined from the orthogonality of \( \sin \frac{n\pi x}{L} \) [see (2.3.39)].

Mathematically, the Fourier series of the initial condition has a rather bizarre behavior at \( x = 0 \) (and at \( x = L \)). In fact, for this problem, the physical situation is not very well defined at \( x = 0 \) (at \( t = 0 \)). This might be illustrated in a space–time diagram, Fig. 3.3.5. We note that Fig. 3.3.5 shows that the domain of our problem is \( t > 0 \) and \( 0 < x < L \). However, there is a conflict that occurs at \( x = 0, t = 0 \) between the initial condition and the boundary condition. The initial condition \( f(t = 0) \) prescribes the temperature to be 100\(^\circ\) even as \( x \to 0 \), whereas the boundary condition \( f(x = 0) \) prescribes the temperature to be 0\(^\circ\) even as \( t \to 0 \). Thus, the physical problem has a discontinuity at \( x = 0, t = 0 \). In the actual physical world, the temperature cannot be discontinuous.

We introduced a discontinuity into our mathematical model by “instantaneously” transporting (at \( t = 0 \)) the rod from a 100\(^\circ\) bath to a 0\(^\circ\) bath at \( x = 0 \). It actually takes a finite time, and the temperature would be continuous. Nevertheless, the transition from 0\(^\circ\) to 100\(^\circ\) would occur over an exceedingly small distance and time. We introduce the temperature discontinuity to approximate the more complicated real physical situation. Fourier’s theorem thus illustrates how the physical discontinuity at \( x = 0 \) (initially, at \( t = 0 \)) is reproduced mathematically. The Fourier sine series of 100\(^\circ\) (which represents the physical solution at \( t = 0 \)) has the nice property that it equals 100\(^\circ\) for all \( x \) inside the rod, \( 0 < x < L \) (thus satisfying the initial condition there), but it equals 0\(^\circ\) at the boundaries, \( x = 0 \) and \( x = L \) (thus also satisfying the boundary conditions). The Fourier sine series of 100\(^\circ\) is a strange mathematical function, but so is the physical approximation it is needed for.
Fourier series computations and the Gibbs phenomenon. Let us gain some confidence in the validity of Fourier series. The Fourier sine series of \( f(x) = 100 \) states that

\[
100 = 400 \left( \frac{\sin \pi x / L}{1} + \frac{\sin 3\pi x / L}{3} + \frac{\sin 5\pi x / L}{5} + \cdots \right).
\]

(3.3.10)

Do we believe (3.3.10)? Certainly, it is not valid at \( x = 0 \) (as well as the other boundary \( x = L \)), since at \( x = 0 \) every term in the infinite series is zero (they cannot add to 100). However, the theory of Fourier series claims that (3.3.10) is valid everywhere except the two ends. For example, we claim it is valid at \( x = L/2 \). Substituting \( x = L/2 \) into (3.3.10) shows that

\[
100 = 400 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \right)
\]

or

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.
\]

At first this may seem strange. However, it is Euler's formula for \( \pi \). It can be used to compute \( \pi \) (although very inefficiently); it can also be shown to be true without relying on the theory of infinite trigonometric series (see Exercise 3.3.17). The validity of (3.3.10) for other values of \( x, 0 < x < L \), may also surprise you. We will sketch the left- and right-hand sides of (3.3.10), hopefully convincing you of their equality. We will sketch the r.h.s. by adding up the contribution of each term of the series. Of course, we cannot add up the required infinite number of terms; we will settle for a finite number of terms. In fact, we will sketch the sum of the first few terms to see how the series approaches the constant 100 as the number of terms increases. It is helpful to know that \( 400/\pi = 127.32395 \ldots \) (although for rough sketching 125 or 130 will do). The first term \((400/\pi) \sin \pi x / L\) by itself is the basic first rise and fall of a sine function; it is not a good approximation to the constant 100 as illustrated in Fig. 3.3.6. On the other hand, for just one term in an infinite series it is not such a bad approximation. The next term to be added is \((400/3\pi) \sin 3\pi x / L\). This is a sinusoidal oscillation, with one-third the amplitude and one-third the period of the first term. It is positive near \( x = 0 \) and \( x = L \), where the approximation needs to be increased, and is negative near \( x = L/2 \), where the approximation needs to be decreased. It is sketched in dashed lines and then added to the first term in Fig. 3.3.7. Note that the sum of the two nonzero terms already seems to be a considerable improvement over the first term. A computer plot of all the partial sums, up to and including the first six nonzero terms, is given in Fig. 3.3.8.

Actually, a lot can be learned from Fig. 3.3.8. Perhaps now it does seem reasonable that the infinite series converges to 100 for \( 0 < x < L \). The worst places (where the finite series differs most from 100) are getting closer and closer to \( x = 0 \) and \( x = L \) as the number of terms increases. For a finite number of terms in the series, the solution starts from zero at \( x = 0 \) and shoots up beyond 100, what we call the primary overshoot. It is interesting to note that Fig. 3.3.8 illustrates the overshoot vividly. We can even extrapolate to guess what happens for 1000 terms. The series should become more and more accurate as the number of terms increases. We might expect the overshoot to vanish as \( n \to \infty \), but put a straight edge on the points of maximum overshoot. It just does not seem to approach 100. Instead, it is far away from that, closer to 118. This overshoot is an example of the Gibbs phenomenon. In general (for large \( n \)), there is an
overshoot (and corresponding undershoot) of approximately 9% of the jump discontinuity. In this case (see Fig. 3.3.4), the Fourier sine series of \( f(x) = 100 \) jumps from \(-100\) to \(+100\) at \( x = 0 \). Thus, the finite series will overshoot by about 9% of 200, or approximately 18. The Gibbs phenomenon occurs only when a finite series of eigenfunctions approximates a discontinuous function.

**Further example of a Fourier sine series.** We consider the Fourier sine series of \( f(x) = x \). \( f(x) = x \) is sketched on the interval \( 0 \leq x \leq L \) in Fig. 3.3.9a. The odd-periodic extension of \( f(x) \) is sketched in Fig. 3.3.9b. The jump discontinuity of the odd-periodic extension at \( x = (2n - 1)L \) shows that, for example, the Fourier sine series of \( f(x) = x \) converges to zero at \( x = L \), while \( f(L) \neq 0 \). We note that the Fourier sine series of \( f(x) = x \) actually equals \( x \) for \(-L < x < L\).

\[
x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad -L < x < L.
\]  

(3.3.11a)

The Fourier coefficients are determined from (3.3.6)

\[
b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} x \sin \frac{n\pi x}{L} \, dx = \frac{2L}{n\pi} \left( -1 \right)^{n+1},
\]  

(3.3.11b)

where the integral can be evaluated by integration by parts (or by a table).

![Figure 3.3.9](image)

**Figure 3.3.9** (a) \( f(x) = x \) and (b) its Fourier sine series.

**Example.** We now consider the Fourier sine series of \( f(x) = \cos \frac{\pi x}{L} \). This may seem to ask for a sine series expansion of an even function, but in applications often the function is only given from \( 0 \leq x \leq L \) and must be expanded in a series of sines due to the boundary conditions. \( \cos \frac{\pi x}{L} \) is sketched in Fig. 3.3.10a. It is an even function, but its odd extension is sketched in Fig. 3.3.10b. The Fourier sine series of \( f(x) \) equals the Fourier series of the odd extension of \( f(x) \). Thus, we repeat the sketch in Fig. 3.3.10b periodically (see Fig. 3.3.11), placing an \( \times \) at the average of the two values at the jump discontinuities. The Fourier sine series representation of \( \cos \frac{\pi x}{L} \) is

\[
\cos \frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 \leq x \leq L,
\]

where with some effort we obtain

\[
b_n = \frac{2}{L} \int_{0}^{L} \cos \frac{\pi x}{L} \sin \frac{n\pi x}{L} \, dx = \begin{cases} 0 & \text{n odd} \\ \frac{4n}{\pi(n^2 - 1)} & \text{n even} \end{cases}
\]  

(3.3.12)

According to Fig. 3.3.11 (based on Fourier’s theorem), equality holds for \( 0 < x < L \), but not at \( x = 0 \) and not at \( x = L \):

\[
\cos \frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.
\]

At \( x = 0 \) and at \( x = L \), the infinite series must converge to 0, since all terms in the series are zero there. Figure 3.3.11 agrees with this. You may be a bit puzzled by an aspect of this problem. You may have recalled that \( \sin \frac{n\pi x}{L} \) is orthogonal to \( \cos \frac{n\pi}{L} \), and thus expected all the \( b_n \) in (3.3.12) to be zero. However, \( b_n \neq 0 \). The subtle point is that you should remember that \( \cos \frac{n\pi}{L} \) and \( \sin \frac{n\pi x}{L} \) are orthogonal on the interval \(-L \leq x \leq L\). Let us develop the basic results. The sine coefficients of a Fourier

![Figure 3.3.10](image)

**Figure 3.3.10** (a) \( f(x) = \cos \frac{\pi x}{L} \) and (b) its odd extension.

![Figure 3.3.11](image)

**Figure 3.3.11** Fourier sine series of \( f(x) = \cos \frac{\pi x}{L} \).

**3.3.2 Fourier Cosine Series**

**Even functions.** Similar ideas are valid for even functions, in which \( f(-x) = f(x) \). Let us develop the basic results. The sine coefficients of a Fourier

Sec. 3.3 Fourier Cosine and Sine Series
series will be zero for an even function,
\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = 0, \]
since \( f(x) \) is even. The Fourier series of an even function is a representation of \( f(x) \) involving an infinite sum of only even functions (cosines):
\[ f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}, \]
(3.3.13a)
if \( f(x) \) is even. The coefficients of the cosines may be evaluated using information about \( f(x) \) only between \( x = 0 \) and \( x = L \), since
\[ a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = \frac{1}{L} \int_{0}^{L} f(x) \, dx \]
(3.3.13b)
\[(n \neq 0) \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \] (3.3.13c)
using the fact that for \( f(x) \) even, \( f(x) \cos \frac{n\pi x}{L} \) is even.

Often, \( f(x) \) is not given as an even function. Instead, in trying to represent an arbitrary function \( f(x) \) using an infinite series of \( \cos \frac{n\pi x}{L} \), the eigenfunctions of the boundary value problem \( d^2\phi/dx^2 = -\lambda \phi \) with \( d\phi/dx(0) = 0 \) and \( d\phi/dx(L) = 0 \), we wanted
\[ f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}, \]
(3.3.14)
only for \( 0 < x < L \). We had previously determined the coefficients \( a_n \) to be the same as given by (3.3.13), but our reason was because of the orthogonality of \( \cos \frac{n\pi x}{L} \). To relate (3.3.14) to a Fourier series, we simply introduce the even extension of \( f(x) \), an example being illustrated in Fig. 3.3.12. If \( f(x) \) is piecewise smooth for \( 0 \leq x \leq L \), then its even extension will also be piecewise smooth, and hence Fourier’s theorem can be applied to the even extension of \( f(x) \). Since the even extension of \( f(x) \) is an even function, the Fourier series of the even extension of \( f(x) \) will have only cosines:
\[ \text{even extension of } f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad -L \leq x \leq L, \]

where \( a_n \) is given by (3.3.13). In the region of interest, \( 0 \leq x \leq L \), \( f(x) \) is identical to the even extension. The resulting series in that region is called the Fourier cosine series of \( f(x) \) (on the interval \( 0 \leq x \leq L \)):
\[ f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L \]
(3.3.15)
\[ a_0 = \frac{1}{L} \int_{0}^{L} f(x) \, dx \]
(3.3.16a)
\[ a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx. \] (3.3.16b)

The Fourier cosine series of \( f(x) \) is exactly the Fourier series of the even extension of \( f(x) \). Since we can apply Fourier’s theorem, we have an algorithm to sketch the Fourier cosine series of \( f(x) \):

1. Sketch \( f(x) \) (for \( 0 < x < L \)).
2. Sketch the even extension of \( f(x) \).
3. Extend as a periodic function (with period \( 2L \)).
4. Mark \( \times \) at points of discontinuity at the average.

**Example.** We consider the Fourier cosine series of \( f(x) = x \). \( f(x) \) is sketched in Fig. 3.3.13a [note that \( f(x) \) is odd!]. We consider \( f(x) \) only from \( x = 0 \) to \( x = L \), and then extend it in Fig. 3.3.13b as an even function. Next, we sketch the Fourier series of the even extension, by periodically extending the even extension (see Fig. 3.3.14). Note that between \( x = 0 \) and \( x = L \) the Fourier cosine series has no jump discontinuities. The Fourier cosine series of \( f(x) = x \) actually equals \( x \), so that
\[ x = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L. \]
(3.3.17a)
The coefficients are given by the following integrals:

\[ a_0 = \frac{1}{L} \int_{-L}^{L} x \, dx = \frac{1}{L^2} \left[ \frac{x^2}{2} \right]_{-L}^{L} = \frac{L}{2} \]  
\[ (3.3.17b) \]

\[ a_n = \frac{2}{L} \int_{0}^{L} x \cos \frac{n\pi x}{L} \, dx = \frac{2L}{(n\pi)^2} \cos n\pi - 1 \]  
\[ (3.3.17c) \]

The latter integral can be evaluated by tables or integration by parts. We omit the details.

### 3.3.3 Representing \( f(x) \) by Both a Sine and Cosine Series

It may be apparent that any function \( f(x) \) (which is piecewise smooth) may be represented both as a Fourier sine series and as a Fourier cosine series. The one you would use is dictated by the boundary conditions (if the problem arose in the context of a solution to a partial differential equation using the method of separation of variables). It is also possible to use a Fourier series (including both sines and cosines). As an example, we consider the sketches of the Fourier, Fourier sine, and Fourier cosine series of

\[ f(x) = \begin{cases} 
\frac{L}{2} \sin \frac{n\pi x}{L} & x < 0 \\
x & 0 < x < \frac{L}{2} \\
L - x & x > \frac{L}{2}.
\end{cases} \]

The graph of \( f(x) \) is sketched for \(-L < x < L\) in Fig. 3.3.15. The Fourier series of \( f(x) \) is sketched by repeating this pattern with period \( 2L \). On the other hand for the Fourier sine (cosine) series, first sketch the odd (even) extension of the function \( f(x) \), before repeating the pattern. These three are sketched in Fig. 3.3.16. Note that for \(-L \leq x \leq L\) only the Fourier series of \( f(x) \) actually equals \( f(x) \). However, for all three cases the series equals \( f(x) \) over the region \( 0 \leq x \leq L \).

### 3.3.4 Even and Odd Parts

Let us consider the Fourier series of a function \( f(x) \) which is not necessarily even or odd:

\[ f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \]  
\[ (3.3.18) \]

where

\[ a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx. \]

It is interesting to see that a Fourier series is the sum of a series of cosines and a series of sines. For example, \( \sum_{n=1}^{\infty} b_n \sin n\pi x/L \) is not in general the Fourier sine series of \( f(x) \), because the coefficients, \( b_n = 1/L \int_{-L}^{L} f(x) \sin n\pi x/L \, dx \), are not in general the same as the coefficients of a Fourier sine series.
The Fourier series of $f(x)$ equals the Fourier sine series of $f_s(x)$ plus the Fourier cosine series of $f_c(x)$, where $f_s(x) = \frac{1}{2} [f(x) + f(-x)]$ and $f_c(x) = \frac{1}{2} [f(x) - f(-x)]$.

Please do not confuse this result with even and odd extensions. For example, the even part of $f(x) = \frac{1}{2} [f(x) + f(-x)]$, while the even extension of $f(x) = \begin{cases} f(x), & x > 0 \\ f(-x), & x < 0. \end{cases}$.

### 3.3.5 Continuous Fourier Series

The convergence theorem for Fourier series shows that the Fourier series of $f(x)$ may be a different function than $f(x)$. Nevertheless, over the interval of interest, they are the same except at those few points where the periodic extension of $f(x)$ has a jump discontinuity. Sine (cosine) series are analyzed in the same way, where instead the odd (even) periodic extension must be considered. In addition to points of jump discontinuity of $f(x)$ itself, the various extensions of $f(x)$ may introduce a jump discontinuity. From the examples in the preceding section, we observe that sometimes the resulting series does not have any jump discontinuities. In these cases the Fourier series of $f(x)$ will actually equal $f(x)$ in the range of interest. Also, the Fourier series itself will be a continuous function.

It is worthwhile to summarize the conditions under which a Fourier series is continuous:

> For piecewise smooth $f(x)$, the Fourier series of $f(x)$ is continuous for $-L < x < L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.

It is necessary for $f(x)$ to be continuous; otherwise, there will be a jump discontinuity and the Fourier series of $f(x)$ will converge to the average. In Fig. 3.3.17 we illustrate the significance of the condition $f(-L) = f(L)$. We illustrate two continuous functions, only one of which satisfies $f(-L) = f(L)$. The condition $f(-L) = f(L)$ insists that the repeated pattern (with period $2L$) will be continuous at the end points. The boxed statement above is a fundamental result for all Fourier series. It explains the following similar theorems for Fourier sine and cosine series.

Consider the Fourier cosine series of $f(x)$ [$f(x)$ has been extended as an even function]. If $f(x)$ is continuous, is the Fourier cosine series continuous? An example that is continuous for $-L < x < L$ is sketched in Fig. 3.3.18. First we extend $f(x)$ evenly and then periodically. It is easily seen that:

> For piecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is continuous for $0 < x < L$ if and only if $f(x)$ is continuous.

We note that no additional conditions on $f(x)$ are necessary for the cosine series to be continuous (besides $f(x)$ being continuous). One reason for this result is...
that if \( f(x) \) is continuous for \( 0 \leq x \leq L \), then the even extension will be continuous for \( -L \leq x \leq L \). Also note that the even extension is the same at \( \pm L \). Thus, the periodic extension will automatically be continuous at the end points.

Compare this result to what happens for a Fourier sine series. Four examples are considered in Fig. 3.3.19, all continuous functions for \( 0 \leq x \leq L \). From the

that if \( f(x) \) is continuous for \( 0 \leq x \leq L \), then the even extension will be continuous for \( -L \leq x \leq L \). Also note that the even extension is the same at \( \pm L \). Thus, the periodic extension will automatically be continuous at the end points.

Compare this result to what happens for a Fourier sine series. Four examples are considered in Fig. 3.3.19, all continuous functions for \( 0 \leq x \leq L \). From the

If \( f(0) \neq 0 \), then the odd extension of \( f(x) \) will have a jump discontinuity at \( x = 0 \), as illustrated in Fig. 3.3.19a and c. If \( f(L) \neq 0 \), then the odd extension at \( x = -L \) will be of opposite sign from \( f(L) \). Thus, the periodic extension will not be continuous at the end points if \( f(L) \neq 0 \) as in Fig. 3.3.19a and b.

**EXERCISES 3.3**

3.3.1. For the following functions, sketch \( f(x) \), the Fourier series of \( f(x) \), the Fourier sine series of \( f(x) \), and the Fourier cosine series of \( f(x) \).

(a) \( f(x) = 1 \)  
(b) \( f(x) = 1 + x \)  
(c) \( f(x) = \begin{cases} 
1 + x & x > 0 \\
0 & x < 0 
\end{cases} \)  
(d) \( f(x) = e^x \)  
(e) \( f(x) = \begin{cases} 
0 & x < 0 \\
1 & x > 0 
\end{cases} \)

3.3.2. For the following functions, sketch the Fourier sine series of \( f(x) \) and determine its Fourier coefficients.

(a) \( f(x) = \cos \frac{\pi x}{L} \)  
(b) \( f(x) = \begin{cases} 
1 & x < L/6 \\
2 & L/6 < x < L/2 \\
0 & x > L/2 
\end{cases} \)  
(c) \( f(x) = \begin{cases} 
0 & x < L/2 \\
1 & x > L/2 
\end{cases} \)  
(d) \( f(x) = e^x \)  
(e) \( f(x) = \begin{cases} 
0 & x < L/2 \\
1 & x > L/2 
\end{cases} \)

3.3.3. For the following functions, sketch the Fourier sine series of \( f(x) \). Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier sine series:

(a) \( f(x) = \cos \frac{\pi x}{L} \)  
(b) \( f(x) = \begin{cases} 
1 & x < L/2 \\
0 & x > L/2 
\end{cases} \)  
(c) \( f(x) = x \)  
(d) \( f(x) = e^x \)  
(e) \( f(x) = \begin{cases} 
0 & x < L/2 \\
1 & x > L/2 
\end{cases} \)

3.3.4. Sketch the Fourier cosine series of \( f(x) = \sin \frac{\pi x}{L} \). Briefly discuss.

3.3.5. For the following functions, sketch the Fourier cosine series of \( f(x) \) and determine its Fourier coefficients:

(a) \( f(x) = x^2 \)  
(b) \( f(x) = \begin{cases} 
1 & x < L/6 \\
2 & L/6 < x < L/2 \\
0 & x > L/2 
\end{cases} \)  
(c) \( f(x) = \begin{cases} 
0 & x < L/2 \\
1 & x > L/2 
\end{cases} \)

Sec. 3.3 Fourier Cosine and Sine Series