Fraunhofer Diffraction

Equipment

Green laser (563.5 nm) on 2-axis translation stage, 1m optical bench, 1 slide holder, slide with four single slits, slide with 3 diffraction gratings, slide with four double slits, slide with multiple slits, pinhole, slide containing an etched Fourier transformed function, screen, tape measure.

Purpose

To understand and test Fraunhofer diffraction through various apertures. To understand Fraunhofer diffraction in terms of Fourier analysis. To gain experience with laser optics.

Theory

Suppose, as depicted in Figure 1, that a laser is shone upon a small slit. The light passing through the slit is then allowed to fall on a screen, positioned at some distance, *L*, from the slit, with its plane perpendicular to the path of the light's propagation. If light traveled in straight lines one would expect the screen to display a single image of the slit with the rest of the screen in shadow. For a sufficiently small slit, however, it is found that this is not the case. Instead, a diffraction pattern is observed, consisting of a central bright fringe along the slit-screen axis with alternating dark and bright fringes on either side of the central fringe. It is evident, based on this observation, that light does not travel in straight lines since bright fringes are seen where shadow would be expected. It follows then, that light has the property of being able to "bend" around corners, a property called diffraction.

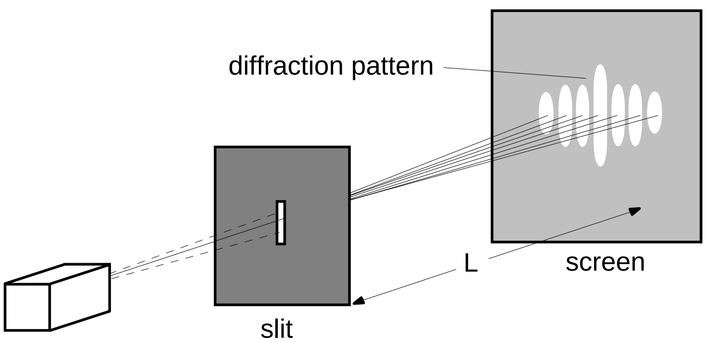


Figure 1

In order to examine diffraction we will assume that light behaves like a wave. With this assumption one can imagine a beam of light to consist of a series of wave fronts aligned perpendicular to the direction of light propagation. As shown in Figure 2, a ray can be drawn perpendicular to the wave fronts, indicating the direction of light propagation. This wave model of light allows us to make use of Huygens' principle which claims that any point on a wave front can itself be regarded as a point source emitting circular wave fronts. Applying this principle to the situation under consideration, every point located in the slit acts as a point source. Consequently, the diffraction pattern must be the result of the constructive and destructive interference of the various waves generated by these point sources.

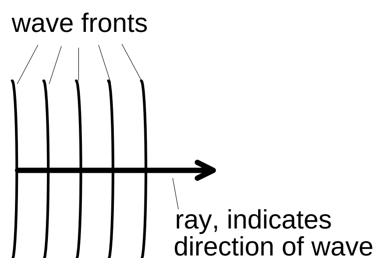
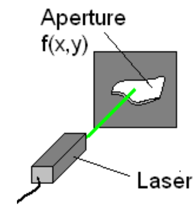


Figure 2

Diffraction phenomena are usually separated into two different types called Fraunhofer and Fresnel diffraction. The latter describes the **near-field** pattern observed when the screen is placed close to the slit (i.e. at small values of *L*), so the wave effects make only a small correction to the geometric shadow of the aperture. Fraunhofer diffraction, on the contrary, describes the diffraction pattern observed in the **far field** (i.e. at large values of *L*) where geometric optics is completely inapplicable. We will postpone Fresnel diffraction until a later date, and concentrate here on the far-field case.

Consider a general aperture illuminated by light as in figure 3. In the plane of the aperture, perpendicular to the plane-wave propagation axis $\hat{z}$, the electric field is given by:



$$E(x,y) = E_0 f(x,y) e^{-i\omega t}, \qquad (1)$$

Figure 3

where $f = (x,y)$ reflects the geometry of the aperture. We treat this field according to Huygens' principle as an antenna (extended oscillator) that generates secondary waves that form the diffraction pattern. To calculate this pattern, we re-express the oscillator as a linear combination of infinite plane antennas:

$$E(x,y) = E_0 \iint_{k_x,k_z} f(k_x,k_y) e^{i(k_x x + k_y y) - i\omega t} dk_x dk_y, \qquad (2)$$

where each antenna is characterized by a pair of wave vectors (k_x, k_y) and $f(k_x,k_y)$ is the **inverse Fourier transform** of the aperture function:

$$f(k_x,k_y) = \Im^{-1}\{f(x,y)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ik_x x - ik_y y} f(x,y) dx dy. \qquad (3)$$

Each antenna in Eq. (2) generates a plane wave

$$e^{i(k_x x + k_y y) - i\omega t} \rightarrow e^{i(k_x x + k_y y + ik_z z) - i\omega t}$$

with $k_y^2 + k_x^2 + k_z^2 = (2\pi/\lambda)^2 = \omega^2/c^2$, so the field behind the aperture can be written as

$$E(x,y,z) = E_0 \iint_{k_x,k_z} f(k_x,k_y) e^{i(k_x x + k_y y + k_z z) - i\omega t} dk_x dk_y.$$

The field produced by each plane wave antenna propagates in the directions defined by angles (θ_x, θ_y) with respect to the *z* axis, such that

$$\sin\theta_x = \frac{k_x}{k}$$
$$\sin\theta_y = \frac{k_y}{k}. \qquad (4)$$

We continue our treatment within the so-called **paraxial approximation**, where we assume that $k_z >> k_x, k_y$, so $k_z \approx k = \frac{2\pi}{\lambda}$, independent of k_x and k_y. The paraxial approximation also implies that $\theta_x \approx \sin\theta_x$ and $\theta_y \approx \sin\theta_y$. If the diffraction pattern is observed on a screen very far away from the aperture (i.e. in the far-field approximation relevant to the Fraunhofer diffraction), we can assume that the irradiance at some point (x', y') on the screen will be determined by the electric field of the plane wave propagating at angles

$$\theta_x = \frac{x'}{L}$$
$$\theta_y = \frac{y'}{L} \qquad (5)$$

with respect to the z axis. Combining Eqs. (5) and (6) we find for the far field amplitude

$$E(x',y') = f(k\frac{x'}{L}, k\frac{y'}{L}) \qquad (6)$$

In other words, in the paraxial approximation **the far-field diffraction pattern is just a scaled inverse Fourier transform of the aperture**. The intensity of the diffraction pattern is found according to

$$I(x',y') = \left\langle \left| E(x',y') \right| \right\rangle^2, \qquad (8)$$

Consider now a single slit, with the aperture function as described in figure 4. The vertical dimension of the slit is sufficiently large so that we may treat the problem as one-dimensional. The aperture function here is the "top-hat" function:

$$f(x) = E_0, \quad x \in [-\frac{a}{2}, \frac{a}{2}]$$
$$f(x) = 0, \quad x \notin [-\frac{a}{2}, \frac{a}{2}]. \qquad (9)$$

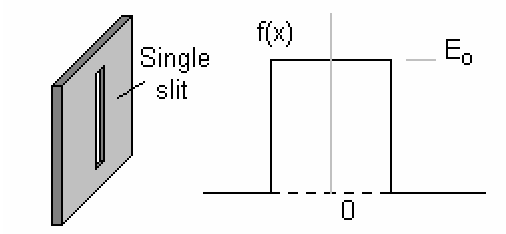


Figure 4

We can then evaluate the Fourier transformation of the aperture function to find the electric field at a point on a screen in the far-field regime

$$E(x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k_x)e^{ik_x x}dx = \frac{1}{2\pi}\int_{-a/2}^{a/2} E_0 e^{ik_x x}dx = E_0\,\frac{e^{ik_x a/2} - e^{-ik_x a/2}}{2\pi ik} = \frac{2aE_0}{2\pi ak}\sin(k_x a/2)$$

or, using equation (4) for k_x and defining:

$$ak_x/2 = ak\sin\theta/2 \equiv \beta \qquad (10)$$

we have:

$$E(\beta) = \frac{aE_0}{2\pi}\frac{\sin\beta}{\beta}. \qquad (11)$$

Recalling that the irradiance at *x'* is just the time-averaged electric field at that point, we can write:

$$I(\theta) = I_0\,[\mathrm{sinc}(\beta)]^2, \qquad (12)$$

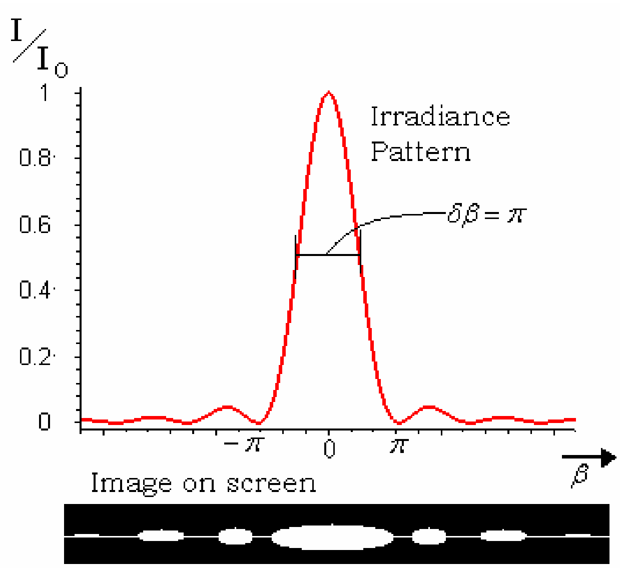


Figure 5

where we've defined I_0 to be $\left(\frac{aE_0}{2\pi}\right)^2$ and used $\mathrm{sinc}(x) \equiv \frac{\sin x}{x}$.

Roots of sin(β) - β	
n	β
1	1.43π
2	2.46π
3	3.47π
4	4.48π
5	5.48π
6	6.48π
7	7.49π

Table 1

Figure 5 displays the resultant irradiance pattern. Note that equation (12) will have minima whenever $\beta = n\pi$, for all non-zero integers *n* (recall that $\lim_{n\to 0}\frac{\sin n}{n} = 1$). Taking the derivative of (12) and setting it equal to zero, we see that sinc β reaches maxima whenever $\beta = \tan\beta$. This is a transcendental equation, for which some roots are listed in Table 1. These values may be used, along with equation (10), to find the angular positions of the maxima of the refracted beams.

Fraunhofer Diffraction from a more complicated apparatus can be calculated by using the fact that the Fourier transform of the convolution two functions $f(x)$ and $g(x)$ is the product of the individual Fourier transforms:

$$\Im^{-1}\{f(x) * g(x)\} = \frac{1}{2\pi}\Im^{-1}\{f(x)\}\cdot\Im^{-1}\{g(x)\}, \qquad (13)$$

where the convolution of functions $f(x)$ and $g(x)$ is given by

$$h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(t)x(x-t)dt\,. \qquad (14)$$

For example, the irradiance pattern for a two slits separated by a distance *b*, with each slit having width *a* may be evaluated using (13) as the convolution of two **Dirac delta functions** and a single slit (Figure 6.)

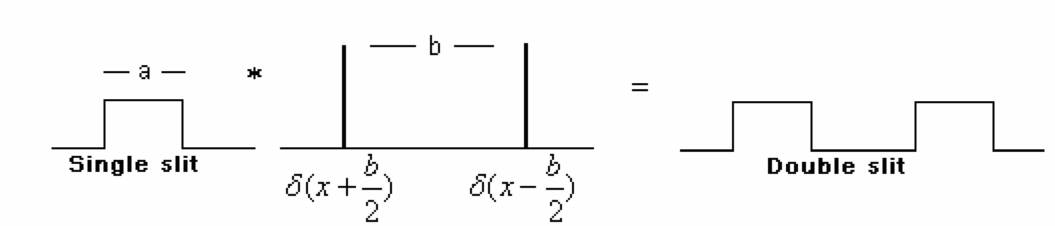


Figure 6

Using equation (13):

$$\Im^{-1}\left\{f_{slit}(x) * \left[\delta(x-\frac{b}{2})+\delta(x+\frac{b}{2})\right]\right\} = \Im^{-1}(f_{slit})\Im^{-1}\left[\delta(x-\frac{b}{2})+\delta(x+\frac{b}{2})\right] \quad (15)$$

with

$$\Im^{-1}(\delta(x-\frac{b}{2})+\delta(x+\frac{b}{2})) = \frac{1}{2\pi}(e^{-ik_x b/2}+e^{ik_x b/2}) = \frac{1}{\pi}\cos(k_x b/2). \quad (16)$$

Using our previous result for the single slit and equation (16), we conclude that

$$I(\theta) \propto \text{sinc}^2\left(\frac{k_x a}{2}\right)\cos^2\left(\frac{k_x b}{2}\right) \quad (19)$$

where the constant of proportionality is unimportant. We see that the single-slit diffraction pattern $\text{sinc}^2(k_x a/2)$ is now modulated by a finer pattern $\cos^2(k_x b/2)$ due to the two-slit diffraction. The distance between the maxima of this pattern is given by

$$\frac{k_x b}{2} = m\pi \quad \text{for } m = 0, \pm 1, \pm 2, ...$$

or, according to Eq. (4),

$$\frac{kb\sin\theta}{2} = m\pi \quad (20)$$

or simply

$$b\sin\theta = m\lambda. \quad (21)$$

The same analysis applies to many slit apertures and in more generality, to an arbitrary collection of similar apertures.

When the number of equidistant slits further increases, the maxima of the diffraction pattern remain at the same positions defined by Eq. (21), but become sharper. An extreme case of such an arrangement is a **diffraction grating** – a repetitive array of slits (or obstacles). Because of their dispersive properties, gratings are commonly used in monochromators and spectrometers. The first artificial diffraction grating was made around 1785 by Philadelphia inventor David Rittenhouse, who strung hairs between two finely threaded screws.

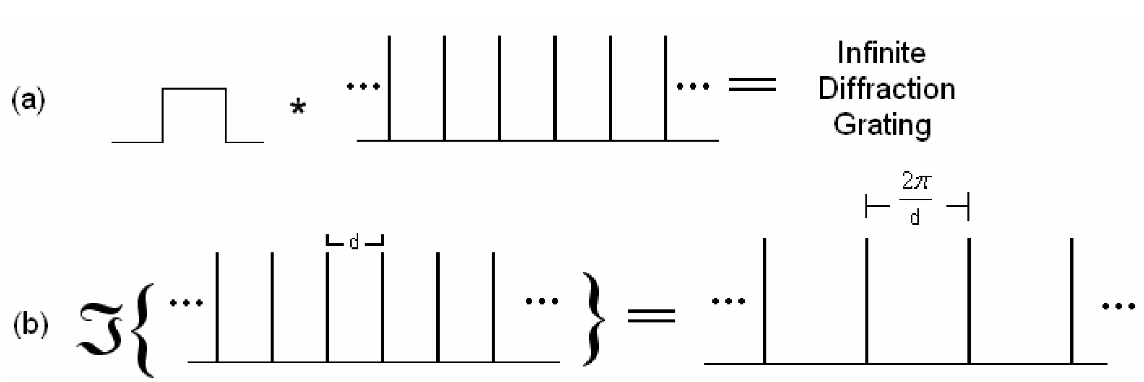


Figure 7

As mentioned, the diffraction grating will create a number of angularly separated maxima, with the angle of diffraction determined by the wavelength of light. The angular position of the m^{th} maximum of the diffracted beam can be obtained by performing the inverse Fourier transform of the grating as illustrated in figure 7. An infinite periodic grating can be considered to be the convolution of a Dirac comb function (an infinite periodic array of delta functions) and a single slit. The Fourier transform of the Dirac comb function of period *d* is also a Dirac comb function of period $2\pi/d$. The m^{th} maximum is thus given by the equation

$$k_x = m\frac{2\pi}{d},$$

from which we find the condition for the maximum

$$d\sin\theta = m\lambda. \tag{22}$$

in analogy to Eq. (22).

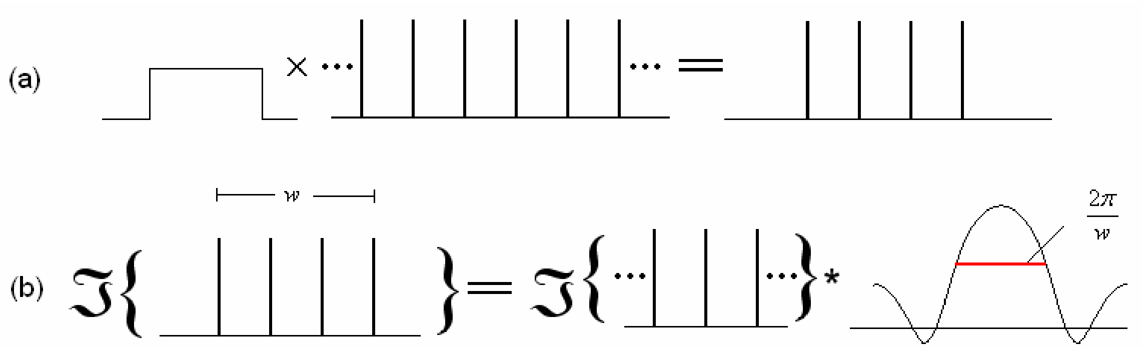


Figure 8

The above treatment is valid for an infinite grating, which produces infinitely sharp diffraction maxima. In a realistic experiment, the grating (more precisely, the illuminated area of the grating) is finite. To find the Fraunhofer diffraction pattern from such a grating, we notice that its aperture function equals to that of an infinite grating times that of a single slit (figure 8). The resultant far field image is then the convolution of the irradiance pattern for a single slit and that of an infinite grating, and the width $\delta\sin\theta$ of a given peak is found according to Eqs. (10) and (11):

$$wk\delta\sin\theta/2 = \pi$$

or

$$\delta\sin\theta = \frac{\lambda}{w}, \tag{23}$$

w being the grating width.

Illuminating a greater portion of the grating will generate sharper peaks. This is important since gratings are typically used to separate light of different wavelengths. As mentioned, a property of the diffraction grating is that the angle of deviation depends on the wavelength of the incident light. However if two wavelengths differ by very little, so will their angular separation. If the peaks of each diffracted beam overlap too much, the resultant image will appear as a single, brighter peak (see figure 9). If $\Delta\lambda_{\min}$ is the smallest possible difference in wavelength which allows for the identification of individual peaks (known as the **limit of resolution**), we define the **chromatic resolving power** of the diffraction grating as:

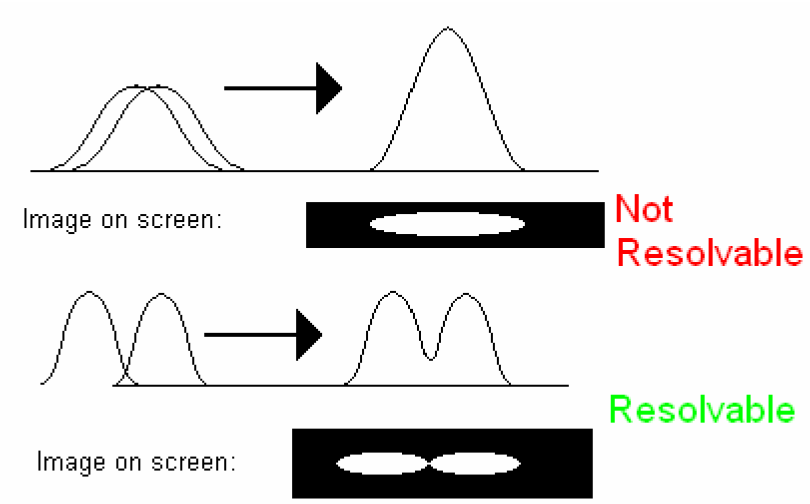


Figure 9

$$\Re \equiv \frac{\lambda}{\Delta\lambda_{\min}}.$$

In order to have a high resolving power, we wish to increase the effective width of the grating, thus making the diffraction peaks sharper. Therefore, for a diffraction grating, it is advantageous to illuminate as large an area as possible.

Two spectral lines separated by an interval $\Delta\lambda$ produce two diffraction peaks whose separation $\Delta \sin\theta$ can be found according to Eq. (22):

$$\Delta\lambda = \frac{d\Delta \sin\theta}{m}.$$

In order to resolve these lines, their separation $\Delta \sin\theta$ should not exceed the individual peak width given by Eq. (23). We find

$$\Delta\lambda = \frac{d\delta \sin\theta}{m} = \frac{d}{mw}\lambda$$

so the resolving power of the m^{th} maximum of the grating is

$$\Re = \frac{\lambda}{\Delta\lambda_{\min}} = \frac{wm}{d} = mN. \qquad (24)$$

The resolution of the grating in the first diffraction order equals the number of lines illuminated.

Figure 10 shows the experimental arrangement that will be used to study diffraction patterns. The laser, mounted on the optical bench by means of the two-axis translation stage, shines on the slide holder which holds one of the various diffraction slides or gratings. The translation stage is used to accurately adjust the x and y position of the laser. The resulting diffraction pattern is observed on the screen.

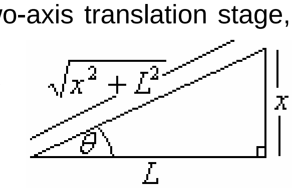


Figure 11

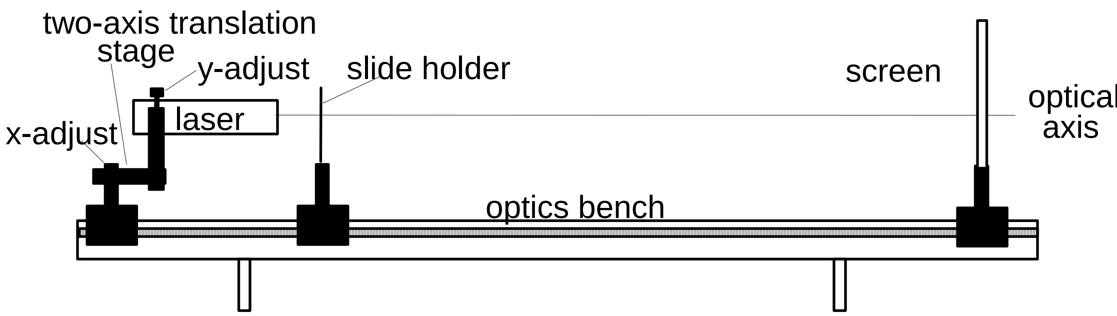


Figure 10

In this experiment, it is frequently required to measure the angle of a diffracted beam by measuring the distance from the aperture to the slide. From basic trigonometry (figure 11) the sine of the diffracted angle may be calculated from the displacement x of the pattern from the primary (undiffracted) beam as follows:

$$\sin\theta = \frac{x}{\sqrt{x^2 + L^2}} \qquad (25)$$

Experimental Procedure

1. Arrange the equipment so that the laser is pointed at a piece of blank paper taped to the wall with the slide holder positioned quite close (≤ 5.00 cm) to the laser. Throughout this experiment it is important to remember that looking directly into the laser can cause retinal damage. As a consequence the optics bench should be positioned so that no one is working in the path of the laser.

2. Using a tape measure, determine the distance from the grating to the wall.

3. Using equations (10) and (12) calculate the values for the maxima and minima of the single slit diffraction pattern for the slit widths of 0.02 mm, 0.04 mm, 0.08 mm, and 0.16 mm.

4. Place the slide consisting of four single slits with openings of 0.02 mm, 0.04 mm, 0.08 mm, and 0.16 mm in the slide holder. Adjust the two-axis translation stage so that the beam of the laser passes through the largest slit. Sketch the resultant intensity pattern, mark the central maximum as x = 0 cm and measure the distance from each maximum to the central fringe. Repeat the process for the remaining three slits. Compare your theoretical results obtained in part 3 to your experimental data.

5. Replace the single slit slide with the slide consisting of four double slits of varying width and slit spacing. The individual spacing of the maxima may be too small to measure accurately, but the distance between three or four maxima is easily measurable. Make a sketch of the resultant pattern and measure the distance from the central fringe to every four or so fringes for as many fringes as possible. Compare the resulting diffraction pattern to the pattern obtained for the single slit and with the theoretical prediction (19).

6. Replace the double slit slide with the multiple slit slide consisting of four multiple slit patterns with the same width and spacing. Compare the resulting diffraction pattern and note, qualitatively, how the number of slits affects the diffraction pattern.

7. Replace the multiple slit slide with the slide consisting of three diffraction gratings. Make sure the grating is positioned perpendicular to the optical axis. This can be accomplished by twisting

the slide holder so that the distance between adjacent maxima of the diffraction pattern is minimized. Adjust the two-axis translation stage so that the laser is incident upon the grating ruled with 2400 lines/inch. (1 inch = 2.54 cm).

8. Using the tape measure determine the distance, *L*, between the central maximum displayed on the wall and the diffraction grating. For at least six maxima on one side of the central maximum measure the displacement, *x*, from the central maximum. To the best of your ability, measure the diameter of the beam incident on the wall. Repeat for the remaining two gratings.

9. Plot a graph of $\sin\theta$ versus *m* with error bars for each order *m* you have measured. From the slope of the best fit line, determine the wavelength of the laser with error and compare to the accepted value, printed on the laser.

10. For several diffraction orders, use the measured peak diameter to determine the resolution of the grating (i.e. if the grating were illuminated by another laser with a slightly different wavelength, how large does the wavelength difference need to be in order to produce a distinct diffraction maximum?). Make an order-of-magnitude estimation of the diameter of the grating area illuminated by the laser and make a prediction for the grating resolving power from Eq. (24). Are the two values consistent?

11. Restrict the diffraction grating field with the *0.16* mm slit and repeat step 9.

12. Replace the diffraction grating with the pinhole slide and sketch the resultant pattern.
    12a. (Bonus) Derive the Fourier transform of a circular aperture and compare to sketch.

13. (Bonus) Place the slide upon which an inverse Fourier transform of a grid is etched in the holder and describe the resulting pattern.

To be included with your lab write-up

1. Values for the distance *L* to the wall at all points in the experiment.
2. Your calculation for part 3.
3. Values for positions of maxima and sketches of patterns for parts 4 and 5.
4. Discussion of observations in part 6.
5. Values obtained in part 8.
6. Plot from part 9 along with value for slope and for resolution of laser.
7. Resolution of gratings from part 10 and 11.
8. Resolutions obtained in part 11.
9. A sketch of the diffraction pattern of the pinhole in step 12.